Moduli Spaces and Cone Complexes Part II: The Return of Analytic Spaces

These notes follow section 5 of [ACP15] and are a continuation of a seminar talk given by Wouter Zomervrucht.

Throughout we'll let $k = \overline{k}$.

Goal: Associate a cone complex to a toroidal embedding.

Definition 0.1. A *toroidal scheme* over k is a pair $U \subset X$ where U is an open subscheme of X that étale locally looks like the inclusion of the torus $T = \mathbb{G}_m^k$ into an affine toric variety. Namely, for all $p \in X$ there exists an étale neighbourhood $\alpha : V \to X$ of p and an étale morphism $\beta" V \to V_\sigma$ where V_σ is an affine toric variety such that $\beta^{-1}T = \alpha^{-1}U$.

Here an affine toric variety is an irreducible affine variety V containing a torus T as a Zariski open subscheme such that the action of T on itself extends to that of V.

Example 0.2. The example to keep in mind is $T = \mathrm{Spec} k[x, y]_{xy}$ inside $\mathbb{A}_k^2$, and analogues in higher dimensions.

Recall from Stefan's talk: Let k be equipped with the trivial valuation, X/k a scheme locally of finite type. The Berkovich analytification, X^{an} of X, is defined as a set follows:

Definition 0.3.

$$X^{an} = \{\mathrm{Spec} L \to X : L/K \text{ is a valued field extension}\}/\sim,$$

where $\sim$ is the equivalence relation given by $(\mathrm{Spec} L' \to \mathrm{Spec} L \xrightarrow{p} X) \sim (\mathrm{Spec} L \xrightarrow{p} X)$.

When $X = \mathrm{Spec} A$ is affine, this is equivalent to the set of seminorms on A that extend the valuation on k.

If T is the torus in $\mathbb{A}_k^r$, we can define the tropicalisation map $\mathrm{Trop} : X^{an} \to \mathbb{R}^n$ by sending a point $p : \mathrm{Spec} L \to X$ to $-(\log|p^\# x_1|, \dots, \log|p^\# x_r|)$.

We now consider an analytic subspace of this space.

Definition 0.4. Let X/k be separated and of finite type. Then $X^\beth$ is the analytic domain in X^{an} whose K-points (for K/k a valued field extension with valuation ring R) are the K-points that extend to $\mathrm{Spec}(R)$.

Remark 0.5. If $X = \mathrm{Spec} A$ is affine, $X^\beth$ is the set of seminorms $||$ extending that of k such that $|f| \leq 1$ for all $f \in A$.

If X/k is proper then $X^\beth = X^{an}$ by the valuative criterion.

Example 0.6. Let S_σ be an fs-monoid with generators $s_1, \ldots, s_n$, and $M = S_\sigma^{gp}$ the associated group. Let $\sigma = \mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0})$ be the dual cone. The affine toric variety $V_\sigma = \mathrm{Spec}k[S_\sigma]$ over k has torus $T = \mathrm{Spec}k[M]$.

Suppose now that $S = \mathbb{N}^n$, so that $V_\sigma = \mathbb{A}^n$.

Let K/k be a valued field extension, $x : \mathrm{Spec}K \to T$ a K-point of V_σ. This extends to an R-point if and only if $\mathrm{Trop}(x)$ is in σ, where $\mathrm{Trop}(x) = (-\log(|x^\#(s_1)|), \ldots, -\log(|x^\#(s_n)|))$

This should indicate that somehow $X^\beth$ is the correct analytic space to work in if we're interested in toroidal embeddings. In general, if $U \subset X$ is a toroidal embedding, Thuillier [Thu07] defines a natural continuous idempotent self map $p_X : X^\beth \to X^\beth$.

Definition 0.7. The skeleton $\overline{\Sigma}(X) \subset X^\beth$ is the image of $X^\beth$ under p_X.

This notation is suggestive. Thuillier showed that if $U \subset X$ is a toroidal embedding without self-intersection, the image of $U^{an} \cap X^\beth$ under p_X is canonically identified with $\Sigma(X)$, the cone complex associated to X. In defining p_X we will also see what $\Sigma(X)$ is.

The dual cone complex

Let $U \subset X$ be a toroidal embedding. We have, for any point $x \in X$, toroidal charts $\alpha : V \to X$ and $\beta : V \to V_\sigma$, where α is Zariski-open and V is affine. This is possible by the assumption we are without self-intersection.

Let S_σ denote the monoid of effective Cartier divisors on V supported in the compliment of $V \cap U$, and $\sigma = \mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0})$. Let $\overline{sigma}$ denote the extended cone associated to σ.

If $x \in X^\beth$ then x corresponds to a point of X over a valuation ring R. It's reduction $\overline{x}$ is a point of X over the residue field of R. Given a toric chart as above with image containing x, $p(x) : S_\sigma \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ takes a divisor $D \in S_\sigma$ with local equation f at x to $p(x)(D) = \mathrm{val}(f)$.

This is a well-defined monoid homomorphism, and so we have an element of $\overline{\sigma}$.

It remains to see why σ is contained in $X^\beth$. For this we will work locally, so we restrict to working on V, where $V^\beth$ is the set of valuations extending the trivial valuation on k that are non-zero on $k[V]$, the coordinate ring of V.

Choose a point $x \in V$ mapping to a point in the closed orbit $\mathcal{O}_\sigma \subset V_\sigma$ of the torus action (i.e. the points of V_σ not in T).

The completion of $\mathcal{O}_{X,x}$ is isomorphic to $k[[x_1, \ldots, x_r]][[S_\sigma]]$, the power series ring in variables x_i with exponents in S_σ, and where r is the dimension of $\mathcal{O}_\sigma$.

Given $f \in k[V]$, write $\sum_{u \in S_\sigma} a_u(f) z^u$ as the image of f in this ring.

Given $v \in \overline{\sigma}$, we obtain $\mathrm{val}_v : k[V] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ via

$$\mathrm{val}_v(f) = \min\{< u, v >: a_u(f) \neq 0\}.$$

As $v \in S_\sigma$, val_v is nonnegative, and so $\overline{\sigma} \subset V^{\daleth}$.

In general we cover X by the V's, such that $V^{\daleth}$ covers $X^{\daleth}$, and the union of the cones $\overline{\sigma}$ for each stratum of X forms $\overline{\Sigma}$.

References

[ACP15] Dan Abramovich, Lucia Caporaso, Sam Payne. (2015) The tropicalization of the moduli space of curves arXiv:1212.0373, 2014.

[Thu07] Amaury Thuillier. (2007) Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d'homotopie de certains schémas formel Manuscripta Math (2007), 381-451.