Motivation Serre's GAGA: algebraic geometry (over $\mathbb{C}$) and analytic geometry are closely related.

If X is a finite type $\mathbb{C}$-scheme, we can 'analytify' X to get a 'complex-analytic space' X^{an} with a sheaf of analytic functions $O_{X^{an}}$. As a set, $X^{an} = X(\mathbb{C})$. For instance, the analytification of $\mathbb{A}^n_{\mathbb{C}}$ is $\mathbb{C}^n$ with the Euclidean topology and with the sheaf of holomorphic functions.

We get:

$$\begin{aligned} X \text{ is proper} &\iff X^{an} \text{ is compact;} \\ X \text{ is separated} &\iff X^{an} \text{ is Hausdorff;} \\ X \text{ is connected} &\iff X^{an} \text{ is connected;} \\ X \text{ is of dimension } n &\iff X^{an} \text{ is of (topological) dimension } n. \end{aligned}$$

We can analytify morphisms of $\mathbb{C}$-schemes, O_X-modules, and coherent sheaves. Serre's GAGA paper provides a dictionary between the language of algebraic geometry on one side and analytic geometry on the other. In many cases it turns out that these languages are the same!

Now, let K be a non-archimedean field. We can try analytifying finite type schemes over K in a similar way. However, the topologies we obtain in this way are not as useful. The induced topology on K^n is totally disconnected: the connected components are singletons. If the absolute value on K is trivial, we even get a discrete space! Berkovich analytification patches the 'holes' in these spaces.

Multiplicative seminorms Let A be a ring. A *multiplicative seminorm* $|\cdot|$ on A is a function $f \mapsto |f|$ such that $|0| = 0$, $|1| = 1$, $|fg| = |f| \cdot |g|$, and $|f + g| \leq |f| + |g|$. The kernel $\ker |\cdot| = \{f \in A : |f| = 0\}$ of a multiplicative seminorm is a prime ideal.

Multiplicative seminorms on fields are the same as absolute values: their kernel is trivial. A *valued field* is a field with an absolute value. A *valued field extension* is a field extension L/K with K and L valued field such that the absolute value on L extends the one on K.

Non-archimedean A multiplicative seminorm $|\cdot|$ is *non-archimedean* if the stronger inequality $|f + g| \leq \max(|f|, |g|)$ holds. A seminorm is non-archimedean if and only if the image of $\mathbb{Z}$ in A is bounded. Extensions of non-archimedean multiplicative seminorms are again non-archimedean.

If $|f| \neq |g|$ then equality $|f + g| = \max(|f|, |g|)$ holds. This shows that the maximum of $|f|$, $|g|$ and $|f + g|$ is achieved at least twice. (Does this remind us of anything?)

From now on, we assume that K is a non-archimedean field with absolute value $|\cdot|_K$.

Berkovich analytification Let X be a K-scheme locally of finite type. To X we assign a space X^{an}, called the *Berkovich analytification* of X, defined as follows.

As a set

$$X^{an} = \{\mathrm{Spec}(L) \to X : L/K \text{ valued field extension}\}/\sim$$

where $\sim$ is the equivalence relation generated by setting

$$(\mathrm{Spec}\, L' \to \mathrm{Spec}\, L \xrightarrow{p} X) \sim (\mathrm{Spec}\, L \xrightarrow{p} X)$$

for valued extensions $L'/L/K$.

Equivalently, points of X^{an} are pairs $(x, |\cdot|)$, consisting of points $x \in X$ and absolute values on $\kappa(x)$ extending $|\cdot|_K$.

Affine case Let $X = \operatorname{Spec} A$ be an affine K-scheme of finite type. We have a natural bijection

$$X^{an} \xrightarrow{\sim} \{\text{multiplicative seminorms on } A \text{ extending } |\cdot|_K\}$$

Given a multiplicative seminorm $\|\cdot\|$ on A, the corresponding point in X^{an} is the point

$$\operatorname{Spec}(\operatorname{Frac}(A/\operatorname{Ker}\|\cdot\|)) \to X.$$

Topology Suppose again that $X = \operatorname{Spec} A$ is affine of finite type. Identify X^{an} with the set of multiplicative seminorms on A extending the absolute value on K. For all $f \in A$ there is an evaluation function

$$\nu_f : X^{an} \to \mathbb{R}_{\geq 0} : \|\cdot\| \mapsto \|f\|.$$

The topology on X^{an} is the coarsest topology such that ν_f is continuous for all $f \in A$.

In general: take opens, take analytifications, and glue.

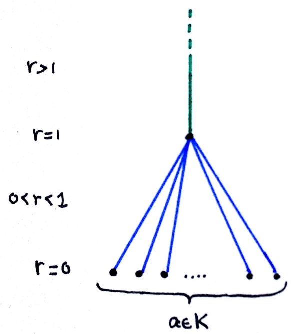
Projection There exists a natural surjective continuous map $X^{an} \to X$ (forgets absolute values). If $X = \operatorname{Spec} A$ is affine and we identify the points of X with multiplicative seminorms, the map $X^{an} \to X$ maps a multiplicative seminorm $\|\cdot\|$ to $\operatorname{Ker}(\|\cdot\|)$.

Functoriality Let $f : X \to Y$ be a morphism of K-schemes of finite type. Then f induces a natural continuous map $f^{an} : X^{an} \to Y^{an}$.

Example: the affine line Suppose for convenience that K is algebraically closed and that the absolute value on K is trivial, and let $X = \mathbb{A}^1_K = \operatorname{Spec} K[t]$. Exercise: the multiplicative seminorms on $K[t]$ extending the absolute value on K are:

- The trivial norm on $K[t]$;
- The multiplicative seminorm $\|f\| = r^{\deg f}$ (where $r > 1$);
- The multiplicative seminorm $\|f\| = r^{\operatorname{ord}_a f}$ (where $0 \leq r < 1$ and $a \in k$).

If we let $r \to 1$ in the second and third case, the corresponding seminorms converge to the trivial absolute valuation. In the third case, $r = 0$ is a special case: this norm is given by $\|f\| = |f(a)|$, and the kernel is nontrivial: it is the ideal generated by $(t - a)$. All the other seminorms have trivial kernel. This inspires us to draw the following picture:



The map $X^{an} \to X$ sends the points at $r = 0$ to the corresponding closed points of the affine line. The points with $r > 0$ are multiplicative seminorms with trivial kernel, so they get mapped to the generic point of X.

Note that the topology on the space in the picture is *not* the topology you'd get by gluing all the branches together in the point corresponding to the trivial absolute value. For every $f \in K[t]$, the evaluation function ν_f is identically 1 on all but finitely many branches. This implies that every open neighbourhood of the trivial absolute value in X^{an} contains infinitely many of such branches.

The affine line in general In the case that the absolute value on K is not trivial, the analytification of the affine line looks way more wild! There will be a lot of branch points.

Properties and analytification Let X be a scheme locally of finite type over K. Then

- X is separated iff X^{an} is Hausdorff;
- X is proper iff X^{an} is compact;
- X is connected iff X^{an} is arcwise connected.

Tropicalization Let $X \subseteq (\mathbb{G}_{m,K})^n$ be a closed subscheme. So X is the spectrum of $A = K[x_1^{\pm}, \ldots, x_n^{\pm}]/I$ with I some ideal in $K[x_1^{\pm}, \ldots, x_n^{\pm}]$. Let $p : \operatorname{Spec} L \to X^{an}$ be a point. That is, we have a valued extension, together with a point in $X(L)$. Then p defines a map on rings $p^{\#} : A \to L$. We define

$$\operatorname{Trop}(p) := (\log |p^{\#} x_1|, \ldots, \log |p^{\#} x_n|) \in \mathbb{R}^n.$$

If we view X^{an} as the set of multiplicative seminorms on A extending the one on K, and p corresponds to the multiplicative $\|\cdot\|$ on A, we have

$$\operatorname{Trop}(p) = (\log \|x_1\|, \ldots, \log \|x_n\|).$$

We get a map $\operatorname{Trop} : X^{an} \to \mathbb{R}^n$, and this map is continuous. The image $\operatorname{Trop}(X)$ of this map is the *tropicalization* of X.

Example Let $f = x + y - 1$ and consider its zero locus $X = V(f) \subseteq \mathbb{G}_m^2$. Let $\|\cdot\|$ be a multiplicative seminorm on $K[x^{\pm}, y^{\pm}]/(x + y - 1)$. As $y = 1 - x$, we see that the maximum of $\log \|x\|$, $\log \|y\|$ and $\log \|x + y\| = \log \|1\| = 0$ occurs at least twice. This shows that

$$\operatorname{Trop}(X) \subseteq \operatorname{Trop}(x \oplus y \oplus 0).$$

In fact, this inclusion is an equality! For example, if $(r, 0)$ with $r \leq 0$ is a point on the horizontal branch of $\operatorname{Trop}(x \oplus y \oplus 0)$, then this point is the tropicalization of the multiplicative seminorm

$$\left\| \sum_i a_i x^i \right\| = \max_i (|a_i| \exp(ri)).$$

A priori this is only defined on $K[x]$ but can be extended to A in a natural way.

Exercise: find multiplicative seminorms that induce points on the other branches of $\operatorname{Trop}(x \oplus y \oplus 0)$.

Kapranov's Theorem The above example is no coincidence!

Let $f = \sum_{\mathbf{i} \in \mathbb{Z}^n} a_{\mathbf{i}} x^{\mathbf{i}} \in K[x_1^{\pm}, \ldots, x_n^{\pm}]$, and set $X = V(f) \subseteq (\mathcal{G}_{m,K})^n$. Define the tropical polynomial

$$f^{\text{trop}} = \bigoplus_{\mathbf{i} \in \mathbb{Z}^n} \log |a_{\mathbf{i}}| \odot x^{\odot \mathbf{i}}.$$

Then

$$\text{Trop}(X) = \text{Trop}(f^{\text{trop}}).$$