Tropical polynomials + plane curves: These notes are taken from lecture notes by Hannah Markwig at the Stockholm Master Class I attended in 2017.

Tropical geometry is "geometry over the max-plus-semiring". It provides a combinatorial framework to study degenerations in alg. geometry.

$$\begin{tikzcd} \text{Alg Varieties} \arrow[r, bend left, "\text{Tropicalization}"] & \text{Trop Varieties} \arrow[l, bend left, "\text{Realization}"] \end{tikzcd}$$

Def: The tropical semiring ~~$(\mathbb{R}\cup\{\infty\}, \max, +)$~~ is $(\mathbb{R}\cup\{\infty\}, \oplus, \otimes)$ w/ operations:

- $x \oplus y = \min(x,y)$

  $x \otimes y = x+y$.

Remark: Tropical geometry is performed over non-Archimedean fields K w/ $v: K^\times \to \mathbb{R}$. The tropical semiring can be viewed as such a non-Arch valuation, where $v(ab)=v(a)+v(b)$ and $v(a+b) \geq \min\{v(a),v(b)\}$.

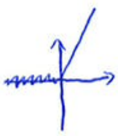
* Equivalently, we can work over the max-plus algebra $(\mathbb{R}\cup\{-\infty\}, \dot{\oplus}, \otimes)$ where $a \dot{\oplus} b = \max\{a,b\}$. HM and GM uses this. ~~We will not~~. We will.

Def: A tropical polynomial is an expression of the form $\bigoplus_\alpha a_\alpha \odot x_1^{\odot \alpha_1} \odot \cdots \odot x_n^{\odot \alpha_n} = \underset{\min}{\max}\{a_\alpha + \alpha_1 x_1 + \cdots + \alpha_n x_n\}$.

Tropical monomials are ~~give~~ p.w. linear fns $\mathbb{R}^n \to \mathbb{R}$.

polynomials give p.w. " " , w/ finitely many pieces.

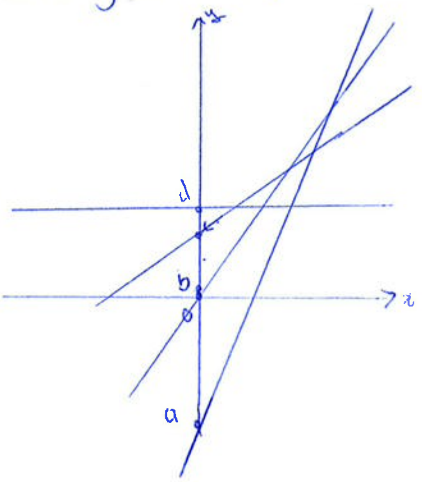
Ex: $0 \oplus x^{\odot 2} = \max\{0, 2x\}$.



We can have extraneous terms, ie.

Ex $f_1 = 0 \oplus x \oplus x^{\odot 2}$ $f_2 = 0 \oplus x^{\odot 2}$ are equivalent as functions, so not unique. rep.

Ex (HM): $f(x) = a \odot x^{\odot 3} \oplus b \odot x^{\odot 2} \oplus c \odot x \oplus d = \max\{d, c+x, b+2x, a+3x\}$



One can show all 4 lines contribute if $d-c \leq c-b \leq b-a$,

and one can then factor f as

$$a \odot (x \oplus (d-c)) \odot (x \oplus (c-b)) \odot (x \oplus (b-a)).$$

The points $(d-c)$, $(c-b)$, and $(b-a)$ are where the maximum value is achieved at least twice.

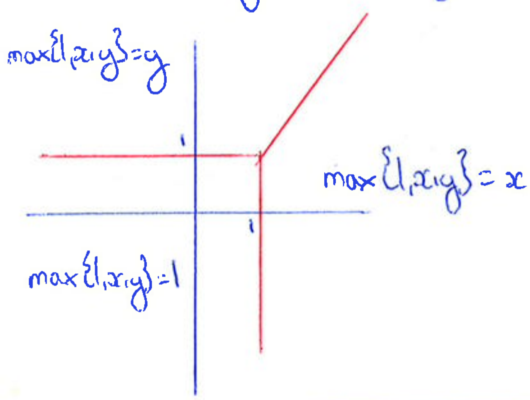
Def: Let f be a tropical polynomial. Then

$$\mathrm{Trop}(f) := \{x \in \mathbb{R}^n : \text{the max of the monomials of } f \text{ is achieved at least twice}\}$$

is the <u>tropical hypersurface</u> of f.

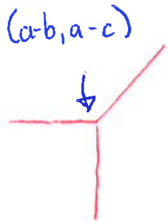
The case $n=2$ is of particular interest and gives tropical plane curves:

Example: $f = 1 \oplus x \oplus y = \max\{1, x, y\}$. $\mathrm{Trop}(f) = \{(a,b) : \max\{1,a,b\} \text{ occurs at least twice}\}$



This is the "mascot" of tropical geometry.

In general, given $f = a \oplus b \odot x \oplus c \odot y = \max\{a, b+x, c+y\}$



Definition: A (polyhedral) cone is a subset of $\mathbb{R}^n$ that is the intersection of finitely many half-spaces passing through 0. A fan is a finite set F of cones in the same space such that

(i) if $\sigma \in F$ and τ is a face of σ, $\tau \in F$.

(ii) if $\sigma, \sigma' \in F$ then $\sigma \cap \sigma'$ is a face of σ and σ'.

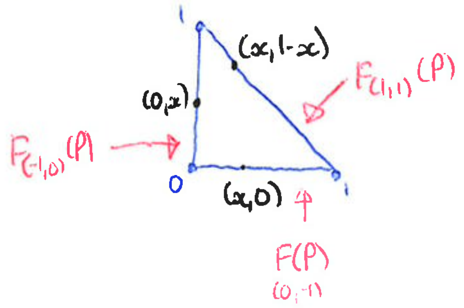
Def: Let $P \in \mathbb{R}^n$ be a polyhedron, $w \in (\mathbb{R}^n)^\vee$. Then $\mathrm{face}_w(P) = \{x \in P : w \cdot x \geq w \cdot y \ \forall y \in P\}$ is the face at which w is maximized.

If P is a polyhedron, the outer normal ~~face of~~ fan is the fan with cones:

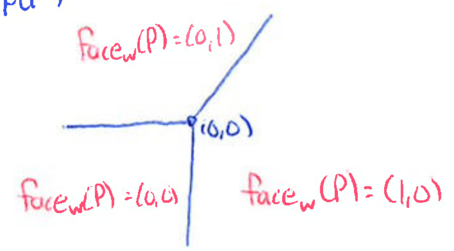
$$N_P(F) = \{ w \in (\mathbb{R}^n)^\vee : \mathrm{face}_w(P) = F \}.$$

for F a face of P.

Example: P:



$N_P(F)$



and in general vertices give top dimensional cones.

Let's see how this relates to tropical curves:

Given a tropical curve $f = \oplus a_\alpha \odot \underline{x}^{\odot\alpha}$, let $\mathrm{Newt}(f) = \mathrm{conv}(\alpha : a_\alpha \neq 0 \text{ or } -\infty)$ be the Newton polytope of f.

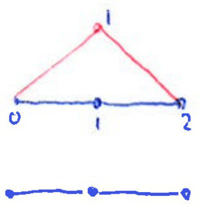
Def:
The dual Newton subdivision of f: ~~is the upper convex hull of~~ First form

$$\{ (\alpha, h) \in \mathrm{Supp}(f) \times \mathbb{R} : h \leq \alpha \}.$$

Its upper faces project to $\mathrm{Newt}(f)$, and the resulting complex is the dual Newton subdivision

Ex: $f = 0 \oplus 1 \odot x \oplus 0 \odot x^2$

$\mathrm{Newt}(f) = [0,2]$.



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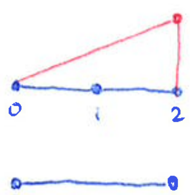
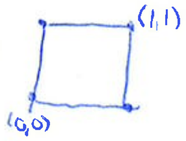
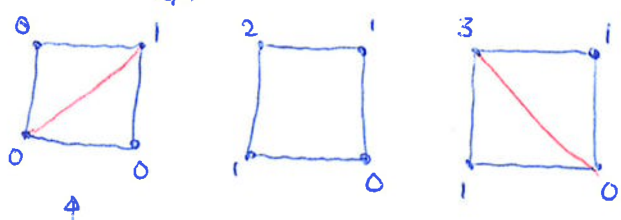


$f = a_0 \oplus a_x \odot x \oplus a_y \odot y \oplus a_{xy} \odot x \odot y$

$\mathrm{Newt}(f) =$





ie. $f = 0 \oplus 0 \odot x \oplus 0 \odot y \oplus 1 \odot x \odot y$

Proposition: Let f be a plane tropical curve. Then $\mathrm{Trop}(f)$ is dual to the dual Newton subdivision:

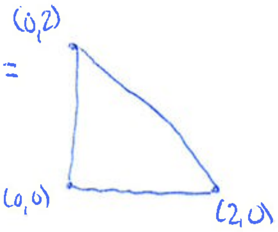
- vertices of $\mathrm{trop}(f) \cong$ polygons of the subdivision
- edges of $\mathrm{trop}(f) \cong$ orthogonal edges of the subdivision.

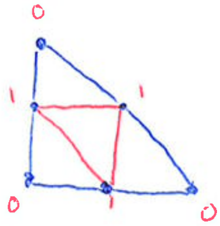
If we have a polygon of the subdivision w/ vertices $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^2$, this gives a vertex (x,y) of $\mathrm{Trop}(f)$ satisfying:

$$a_{\alpha_1} + (x,y)\cdot\alpha_1 = a_{\alpha_2} + (x,y)\cdot\alpha_2 + a_{\alpha_3} + (x,y)\cdot\alpha_3 \geq a_{\alpha_r} + (x,y)\cdot\alpha_r \quad \forall \text{ other } \alpha \in \mathrm{Supp}(f).$$

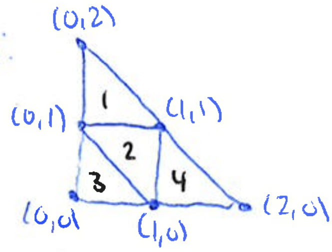
Example: $f = 0 \oplus 0 \odot x^{\odot 2} \oplus 0 \odot y^{\odot 2} \oplus 1 \odot x \oplus 1 \odot y \oplus 1 \odot x \odot y$.

Newt(f) =





Dual Newton subdivision:



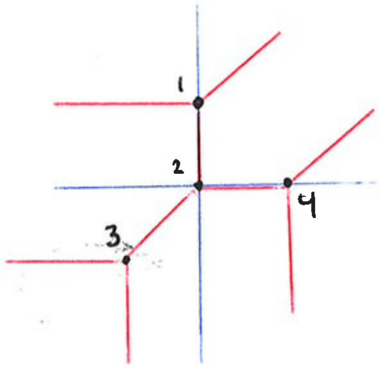
By the Proposition, Trop(f) has 4 vertices:

1: $2y = 1+y = 1+x+y \Rightarrow x = 0, y = 1.$

2: $1+x = 1+y = 1+x+y \Rightarrow x = y = 0.$

3: $0 = 1+x = 1+y \Rightarrow x = y = -1$

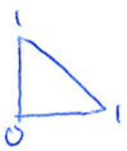
4: $2x = 1+x+y = 1+x \Rightarrow y = 0, x = 1.$

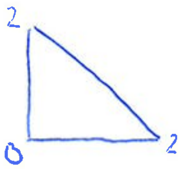


with the edges determined by the subdivision.

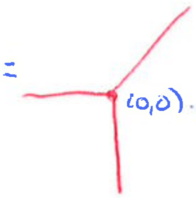
However! Different dual subdivisions can yield the same tropical curves.

$f_1 = 0 \oplus x \oplus y$ $\quad f_2 = 0 \oplus x^{\odot 2} \oplus y^{\odot 2}$





but $\mathrm{Trop}(f_i)$ =



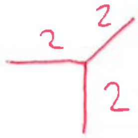
So we add more information by adding weights to Trop(f).

Def: Let Trop(f) be a plane tropical curve, and let e be an edge of Trop(f). The <u>weight</u> of e is the lattice length of the dual edge in the dual Newton subdivision of f.

(i.e. lattice pts we pass through).

Example:

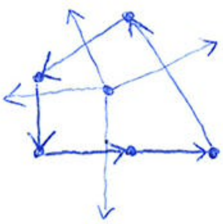
Trop(f_1) = 

Trop(f_2) = 

Theorem: Let Trop(f) be a plane tropical curve, p a vertex, $e_1, \dots, e_r$ the incident edges with weights $w_1, \dots, w_r$, and let $v_1, \dots, v_r$ denote the primitive integer vectors in the directions of the e_i's <u>away</u> from p. Then

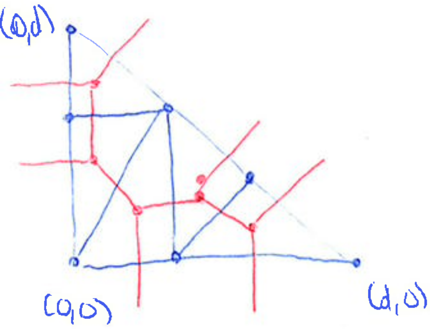
$$\sum_{i=1}^{r} m_i \vec{v}_i = 0.$$

Proof idea: Let Δ_p denote the polygon in the dual Newton ~~poly~~ subdivision corr to p. Orient the sides counterclockwise:



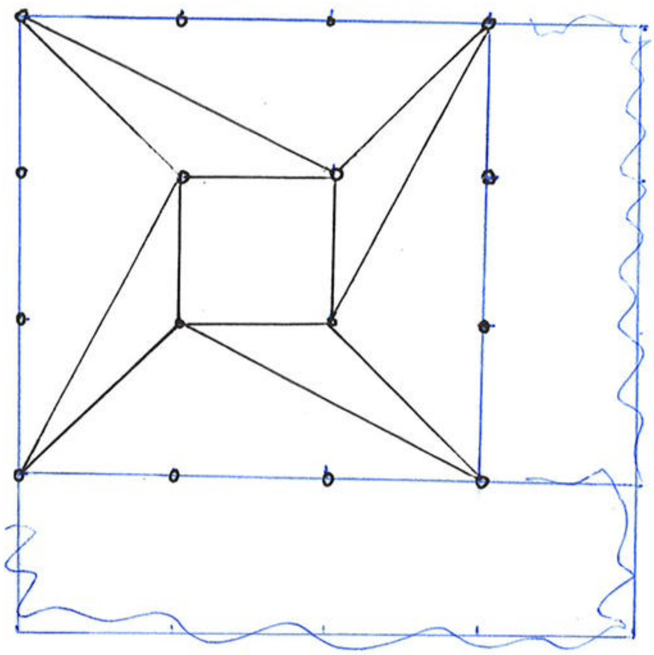
The vectors $w_i v_i$ correspond to the edges of Δ_p rotated clockwise by $\frac{\pi}{2}$, and hence sum to 0.

Def: A tropical curve has degree d if it is dual to a subdivision of $\mathrm{Conv}\{(0,0),(d,0),(0,d)\}$.



Next time: Abstract tropical curves (weighted graphs)!

Can this be the dual Newton subdivision of a tropical polynomial in 2 variables?



A: No. For example, each vertex in the middle square would have to be higher than the previous one (counter-clockwise).