Counting curves

Rosa Winter
21/03/2018

These are notes for the Tropical Geometry seminar at Leiden University in the spring of 2018, see `http://pub.math.leidenuniv.nl/~winterrl/TropicalGeometry.html`.

1 Motivation

In complex enumerative geometry, people are intereseted in finding the number $N(g,d)$ - the number of curves in $\mathbb{P}^2_{\mathbb{C}}$ of genus g and degree d through $3d+g-1$ points.

Example 1.1. $N(0,1) = N(0,2) = 1$, $N(0,3) = 12$.

In 1994, Kontsevich found a recursive fromula for $N(0,d)$. In 1998, Caporaso and Harris found recursive formulas for general genus.

In 2005, Mikhalkin computed these numbers using tropical geometry ([Mik05]). He showed that we can obtain these numbers by counting certain lattice paths in polygons, which makes the computations much easier. In this talk we will define $N^{trop}(g,d)$. Mikhalkin's Correspondence Theorem states that $N(g,d) = N^{trop}(g,d)$.

2 Counting tropical curves

Recall:
$M^{trop}_{g,n}(\Delta)$ is the set of isomorphism classes of connected n-marked parametrized tropical curves of degree Δ and genus at most g.

Remark 2.1. We will restrict ourselves to connected n-marked parametrized tropical curves in this talk. However, everything can be defined for disconnected curves as well.

Each tropical plane curve is the image of a parametrized tropical curve because of the balancing condition. We therefore want to use $M^{trop}_{g,n}(\Delta)$ to count complex curves. However, the whole space will be too big; first we need to know which elements in $M^{trop}_{g,n}(\Delta)$ are such that their image in $\mathbb{R}^2$ is a plane tropical curve.

Definition 2.2. The combinatorial type of an n-marked parametrized tropical curve is the combinatorial type of the n-marked abstract tropical curve, together with the data of the slopes of all the images of the edges. By $M^{trop,\alpha}_{g,n}(\Delta)$ we denote the subset of $M^{trop}_{g,n}(\Delta)$ of curves of type α.

Lemma 2.3. *Let α be a combinatorial type with m bounded edges. Then the image of the map*

$$M^{trop,\alpha}_{g,n}(\Delta) \longrightarrow \mathbb{R}^2,\ (\Gamma, x_1, \ldots, x_n, h) \longmapsto (h(x_1), \|h(b_1)\|, \ldots, \|h(b_m)\|),$$

where $\{b_1, \ldots, b_n\}$ is the set of bounded edges of Γ, is an open convex polyhedron in $\mathbb{R}^{2+\#\Gamma_0}$.

Proof. See [Mar06], Lemma 4.21. □

Definition 2.4. An n-marked parametrized tropical curve of combinatorial type α and genus $g' \leq g$ is regular if $M_{g,n}^{trop,\alpha}(\Delta)$ has dimension $2 + \#\Gamma_0 - 2g'$. Otherwise it is called superabundant.

Theorem 2.5. *(Speyer's Theorem) For every regular parametrized curve (Γ, h), there is a complex curve that can be associated to a tropical curve equal to $h(\Gamma)$.*

Proof. See [Mar06], Theorem 4.27. □

Remark 2.6. To see how complex curves relate to plain tropical curves, see for example [Mik05], 3.5.

From this theorem is follows that it makes sense to only count regular curves, as they relate to complex curves.

Remark 2.7. If α is a regular n-marked parametrized tropical curve of genus g with all vertices 3-valent, then it has $n+\#\Delta$ unbounded edges, and $n+\#\Delta-3+3g$ bounded edges. Then the dimension of $M_{g,n}^{trop,\alpha}(\Delta)$ is equal to $2+n+\#\Delta-3+3g-2g = n+\#\Delta+g-1$. So to count a finite number of curves trough n points, we want $n = \#\Delta + g - 1$.

From now on, we will set $g = 0$ so that the definitions become easier. However, all of the following can be done for arbitrary genus. We also set $n = \#\Delta - 1$ from now on.

Definition 2.8. An n-marked parametrized tropical curve is called relevant if it has no contracted bounded edges in the image in $\mathbb{R}^2$, and all vertices that have double edges adjacent to them are at least 4-valent (this is enough for what we need in this talk; for the full definition, see [Mar06], 4.36).
By $\widetilde{M}_{0,n}^{trop}(\Delta)$ we denote the subset of $M_{0,n}^{trop}(\Delta)$ of relevant curves.

Remark 2.9. By restricting to counting only 3-valent relevant curves, we do not lose any complex curves; every complex curve corresponds to a plane tropical curve that can be parametrized by a relevant curve (see Remark 4.37 in [Mar06]).

Theorem 2.10. *All 3-valent curves in $\widetilde{M}_{0,n}^{trop}(\Delta)$ are regular.*

Proof. See [Mar06], Corollary 4.42. □

We now know that the image in $\mathbb{R}^2$ of a 3-valent n-marked parametrized tropical curve in $\widetilde{M}_{0,n}^{trop}$ relates to a complex curve over K. But of course, different complex curves can define the same tropical plane curve. Therefore we want to define a multiplicity for elements in $\widetilde{M}_{0,n}^{trop}$.

Recall: we have an i^{th} evaluation map

$$\mathrm{ev}_i : \widetilde{M}_{0,n}^{trop}(\Delta) \longrightarrow \mathbb{R}^2,\ (\Gamma, x_1, \ldots, x_n, h) \longmapsto h(x_i).$$

Definition 2.11. By ev we denote the evaluation map

$$\mathrm{ev} = \mathrm{ev}_1 \times \cdots \times \mathrm{ev}_n : \widetilde{M}_{0,n}^{trop}(\Delta) \longrightarrow \mathbb{R}^{2n},\ (\Gamma, x_1, \ldots, x_n, h) \longmapsto (h(x_1), \ldots, h(x_n)).$$

Lemma 2.12. *For α a 3-valent combinatorial type in $\widetilde{M}_{0,n}^{trop}(\Delta)$, the evaluation map restricted to $M_{0,n}^{trop,\alpha}(\Delta)$ is given by a square matrix.*

Proof. See [Mar06], Definition 4.63 and Remark 4.69. □

Definition 2.13. The multiplicity of a 3-valent curve in $\widetilde{M}_{0,n}^{trop}(\Delta)$ is defined to be the absolute value of the determinant of the evaluation map restricted to $M_{0,n}^{trop,\alpha}(\Delta)$.

Now that we defined the multiplicity of a curve in $\widetilde{M}_{0,n}^{trop}(\Delta)$, we can define the number of curves through n points in tropical general position.

Definition 2.14. A set of points $\mathcal{P} = (P_1, \ldots, P_n)$ in $\mathbb{R}^{2n}$ is in tropical general position if the combinatorial types of all curves in $\mathrm{ev}^{-1}(\mathcal{P})$ are 3-valent.

Definition 2.15. The number of tropical curves through $\mathcal{P} = (P_1, \ldots, P_n)$ points in general position of degree Δ and genus 0 counted with multiplicity is defined as

$$N_{irr}^{trop}(0, \Delta, \mathcal{P}) = \sum_{C \in \mathrm{ev}^{-1}(\mathcal{P})} \mathrm{mult}(C)$$

Remark 2.16. The above definition is well defined, since the sum on the right is finite: there are only finitely many combinatorial types α in $\widetilde{M}_{0,n}^{trop}(\Delta)$ ([Mar06], Lemma 4.30), and a combinatorial type with infinitely many preimages of the evaluation map has multiplicity zero.

Theorem 2.17. *For any choice of $\mathcal{P}$ in tropical general position, we have*

$$N_{irr}^{trop}(0, \Delta_d, \mathcal{P}) = N_{irr}(0, d).$$

Furthermore, there exists a configuration $\mathcal{Q} \subset (\mathbb{C}^*)^2$ of $3d+1$ points in general position such that for every tropical curve C of genus g and degree d passing through $\mathcal{P}$ we have mult(C) distinct complex curves of genus g and degree d passing through Q. These curves are distinct for distinct C

Proof. See [Mik05], Theorem 1. □

.

3 Counting lattice paths

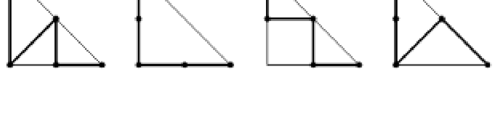
We will now show what makes Theorem 2.17 so powerful; we can determine $N^{trop}(g, \Delta_d)$ very easily using lattice paths.

Definition 3.1. A path $\gamma : [0, n] \longrightarrow \mathbb{R}^2$ is a lattice path if $\gamma|_{[j-1,j]}$ is affine-linear for $j = 1, \ldots, n$, and $\gamma(j) \in \mathbb{Z}^2$ for all $j = 0, \ldots, n$.

We now set $\varepsilon > 0$, and $\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}, (x, y) \longmapsto x - \varepsilon y$.

Definition 3.2. A lattice path γ is λ-increasing if $\lambda \circ \gamma$ is strictly increasing.

Example 3.3. This example is in [Mik05], below Definition 7.1. These are all λ-increasing paths for a triangle with vertices $(0, 0), (2, 0)$, and $(0, 2)$.



Definition 3.4. For a convex polygon Δ in $\mathbb{R}^2$, let $p, q \in \Delta$ be the vertices where $\lambda|_\Delta$ reaches its minimum and maximum respectively. The points p, q divide the boundary $\partial\Delta$ into two increasing lattice paths

$$\alpha_+ : [0, n_+] \longrightarrow \partial\Delta, \quad , \alpha_- : [0, n_-] \longrightarrow \partial\Delta,$$

where $\alpha_+(0) = \alpha_-(0) = p$, $\alpha_+(n_+) = \alpha_-(n_-) = q$, and α_+ goes clockwise, α_- counterclockwise.

Definition 3.5. Let $\gamma : [0, n] \longrightarrow \Delta \subset \mathbb{R}^2$ be an increasing lattice path such that $\gamma(0) = p$ and $\gamma(n) = q$. The path γ divides Δ into two closed regions: Δ_+ enclosed by γ and α_+, and Δ_- enclosed by γ and α_-.

Definition 3.6. We define the positive (resp. negative) multiplicity $\mu_\pm(\gamma)$ of the path γ inductively.

- We set $\mu_\pm(\alpha_\pm) = 1$.
- If $\gamma \neq \alpha_\pm$ take $1 \leq k \leq n$ to be the smallest number such that $\gamma(k)$ is a vertex of $\Delta_\pm$ with the angle less than π. If such k does not exist, we set $\mu_\pm = 0$
- If k exists, we consider two other increasing lattice paths connecting p and q,

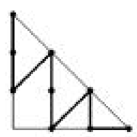
  $$\gamma' : [0, n-1] \longrightarrow \Delta \text{ and } \gamma'' : [0, n] \longrightarrow \mathbb{R}^2.$$

  We define γ' by $\gamma'(j) = \gamma(j)$ if $j < k$ and $\gamma'(j) = \gamma(j+1)$ if $j \geq k$.
  We define γ'' by $\gamma''(j) = \gamma(j)$ if $j \neq k$ and $\gamma''(k) = \gamma(k-1) + \gamma(k+1) - \gamma(k) \in \mathbb{Z}^2$.
  We set

  $$\mu_\pm(\gamma) = 2 \cdot \text{Area}(T)\mu_\pm(\gamma') + \mu_\pm(\gamma''),$$

  where T is the triangle with the vertices $\gamma(k-1), \gamma(k), \gamma(k+1)$.
- If $\gamma''(k) \notin \Delta$, then we set $\mu_\pm(\gamma'') = 0$.
- The multiplicity $\mu(\gamma)$ is then the product $\mu_+(\gamma) \cdot \mu_-(\gamma)$

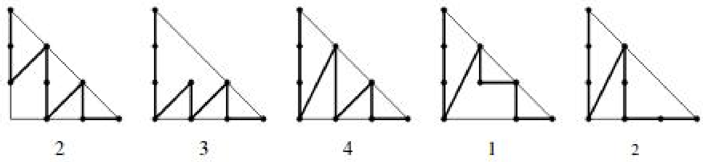
Example 3.7. This is Example 7.2 in [Mik05]. The following lattice path has multiplicity 2.



Theorem 3.8. . *The number $N^{trop}(g, \Delta)$ is equal to the number (counted with multiplicities) of λ-increasing lattice paths $[0, \#\Delta + g - 1] \longrightarrow \Delta$ connecting p and q. Furthermore, there exists a configuration $\mathcal{P} \in \mathbb{R}^2$ of $\#\Delta + g - 1$ points in tropical general position such that each λ-increasing lattice path encodes a number of tropical curves of genus g and degree Δ passing via $\mathcal{P}$ of total multiplicity $\mu(\lambda)$. These curves are distinct for distinct paths.*

Proof. See [Mik05], Theorem 2. ☐

EXAMPLE 3.9. This is Example 7.4 in [Mik05]. The following are all λ-increasing lattice paths in Δ_3 connecting p and q, with their multiplicities. We see that their sum gives the result $N(0, \Delta_3) = 12$.



4 References

[Mar06] Hannah Markwig. The enumeration of plane tropical curves. 2006. PhD dissertation.

[Mik05] G. Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.*, 18:313–377, 2005.