NOTES ON THE MINICOURSE "RUNGE APPROXIMATION, UNIQUE CONTINUATION AND THE FRACTIONAL CALDERÓN PROBLEM".

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ABSTRACT. These notes are lecture notes of a minicourse held at the Spring School "Local and Nonlocal Elliptic and Geometric Problems" at AIMS Senegal, Mbour (18.2.-22.2.19). They are based on recent works on the fractional Calderón problem, Runge approximation results and unique continuation properties.

1. LECTURE 1: THE FRACTIONAL CALDERÓN PROBLEM – SET-UP

1.1. **Motivation: The classical Calderón problem.** The classical Calderón problem is a prototypical inverse problem that became known through Calderón's work which had been motivated by his experience as an oil engineer in Argentina. In the context of geoprospection he had been interested in means of detecting for instance oil in an efficient way. To this end, he suggested to study an associated inverse problem using that different materials have different conducting properties. We will formulate the problem in the sequel and take it as a motivation for our study of the nonlocal Calderón problem which will be the main part of this minicourse.

The *direct* problem associated with the classical Calderón problem is given by the following elliptic PDE, which is also known as the *conductivity equation*:

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Here $\gamma : \Omega \rightarrow \mathbb{R}^{n\times n}$ is a *given* uniformly elliptic, bounded metric. For data $f \in H^{\frac{1}{2}}(\partial\Omega)$ this problem is well-posed and it is possible to define the Dirichlet-to-Neumann map

$$\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega), \ f \mapsto \gamma\nu \cdot \nabla u|_{\partial\Omega}.$$

Physically this system describes a conducting medium without interior sources or sinks of charge with possibly inhomogeneous, anisotropic conductivitiy γ for applied voltages f on the boundary and resulting currents $\gamma\nu \cdot \nabla u|_{\partial\Omega}$ on the boundary. The Dirichlet-to-Neumann map hence describes the *voltage*-to-*current* map at the boundary.

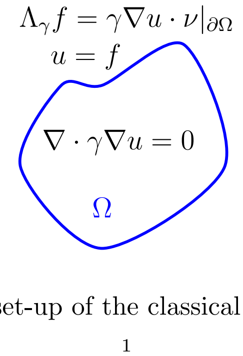

FIGURE 1. The set-up of the classical Calderón problem.

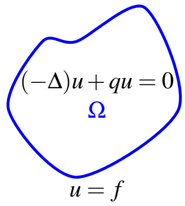

FIGURE 2. The set-up of the inverse problem for the Schrödinger equation.

The associated *inverse* problem deals with the following question:

Q. Assume that $\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is given, is it then possible to recover γ from this information?

With respect to the classical problem we hence reverse the question and assume that Λ_γ but *not* γ is given to us. We seek to deduce the function γ.

This problem turns out to be quite hard, due to

- its nonlinearity,
- the fact that γ is in the principal part of the operator.

Thus, here and in the sequel, we simplify the problem and assume that γ is isotropic (excluding interesting scenarios such as muscle tissue), i.e.

$$\gamma : \Omega \to \mathbb{R},\ 0 < \gamma \leq C < \infty.$$

This allows us to carry out a Liouville transform, i.e. to consider the function $w = \gamma^{\frac{1}{2}} u$ which transforms the conductivity equation (1) for u into a *Schrödinger equation* for w

$$\begin{aligned} (-\Delta + q)w &= 0 \text{ in } \Omega, \\ w &= g \text{ on } \partial\Omega, \end{aligned} \tag{2}$$

where $q = \frac{\Delta \gamma^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}}$. Again we can define (under mild conditions) the associated Dirichlet-to-Neumann map

$$\Lambda_q : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega),\ g \mapsto \nu \cdot \nabla w|_{\partial\Omega}.$$

Again, we can also consider an associated inverse problem:

Q. Assume that $\Lambda_q : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is given, is it then possible to recover q from this information?

This is the form of the problem which we seek to consider (for a nonlocal equation) in the sequel. The problem formulation (2) now has the advantage that the unknown is in the *lower order* part of the operator which simplifies the mathematical treatment of the problem.

The question on the recovery of q can be split into several smaller subquestions:

- *Uniqueness*: Is it true that if $\Lambda_{q_1} = \Lambda_{q_2}$, that then $q_1 = q_2$? For the classical Calderón problem this was first proved in [SU87].
- *(Conditional) Stability*: Is it true that if $\Lambda_{q_1} \sim \Lambda_{q_2}$, that then roughly $q_1 \sim q_2$ (in a sense that is to be made precise)? In general, this is wrong. The inverse problems which we are going to consider are extremely *ill-posed* and one can only expect stability under a priori conditions (that is why the property is called *conditional* stability). That this is indeed the case was first proved in [Ale88].
- *Recovery*: Given Λ_q, is it possible to algorithmically reconstruct q? This was first studied and proved in [Nac88].

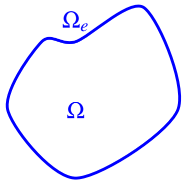

FIGURE 3. The fractional Schrödinger equation is posed on a bounded domain. As the fractional Laplacian is a nonlocal operator, we have to impose boundary data in the whole complement.

Motivated by the classical Calderón problem, in the sequel, we are going to study analogous problems for a non-local problem. For more information on the (classical) Calderón problem we refer to the survey article [Uhl09] and the references therein.

Exercise 1.1. *Assume that*

$$(\gamma(x)u'(x))' = 0 \text{ in } (0,1)$$

and that

$$u(0), u(1), \gamma(0)u'(0), \gamma(1)u'(1)$$

are given. Determine $\gamma(x)$*!*

1.2. **The fractional Calderón problem.** In this section, we follow the presentation of the articles [GSU16] and [RS17a]. We will set-up and solve the uniqueness question for the fractional Calderón problem.

1.2.1. *Formal question.* Before formulating the problem precisely, let us give a formal version of the question that we will study in the sequel. In analogy to (2) we first consider the following direct problem

$$\begin{aligned} ((-\Delta)^s + q)u &= 0 \text{ in } \Omega, \\ u &= f \text{ in } \Omega_e = \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \tag{3}$$

In the sequel, we will always work under the following assumptions

Ω is open, bounded and Lipschitz, $q \in L^\infty(\Omega)$, $s \in (0,1)$.

Under mild conditions on the potential q the direct problem (3) is well-posed and we will be able to define the analogue of the Dirichlet-to-Neumann map as

$$\Lambda_q : H^s(\Omega_e) \to (H^s(\Omega_e))^*, \ f \mapsto (-\Delta)^s u|_{\Omega_e}.$$

The inverse problem that we are going to study will be the following question:

Q. Assume that $\Lambda_q : H^s(\Omega_e) \to (H^s(\Omega_e))^*$ is given, is it then possible to recover q from this information?

In the sequel, we phrase our question in more careful terms.

1.2.2. *The direct problem.* Before turning to the inverse problem, we discuss the direct problem (3). In particular, we consider its well-posedness.

To this end, we start by defining the associated functional analytic set-up:

Definition 1.2. *We will use the following definitions:*

- *For $u \in \mathcal{S}$, we define*
$$(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u)(x),$$
where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{S}$ denotes the Schwartz functions.
- $H^{\mu}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} : \int\limits_{\mathbb{R}^n} (1+|\xi|^2)^{\mu} |\mathcal{F} u|^2 d\xi < \infty\}$*, $\mu \in \mathbb{R}$. In particular, we obtain that $(-\Delta)^s : H^{\mu}(\mathbb{R}^n) \to H^{\mu-2s}(\mathbb{R}^n)$.*
- $H^{\mu}(\Omega) := \{u|_{\Omega} : u \in H^{\mu}(\mathbb{R}^n)\}$*, with* $\|u\|_{H^{\mu}(\Omega)} := \inf\{\|W\|_{H^{\mu}(\mathbb{R}^n)},\ W|_{\Omega} = u\}$,
- $\tilde{H}^{\mu}(\Omega) =$ *closure of $C^{\infty}_c(\Omega)$ in $H^{\mu}(\mathbb{R}^n)$.*
For Lipschitz domains and $\mu \geq 0$ this is also equal to $H^{\mu}_{\overline{\Omega}} := \{u : \mathbb{R}^n \to \mathbb{R} :\ supp(u) \subset \overline{\Omega}\}$.
- *We have*
$$(\widetilde{H}^{\mu}(\Omega))^* = H^{-\mu}(\Omega),\ (H^{\mu}(\Omega))^* = \widetilde{H}^{-\mu}(\Omega),\ \mu \in \mathbb{R}.$$

For more information on these function spaces we refer to the book [McL00].

With this notation fixed, we study the solvability of the Dirichlet problem
$$\begin{aligned} ((-\Delta)^s + q)u &= F \text{ in } \Omega,\\ u &= f \text{ in } \Omega_e. \end{aligned} \tag{4}$$

Proposition 1.3 (Well-posedness)**.** *Let $B_q : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$ be given by*
$$B_q(u,v) := ((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v)_{\mathbb{R}^n} + (qu,v)_{\Omega}.$$
Then, there exists a countable set $\Sigma := \{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{R}$ with
$$\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$
with the following property: If $\lambda \in \mathbb{R} \setminus \Sigma$, then for any $F \in (\widetilde{H}^s(\Omega))^*$, $f \in H^s(\mathbb{R}^n)$ there exists a unique $u \in H^s(\mathbb{R}^n)$ such that
$$B_q(u,w) - \lambda(u,w)_{\Omega} = F(w) \text{ for all } w \in \widetilde{H}^s(\Omega),\ u - f \in \widetilde{H}^s(\Omega). \tag{5}$$
Moreover,
$$\|u\|_{H^s(\mathbb{R}^n)} \leq C(\|F\|_{(\widetilde{H}^s(\Omega))^*} + \|f\|_{H^s(\mathbb{R}^n)}),\ u - f \in \widetilde{H}^s(\Omega). \tag{6}$$

Remark 1.4. *The equation (5) is the weak form of the equation (4) with potential $q - \lambda$. We emphasise that the proof of Proposition 1.3 shows that the solution is independent of the values of $f|_{\Omega}$. In particular, it is possible to strengthen the a priori estimate (6) to infer*
$$\|u\|_{H^s(\mathbb{R}^n)} \leq C(\|F\|_{(\widetilde{H}^s(\Omega))^*} + \|f\|_{H^s(\Omega_e)}),\ u - f \in \widetilde{H}^s(\Omega). \tag{7}$$

Exercise 1.5. *Discuss why the definition of a weak solution as above is reasonable (by comparing it to the definition of a weak solution to a second order elliptic PDE).*
Prove the well-posedness result by
- *showing that for a properly chosen constant $\nu \geq 0$ the bilinear form $B_q(\cdot,\cdot) + \nu(\cdot,\cdot)_{\Omega}$ defines a scalar product,*
- *and applying the Riesz representation theorem.*

In the sequel, we will always agree on the following convention:

Convention 1.6. *We assume that $0 \notin \Sigma$.*

As a consequence of this convention we will have well-posedness of (4) (in its weak form). This will allow us to define the Dirichlet-to-Neumann map for this problem as a graph. We remark that even if we drop this condition and allow for $0 \in \Sigma$, then it is possible to formulate the associated inverse problem using the associated *Cauchy data* (see [RS18] and [LL17]).

1.2.3. *The Dirichlet-to-Neumann map.* We use the bilinear form from above to define the Dirichlet-to-Neumann map associated with our problem:

Lemma 1.7. *The map*

$$\Lambda_q : H^s(\Omega_e) \to (H^s(\Omega_e))^*$$

defined by $(\Lambda_q(f), g) = B_q(u_f, e_g)$, *where* $f, g \in H^s(\Omega_e)$ *and* $e_g \in H^s(\mathbb{R}^n)$ *is an extension of* g *into* $\mathbb{R}^n$ *and where* u_f *is a weak solution to* (3), *is well-defined, linear and bounded. Moreover, it is symmetric*

$$(\Lambda_q(f), g) = (f, \Lambda_q(g)).$$

We view the map Λ_q as the generalisation of the Dirichlet-to-Neumann map of the classical Calderón problem.

Proof. We start by discussing the well-definedness. To this end, let $\varphi, \psi \in \widetilde{H}^s(\Omega)$. Then, by definition of a weak solution to the Dirichlet problem and the fact that $0 \notin \Sigma$

$$B_q(u_{f+\varphi}, g + \psi) = B_q(u_{f+\varphi}, g) = B_q(u_f, g).$$

More precisely, the first equality is a consequence of the fact that $u_{f+\varphi}$ is a weak solution, and the second one follows from the observation that the solution is independent of the restriction of the data to Ω.

In order to observe the boundedness, we choose e_g to be an extension of g such that $\|e_g\|_{H^s(\mathbb{R}^n)} \leq C\|g\|_{H^s(\Omega_e)}$ and note

$$\begin{aligned}|(\Lambda_q f, g)| = |B_q(u_f, e_g)| &\leq |((-\Delta)^{\frac{s}{2}} u_f, (-\Delta)^{\frac{s}{2}} e_g)| + |(q u_f, e_g)| \\ &\leq \|(-\Delta)^{\frac{s}{2}} u_f\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} e_g\|_{L^2(\mathbb{R}^n)} + \|q\|_{L^\infty(\Omega)} \|u_f\|_{L^2(\Omega)} \|e_g\|_{L^2(\Omega)} \\ &\leq C\|f\|_{H^s(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)}.\end{aligned}$$

Taking the infimum over all possible extensions then yields

$$|(\Lambda_q f, g)| \leq C\|f\|_{H^s(\Omega_e)} \|g\|_{H^s(\Omega_e)}.$$

The symmetry follows by choosing $e_g = u_g$, where u_g is a solution to the Dirichlet problem with data g. □

With the definition of the Dirichlet-to-Neumann map in hand, we easily infer Alessandrini's indentity. For the inverse problem this will play a crucial role, as it will allow us to relate the given data to the unknown potentials.

Lemma 1.8 (Alessandrini's identity)**.** *Let* $q_1, q_2 \in L^\infty(\Omega)$. *For all* $f_1, f_2 \in H^s(\Omega_e)$ *we have*

$$((\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2)_{\Omega_e} = ((q_1 - q_2) u_1, u_2)_\Omega,$$

where the functions u_j *are (weak) solutions to*

$$\begin{aligned}((-\Delta)^s + q_j) u_j &= 0 \text{ in } \Omega, \\ u_j &= f_j \text{ in } \Omega_e.\end{aligned}$$

Proof. We have

$$\begin{aligned}((\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2)_{\Omega_e} &= (\Lambda_{q_1} f_1, f_2) - (f_2, \Lambda_{q_2} f_2)_{\Omega_e} \\ &= B_{q_1}(u_1, u_2) - B_{q_2}(u_1, u_2) = ((q_1 - q_2) u_1, u_2)_\Omega.\end{aligned}$$

□

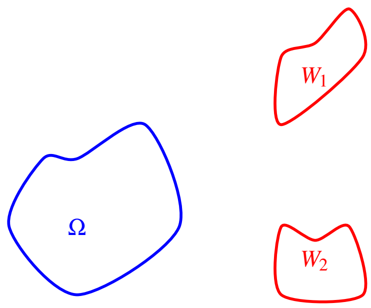

FIGURE 4. The set-up of Theorem 1. We compute the generalised Dirichlet-to-Neumann map on smooth functions supported in W_1 and only evaluate it in the domain W_2.

Remark 1.9. *We remark that as indicated in the introduction on the fractional Calderón problem, under regularity assumptions it is possible to compute the Dirichlet-to-Neumann map explicitly: Under suitable regularity it holds that*

$$\Lambda_q f = (-\Delta)^s u_f|_{\Omega}.$$

Indeed, we have (at least formally)

$$\begin{aligned}(\Lambda_q f, g) &= ((-\Delta)^{\frac{s}{2}} u_f, (-\Delta)^{\frac{s}{2}} e_g)_{\mathbb{R}^n} + (q u_f, e_g)_{\Omega} = ((-\Delta)^s u_f, e_g)_{\mathbb{R}^n} + (q u_f, e_g)_{\Omega}\\ &= ((-\Delta)^s u_f, e_g)_{\Omega} + ((-\Delta)^s u_f, e_g)_{\Omega_e} + (q u_f, e_g)_{\Omega}\\ &= ((-\Delta)^s u_f, e_g)_{\Omega_e} = ((-\Delta)^s u_f, g)_{\Omega_e}.\end{aligned}$$

In this sense, the Dirichlet-to-Neumann map is explicit and should rather be understood as a "Dirichlet-to-adjoint Dirichlet" map.

Exercise 1.10. *Discuss at which points in the previous identification of Λ_q regularity was used!*

With the previous results in hand, we now have a precise version of the problem from **Q** at hand. Moreover, we can formulate our first main result:

Theorem 1. *Let $q_1, q_2 \in L^\infty(\Omega)$, $W_1, W_2 \subset \Omega_e$ be open and let $\Lambda_{q_1}, \Lambda_{q_2}$ be as above.*
If for all $f \in C_c^\infty(W_1)$ we have

$$\Lambda_{q_1}(f)|_{W_2} = \Lambda_{q_2}(f)|_{W_2},$$

then $q_1 = q_2$ in Ω.

The answers the uniqueness question in the case of the fractional Calderón problem. Moreover, it only requires partial data. It is possible to strengthen it to larger classes of potentials.

Exercise 1.11 (Dimension counting)**.**

(i) *In the local Calderón problem use that*

$$\Lambda_q f(x) = \int\limits_{\partial\Omega} k_q(x,y) f(y) dy$$

is known for every $f \in C_c^\infty(\partial\Omega)$ to argue that the inverse problem is

$$\begin{cases} \text{underdetermined } \textit{for } n = 1,\\ \text{determined } \textit{for } n = 2,\\ \text{overdetermined } \textit{for } n \geq 3.\end{cases}$$

(ii) *In the nonlocal Calderón problem: Arguing similarly as in (i), show that the problem is always overdetermined.*

In the next lecture we are going to discuss the proof of Theorem 1. Key ingredients will be the Alessandrini identity and approximation results.

Remark 1.12. *It is possible to strengthen the statement of Theorem 1 substantially by for instance allowing irregular potentials in suitable multiplier spaces. This for instance allows one to (essentially) treat scaling critical potentials, e.g.* $q \in L^{\frac{n}{2s}}(\Omega)$ *or even* $q \in W^{-1,\frac{n}{s}}(\Omega)$*. For more information on this we refer to* [RS17a]*.*

2. Lecture 2: Runge Approximation and Global Unique Continuation

In this lecture we are going to prove the result from Theorem 1. Using Alessandrini's identity (Lemma 1.8), we are lead to proving density statements for solutions of our fractional Schrödinger equation (3).

2.1. Global Uniqueness and Approximation.

In the course of proving the uniqueness result from Theorem 1 we will rely on the following global uniqueness result:

Theorem 2. *Let* $u \in H^r(\mathbb{R}^n)$ *for some* $r \in \mathbb{R}$*. If*

$$u \equiv 0 \text{ and } (-\Delta)^s u \equiv 0 \text{ in the same open set,}$$

then $u \equiv 0$ *in* $\mathbb{R}^n$*.*

Exercise 2.1. *The result of Theorem 2 is an extremely nonlocal statement. Discuss why this cannot hold for* $s = 1$*!*

Theorem (2) can be obtained as a consequence of nonlocal unique continuation arguments for the fractional Laplacian [FF14, Seo14, Rül15, GFR19]. We postpone its proof to the next lecture and first turn to its consequences.

It will turn out that Theorem (2) is essentially dual to the following approximation property.

Theorem 3. *Let* $\Omega \subset \mathbb{R}^n$ *be open, bounded, Lipschitz,* $\Omega_1 \subset \mathbb{R}^n$ *open such that* $\Omega \subset \Omega_1$ *and* $\text{int}(\Omega_1 \setminus \Omega) \neq \emptyset$*. Then for any* $f \in L^2(\Omega)$ *and for any* $\epsilon > 0$ *there exists a function* $u \in H^s(\mathbb{R}^n)$ *such that*

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \ \ supp(u) \subset \Omega_1$$

and

$$\|u - f\|_{L^2(\Omega)} < \epsilon.$$

Remark 2.2. *The validity of such an approximation property was first proved in* [DSV14] *(in* C^k *topologies, to which this result could also be upgraded, see Section 7 in* [GSU16]*). In the form that it is stated here it goes back to* [GSU16]*. Quantitative versions which also hold for a significantly larger set of potentials are studied in* [RS17b]*.*

Recently the results have also been extended significantly in that it was shown that they remain valid for a large class of operators. We refer to [DSV16, CDV18a, CDV18b, Kry18] *for qualitative and to* [RS17a] *to quantitative results.*

We reformulate the approximation result. To this end, we define the *Poisson* operator

$$P_q : H^s(\Omega_e) \to H^s(\mathbb{R}^n), \ f \mapsto u,$$

where u is a weak solution to (3). With this notation in hand, we seek to prove the following approximation result:

Lemma 2.3. *The set*

$$\mathcal{R} := \{u|_\Omega : \ u = P_q f, \ f \in C_c^\infty(\Omega)\}$$

is dense in $L^2(\Omega)$.

We remark that Lemma 2.3 entails the statement of Theorem 3.

Proof. We argue by using the Theorem of Hahn-Banach: Using the Hahn-Banach Theorem, it suffices to show that for any $v \in L^2(\Omega)$ such that $(v, w)_\Omega = 0$ for all $w \in \mathcal{R}$, we have $v \equiv 0$.

Let us assume that $v \in L^2(\Omega)$ is orthogonal to all of $\mathcal{R}$, i.e.

$$(v, P_q f)_\Omega = 0 \text{ for all } f \in C_c^\infty(W). \tag{8}$$

We claim that

$$(v, P_q f) = -B_q(\varphi, f) \text{ for all } f \in C_c^\infty(W), \tag{9}$$

where $\varphi \in H^s(\mathbb{R}^n)$ is a solution to the dual problem

$$\begin{aligned} ((-\Delta)^s + q)\varphi &= v \text{ in } \Omega, \\ \varphi &= 0 \text{ in } \Omega_e. \end{aligned}$$

Indeed, this follows from the definition of a solution to the problem (3) and the dual equation:

$$(v, P_q f)_\Omega = (v, u_f - f)_\Omega = B_q(\varphi, u_f - f) = -B_q(\varphi, f).$$

Here the first identity follows from the fact that $\text{supp}(f) \subset W$, the second from the definition of a weak solution (for φ) and the third from the definition of a weak solution (for u_f).

Combining (8) and (10), we obtain

$$B_q(\varphi, f) = 0 \text{ for all } f \in C_c^\infty(W),$$

where φ is as above. Since $f|_\Omega = 0$, we hence infer that $B_q(\varphi, f) = 0$ is equivalent to

$$0 = ((-\Delta)^{\frac{s}{2}}\varphi, (-\Delta)^{\frac{s}{2}} f)_{\mathbb{R}^n} = ((-\Delta)^s \varphi, f)_{\mathbb{R}^n} \text{ for all } f \in C_c^\infty(W).$$

As a consequence, $(-\Delta)^s \varphi|_W = 0$ in W. As $W \subset \Omega_e$, we also have $\varphi|_W = 0$. As a result, Theorem 2 implies that $\varphi = 0$, whence we also obtain that $v = 0$. By the theorem of Hahn-Banach, this concludes the density proof. ☐

We have now shown that the uniqueness result from Theorem 2 implies the approximation result from Theorem 3. In fact a stronger statement holds: The unique continuation property for the dual equation is equivalent to the approximation result for the original equation.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be as above and $W \subset \Omega_e$ open. Then the following are equivalent*

(a) *For any $\epsilon > 0$ and for any $v \in L^2(\Omega)$ there exists $f \in \widetilde{H}^s(W)$ such that*

$$\|v - P_q f\|_{L^2(\Omega)} \leq \epsilon.$$

(b) *Let $v \in L^2(\Omega)$ and assume that $w \in \widetilde{H}^s(\Omega)$ is a solution to*

$$\begin{aligned} ((-\Delta)^s + q)w &= v \text{ in } \Omega, \\ w &= 0 \text{ in } \Omega_e. \end{aligned} \tag{10}$$

Assume that $(-\Delta)^s w = 0$ in W. Then $w \equiv 0$ and $v \equiv 0$.

Such duality relations are well-known in the context of control theory problems. In this context the approximation result would be viewed as an approximate controllability result from the exterior. In the setting of the nonlocal problems studied here the duality from above was first formulated in [GRSU18].

Proof. The implication (b) $\Rightarrow$ (a) is a consequence of the Theorem of Hahn-Banach as explained in the proof of Theorem 3.

It remains to discuss the implication (a) $\Rightarrow$ (b). To this end let $v \in L^2(\Omega)$ and $w \in \tilde{H}^s(\Omega)$ be a solution of (10). Assume that $(-\Delta)^s w = 0$ in W. We need to show that $v \equiv 0$ and $w \equiv 0$. By the assumed approximation property (a), we know that for any $\psi \in L^2(\Omega)$ there exists $f \in \widetilde{H}^s(W)$ such that $\|\psi - P_q f\|_{L^2(\Omega)} \leq \epsilon$. Thus,

$$\begin{aligned}(v,\psi)_\Omega &= (v, \psi - P_q f)_\Omega + (v, P_q f)_\Omega \\ &= (v, \psi - P_q f)_\Omega - ((-\Delta)^s w, f)_W \\ &= (v, \psi - P_q f).\end{aligned}$$

Here we used the dual equation to infer that $(v, P_q f)_\Omega = -((-\Delta)^s w, f)_W$ and the vanishing of $(-\Delta)^s w|_W$.

By the approximation condition (a) this however implies

$$|(v,\psi)_\Omega| \leq \|v\|_{L^2(\Omega)} \|v - P_q f\|_{L^2(\Omega)} \leq \epsilon \|v\|_{L^2(\Omega)}.$$

Passing to the limit $\epsilon \to 0$, we hence obtain $(v, \psi) = 0$ for all $\psi \in L^2(\Omega)$, whence $v = 0$. By well-posedness this also implies that $w = 0$. ☐

2.2. **Proof of Theorem 1.** With the above approximation results in hand, we discuss the proof of Theorem 1:

Proof. By the Alessandrini identity from Lemma 1.8, we have that for all functions u_j solving (3) with data $f_j \in C_c^\infty(W_j)$ we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0. \tag{11}$$

Choosing $f \in L^2(\Omega)$ arbitrarily, Theorem 3 (or rather Lemma 2.3) implies that there exist sequences $\{u_j^k\}_{k\in\mathbb{N}} \subset H^s(\mathbb{R}^n)$ such that

$$((-\Delta)^s + q_j) u_j^k = 0 \text{ in } \Omega,$$

and the functions u_j^k have exterior values in $C_c^\infty(W_j)$ such that

$$u_1^k = f + r_1^k, \ u_2^k = 1 + r_2^k,$$

with $r_1^k, r_2^k \to 0$ in $L^2(\Omega)$ as $r_k \to 0$. Inserting this into the identity (11) we obtain

$$0 = \lim_{k\to\infty} \int_\Omega (q_1 - q_2)(f + r_1^k)(1 + r_2^k) dx = \int_\Omega (q_1 - q_2) f dx.$$

As this holds for all $f \in L^2(\Omega)$, we conclude that $q_1 = q_2$. ☐

3. Lecture 3: Unique Continuation and Carleman Estimates

In this lecture, we seek to discuss some background on the uniqueness result in Theorem (1). Our main tools here will be the Caffarelli-Silvestre extension operator as well as a Carleman estimate for solutions to the Caffarelli-Silvestre extension.

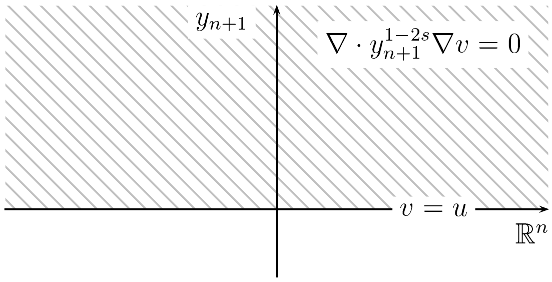

FIGURE 5. An illustration of the extension problem of Caffarelli-Silvestre.

3.1. **The Caffarelli-Silvestre Extension.** The Caffarelli-Silvestre extension allows us to localize the problem at hand. Instead of working with a nonlocal equation, we may consider the following local problem.

Theorem 4 ([CS07])**.** *Let $u \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. Then there exists an extension operator*

$$E_s : H^r(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n),\ u \mapsto \tilde{u} := E_s(u),$$

where $\tilde{u}$ is a solution to

$$\begin{aligned} \nabla \cdot x_{n+1}^{1-2s}\nabla \tilde{u} &= 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \tilde{u} &= u \text{ in } \mathbb{R}^n \times \{0\}, \end{aligned}$$

(where the boundary data are attained in an $H^r(\mathbb{R}^n)$ sense as $x_{n+1} \to 0$) and for some constant $c_{n,s} \neq 0$

$$\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{x_{n+1}}\tilde{u} = c_{n,s}(-\Delta)^s u \text{ in } H^{r-2s}(\mathbb{R}^n).$$

Moreover, if $u|_{B'_{2r}} \in C^\infty(B'_{2r})$, then

$$\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{x_{n+1}}\tilde{u} = c_{n,s}(-\Delta)^s u \text{ in } C^\infty(B'_r).$$

The last estimate is a consequence of the *pseudolocality* of the fractional Laplacian (see Lemma 4.3 in [RS17a]).

We seek to prove Theorem 2 with the help of this result. By means of the extension, it is possible to reformulate the global uniqueness from Theorem 2 as a *weak boundary unique continuation result.*

In order to in exploit this, we first show that in the setting of Theorem 2 to the infinite order of vanishing of $\tilde{u}$ at boundary points $x \in W \times \{0\}$:

Proposition 3.1. *Let $u \in H^s(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ with*

$$u = 0 \text{ and } (-\Delta)^s u = 0 \text{ in } B'_1.$$

Let $\tilde{u} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be the associated Caffarelli-Silvestre extension of u. Then, for any $m \in \mathbb{N}$ we have

$$\lim_{r\to 0} r^{-m} \int_{B_r^+(0)} x_{n+1}^{1-2s}\tilde{u}^2 dx = 0.$$

We omit the proof of this result. The main idea is to view the boundary data as Dirichlet and Neumann data for the Caffarelli-Silvestre extension, which is thus overdetermined in an open set. More precisely, in the tangential directions the vanishing follows from the boundary data, while in the normal directions the vanishing is obtained from the vanishing of the weighted normal derivative of the Caffarelli-Silvestre extension and by bootstrapping the equation.

Although there are many simpler proofs in this special case, it is an instructive exercise to prove the claim for $s = \frac{1}{2}$:

Exercise 3.2. *Prove the statement of Proposition 3.1 in the case* $s = \frac{1}{2}$.

With Proposition 3.1 in hand, it suffices to exclude non-trivial solutions vanishing of infinite order at the boundary. A priori such functions could occur, e.g. $f(x) = e^{-\frac{1}{|x|}}$. In order to exclude this, we introduce a *Carleman estimate* for this problem.

3.2. A Carleman Estimate.

In this section we conclude the proof of Theorem 2 by invoking a Carleman estimate:

Theorem 5. *Let* $\bar{u}$ *be a solution to*

$$\begin{aligned} \nabla \cdot x_{n+1}^{1-2s} \nabla \bar{u} &= 0 \text{ in } B_4^+, \\ \bar{u} &= 0 \text{ on } B_4' \end{aligned} \tag{12}$$

with $supp(\bar{u}) \subset B_4^+ \setminus \{0\}$. *Then for*

$$\phi(x) = \psi(\ln(|x|)) = -\ln(|x|) + \frac{1}{10}\left(\ln(|x|)\arctan(\ln(|x|)) - \frac{1}{2}\ln(1+\ln^2(|x|))\right),$$

there is a constant $C > 1$ *such that for all* $\tau \geq \tau_0 > 1$ *it holds*

$$\begin{aligned} &\tau^{\frac{3}{2}} \| e^{\tau\phi}(1+\ln(|x|))^{-\frac{1}{2}} |x|^{-1} x_{n+1}^{\frac{1-2s}{2}} \bar{u} \|_{L^2(\mathbb{R}^{n+1}_+)} + \tau^{\frac{1}{2}} \| e^{\tau\phi}(1+\ln^2(|x|))^{-\frac{1}{2}} x_{n+1}^{\frac{1-2s}{2}} \nabla \bar{u} \|_{L^2(\mathbb{R}^{n+1}_+)} \\ &\leq C \| e^{\tau\phi} |x| x_{n+1}^{\frac{2s-1}{2}} f \|_{L^2(\mathbb{R}^{n+1}_+)}. \end{aligned} \tag{13}$$

Remark 3.3. *We remark that the estimate (13) is uniform in* τ. *Morally, one should think of the weights as* $e^{\tau\phi(x)} \sim |x|^{-\tau}$. *As a consequence, the estimate allows to obtain extremely precise information at small radii (by choosing* τ *to be large).*

Remark 3.4. *The Carleman estimate from above can be strengthened considerably. For instance it is possible to consider variable coefficients* [Rül15, GFR19] *or non-trivial boundary conditions (which allows one to prove for instance strong unique continuation results).*

Relying on the estimates from Theorem 5, we provide the proof of Theorem 2:

Proof of Theorem 2. We split the argument into three steps:

Step 1: Cut-off argument. After mollification we may assume that $u \in H^s(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$. Let $\tilde{u}$ be the Caffarelli-Silvestre extension of u. Define $\bar{u} := \eta_\epsilon \tilde{u}$ with η_ϵ a smooth radial cut-off function such that

$$\text{supp}(\eta_\epsilon) \subset B_4^+ \setminus B_\epsilon^+, \ \eta_\epsilon = 1 \text{ in } B_3^+ \setminus B_{2\epsilon}^+, \ |\nabla^\alpha \eta_\epsilon(x)| \leq C|x|^{-\alpha} \text{ for } \alpha \in \mathbb{N}^n.$$

Thus, $\bar{u}$ is a solution to (12) with

$$f(x) = 2x_{n+1}^{1-2s} \nabla \eta_\epsilon \cdot \nabla \tilde{u} + \tilde{u} \nabla \cdot x_{n+1}^{1-2s} \nabla \eta_\epsilon.$$

Inserting this into the Carleman estimate (13) from Theorem 5, we obtain

$$
\begin{aligned}
&\tau^{\frac{3}{2}}\|e^{\tau\phi}(1+\ln(|x|))^{-\frac{1}{2}}|x|^{-1}x_{n+1}^{\frac{1-2s}{2}}\eta_\epsilon\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)} + \tau^{\frac{1}{2}}\|e^{\tau\phi}(1+\ln(|x|))^{-\frac{1}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla(\eta_\epsilon\tilde{u})\|_{L^2(\mathbb{R}^{n+1}_+)}\\
&\leq C\|e^{\tau\phi}|x|x_{n+1}^{\frac{2s-1}{2}}(2x_{n+1}^{1-2s}\nabla\eta_\epsilon\cdot\nabla\tilde{u} + \tilde{u}\nabla\cdot x_{n+1}^{1-2s}\nabla\eta_\epsilon)\|_{L^2(\mathbb{R}^{n+1}_+)}\\
&\leq C(\|e^{\tau\phi}|x|x_{n+1}^{\frac{1-2s}{2}}\nabla\eta_\epsilon\cdot\nabla\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)} + \|e^{\tau\phi}|x|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\Delta\eta_\epsilon\|_{L^2(\mathbb{R}^{n+1}_+)}\\
&\quad + \|e^{\tau\phi}|x|x_{n+1}^{\frac{2s-1}{2}}x_{n+1}^{-2s}\tilde{u}\eta_\epsilon'\frac{x_{n+1}}{|x|}\|_{L^2(\mathbb{R}^{n+1}_+)}).
\end{aligned}
$$

Step 2: Limit $\epsilon \rightarrow 0$. We only consider the contributions on the right hand side in the last estimate from Step 1. By the estimates for $|\nabla^\alpha \eta_\epsilon|$ we have

$$
\begin{aligned}
&\|e^{\tau\phi}|x|x_{n+1}^{\frac{1-2s}{2}}\nabla\eta_\epsilon\cdot\nabla\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)} + \|e^{\tau\phi}|x|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\Delta\eta_\epsilon\|_{L^2(\mathbb{R}^{n+1}_+)}\\
&\quad + \|e^{\tau\phi}|x|x_{n+1}^{\frac{2s-1}{2}}x_{n+1}^{-2s}\tilde{u}\eta_\epsilon'\frac{x_{n+1}}{|x|}\|_{L^2(\mathbb{R}^{n+1}_+)}\\
&\leq C(\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|_{L^2(B_{2\epsilon}^+\setminus B_\epsilon^+)} + \|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|^{-1}\tilde{u}\|_{L^2(B_{2\epsilon}^+\setminus B_\epsilon^+)}\\
&\quad + \|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|_{L^2(B_4^+\setminus B_3^+)} + \|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|^{-1}\tilde{u}\|_{L^2(B_4^+\setminus B_3^+)}).
\end{aligned}
$$

Invoking Caccioppoli's estimate, we further bound the gradient terms

$$
\begin{aligned}
&\|e^{\tau\phi}|x|x_{n+1}^{\frac{1-2s}{2}}\nabla\eta_\epsilon\cdot\nabla\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)} + \|e^{\tau\phi}|x|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\Delta\eta_\epsilon\|_{L^2(\mathbb{R}^{n+1}_+)}\\
&\quad + \|e^{\tau\phi}|x|x_{n+1}^{\frac{2s-1}{2}}x_{n+1}^{-2s}\tilde{u}\eta_\epsilon'\frac{x_{n+1}}{|x|}\|_{L^2(\mathbb{R}^{n+1}_+)}\\
&\leq C(\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|^{-1}\tilde{u}\|_{L^2(B_{3\epsilon}^+\setminus B_{\frac{1}{2}\epsilon}^+)} + \|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|^{-1}\tilde{u}\|_{L^2(B_5^+\setminus B_{\frac{7}{4}}^+)})\\
&\leq C(\epsilon^{-2\tau}\|x_{n+1}^{\frac{1-2s}{2}}|x|^{-1}\tilde{u}\|_{L^2(B_{3\epsilon}^+\setminus B_{\frac{1}{2}\epsilon}^+)} + \|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|^{-1}\tilde{u}\|_{L^2(B_5^+\setminus B_{\frac{7}{4}}^+)}).
\end{aligned}
$$

Passing to the limit $\epsilon \rightarrow 0$ and using the infinite order of vanishing of $\tilde{u}$ (see Proposition 3.1) then yields that

(14)
$$
\begin{aligned}
&\tau^{\frac{3}{2}}\|e^{\tau\phi}(1+\ln(|x|))^{-\frac{1}{2}}|x|^{-1}x_{n+1}^{\frac{1-2s}{2}}\eta_0\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)} + \tau^{\frac{1}{2}}\|e^{\tau\phi}(1+\ln(|x|))^{-\frac{1}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla(\eta_0\tilde{u})\|_{L^2(\mathbb{R}^{n+1}_+)}\\
&\leq C\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|^{-1}\tilde{u}\|_{L^2(B_5^+\setminus B_{\frac{7}{4}}^+)}.
\end{aligned}
$$

Step 3: Conclusion. Using (14) together with the monotonicity of $\phi(x)$ allows us to further conclude that

$$
\begin{aligned}
&e^{\tau\psi(\ln(1/2))}\|(1+\ln(|x|))^{-\frac{1}{2}}|x|^{-1}x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(B_{1/2}^+\setminus B_{1/2}^+)}\\
&\leq Ce^{\tau\psi(\ln(7/4))}\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|^{-1}\tilde{u}\|_{L^2(B_5^+\setminus B_{\frac{7}{4}}^+)}.
\end{aligned}
\tag{15}
$$

Passing to the limit $\tau \rightarrow 0$, we obtain a contradiction unless $\tilde{u}|_{B_{1/2}^+\setminus B_{1/4}^+} = 0$. Using the weak unique continuation property for uniformly elliptic equations (see Nicola's lecture), this concludes the proof. $\square$

Remark 3.5. *We remark that our argument was a very qualitative one. It would have been possible to obtain much more precise information, e.g. it is possible to obtain*

- *Doubling estimates,*
- *three spheres inequalities,*
- *homogeneous blow-ups.*

This allows to infer the strong unique continuation property for fractional Schrödinger equations as well as the unique continuation property from measurable sets (see [FF14] *for a frequency function argument for this and* [Rül15, GRSU18, GFR19] *for Carleman arguments for this). Moreover a blow-up classification is also possible (see* [KRS16]*). Further it is possible to consider variable coefficient equations and higher order operators* [Rül15]*, Section 7 and* [GFR19]*.*

Last but not least, we discuss the argument for Theorem 5:

Proof of Theorem 5. Again we agree in two steps:

Step 1: Introduction of conformal polar coordinates. We consider the conformal polar coordinates $x = e^t\theta$, where $\theta \in S^n_+$, $t \in \mathbb{R}$. Setting $\overline{w}(t,\theta) = e^{\frac{n-2s}{2}t}\tilde{u}(e^t\theta)$, we obtain the new equation

$$(\theta_n^{1-2s}\partial_t^2 - \theta_n^{1-2s}\frac{(n-2s)^2}{4} + \nabla_{S^n_+}\cdot\theta_n^{1-2s}\nabla_{S^n_+})\overline{w} = \overline{f} \text{ in } \mathbb{R}\times S^n_+, \qquad (16)$$
$$\overline{w} = 0 \text{ on } \mathbb{R}\times S^n_+.$$

Here $\overline{f}(t,\theta) = e^{\frac{n+2+2s}{2}t}f(e^t\theta)$, $\theta_n = \frac{x_{n+1}}{|x|}$, and all equations are interpreted weakly.

In conformal coordinates the desired estimate turns into

$$\tau^{\frac{3}{2}}\|e^{\tau\varphi}\theta_n^{\frac{1-2s}{2}}(\psi'')^{\frac{1}{2}}\overline{w}\|_{L^2(\mathbb{R}\times S^n_+)} \le C\|e^{\tau\psi}\theta_n^{\frac{2s-1}{2}}\overline{f}\|_{L^2(\mathbb{R}\times S^n_+)}.$$

Setting $\tilde{w}(t,\theta) := \theta_n^{\frac{1-2s}{2}}\overline{w}(t,\theta)$ and multiplying the equation (16) by $\theta_n^{\frac{2s-1}{2}}$ leads to the new bulk equation

$$(\partial_t^2 - \frac{(n-2s)^2}{4} + \tilde{\Delta}_{S^n_+})\tilde{w} = \theta_n^{\frac{2s-1}{2}}\bar{f}(t,\theta),$$

where $\tilde{\Delta}_{S^n_+} := \theta_n^{\frac{2s-1}{2}}\nabla_{S^n}\cdot\theta_n^{1-2s}\nabla_{S^n}\theta_n^{\frac{2s-1}{2}}$. The desired estimate then turns into

$$\tau^{\frac{3}{2}}\|e^{\tau\varphi}\tilde{w}\|_{L^2(\mathbb{R}\times S^n_+)} \le C\|e^{\tau\varphi}\tilde{f}\|_{L^2(\mathbb{R}\times S^n_+)}. \qquad (17)$$

Step 2: Conjugation. We view (17) as an estimate for the function $v := e^{\tau\varphi}\tilde{w}$. This equation solves a new bulk equation, which is determined by the conjugated operator

$$L_\psi := e^{\tau\psi}Le^{-\tau\psi} = \partial_t^2 + \tau^2|\psi'|^2 - \frac{(n-2s)^2}{4} + \tilde{\Delta} - 2\tau\psi'\partial_t - \tau\psi'',$$

where $L = \partial_t^2 - \frac{(n-2s)^2}{4} + \tilde{\Delta}_{S^n_+}$. We split the operator L_ψ into its (formally) symmetric and antisymmetric parts:

$$L_\psi = S_\psi + A_\psi,$$
$$S_\psi = \partial_t^2 + \tau^2|\psi'|^2 - \frac{(n-2s)}{4} + \tilde{\Delta}_{S^n_+},$$
$$A_\psi = -2\tau\psi'\partial_t - \tau\psi''.$$

We notice that as operators in t, θ, τ both operators S_ψ, A_ψ have a non-trivial characteristic set whose intersection is of co-dimension one. On this set, one cannot hope to obtain positivity from

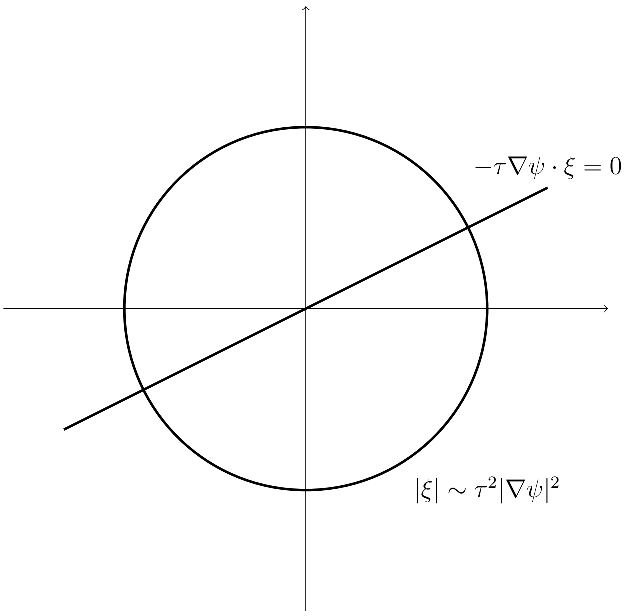

FIGURE 6. The characteristic sets of the conjugated symbols A_ψ (plane) and S_ψ (circle). Their intersection is a co-dimension two object. In order to obtain positivity in our estimates we thus have to deduce positivity from the commutator.

S_ψ, A_ψ, here positivity has to come from the next order expansion, the commutator. We make this more precise:

$$\begin{aligned}\|L_\psi v\|_{L^2}^2 &= \|(S_\psi + A_\psi)v\|_{L^2}^2\\ &= \|A_\psi v\|_{L^2}^2 + \|S_\psi v\|_{L^2}^2 + 2(S_\psi v, A_\psi v)\\ &= \|A_\psi v\|_{L^2}^2 + \|S_\psi v\|_{L^2}^2 + ([S_\psi, A_\psi]v, v),\end{aligned}$$

where $[S_\psi, A_\psi] := S_\psi A_\psi - A_\psi S_\psi$ denotes the commutator between S_ψ and A_ψ.

Computing the commutator, we obtain

$$([S_\psi, A_\psi]v, v) = 4\tau(\psi''\partial_t v, \partial_t v) + 4\tau^3(\psi''(\psi')^2 v, v) - 2\tau(\psi'''\partial_t v, v) - \tau(\psi''''v, v).$$

Using our explicit form of the weight ψ, we infer that the first two contributions in the commutator are positive and – for sufficiently large choice of τ – control the third and fourth contribution. As a consequence, if $\tau_0 > 1$ is sufficiently large, we conclude that

$$\begin{aligned}\|L_\psi v\|_{L^2}^2 &= \|A_\psi v\|_{L^2}^2 + \|S_\psi v\|_{L^2}^2 + ([S_\psi, A_\psi]v, v)\\ &= \|A_\psi v\|_{L^2}^2 + \|S_\psi v\|_{L^2}^2 + 2\tau(\psi''\partial_t v, \partial_t v) + 2\tau^3(\psi''(\psi')^2 v, v).\end{aligned}$$

Inserting $\psi(t) = t + \frac{c}{1+t^2}$ and undoing our change of coordinates, this concludes the argument. □

4. Lecture 4: Single Measurement Uniqueness and Algorithmic Reconstruction

Last but not least, we discuss the (algorithmic) reconstruction problem for the fractional Laplacian. Here we follow the presentation in the article [GRSU18] and discuss the following main result:

Theorem 6. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, bounded, open, Lipschitz. Let $s \in (0,1)$ and assume that $W_1, W_2 \subset \Omega_e$ are open sets such that $\overline{\Omega} \cap \overline{W}_j = \emptyset$. Assume further that either*

- *$s \in [1/4, 1)$ and $q \in L^{\infty}(\Omega)$,*
- *$q \in C^0(\overline{\Omega})$ (and $s \in (0,1)$ arbitrary),*

and that 0 is not a Dirichlet eigenvalue of $(-\Delta)^s + q$ in Ω. For any $f \in \widetilde{H}^s(W_1) \setminus \{0\}$, the potential q is then uniquely and algorithmically determined from the knowledge of $\Lambda_q(f)|_{W_2}$.

Remark 4.1.

- *The result of Theorem 6 is a single measurement uniqueness result. In contrast to the situation in Theorem 2, we do not need the full knowledge of the infinite dimensional Dirichlet-to-Neumann maps, but only a single evaluation of it.*
- *Dimension counting arguments as in Exercise 1.5 show that in the local case, a single measurment cannot be true. Also in the nonlocal case, the single measurement problem is exactly determined and no longer overdetermined.*
- *The restrictions on the regularity of q are a consequence of the availability of unique continuation results from measureable sets.*

The key idea leading to Theorem 6 consists in turning the global uniqueness results from Theorem 2 into a reconstruction algorithm.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, bounded, open, Lipschitz. Let $s \in (0,1)$ and assume that $W \subset \Omega_e$ is an open set such that $\overline{\Omega} \cap \overline{W} = \emptyset$. Then, any $v \in H^s(\mathbb{R}^n)$ with supp$(v) \subset \overline{\Omega}$ is uniquely determined by the knowledge of $(-\Delta)^s v|_W =: h$. The function v can be reconstructed from h as*

$$v = \lim_{\alpha \to 0} v_\alpha \text{ (in } H^s(\mathbb{R}^n)),$$

where v_α with $\alpha > 0$ is the unique solution to

$$v_\alpha = \operatorname{argmin}_{w \in \widetilde{H}^s(\Omega)} (\|(-\Delta)^s w - h\|^2_{H^{-s}(W)} + \alpha \|w\|^2_{H^s(\mathbb{R}^n)}).$$

Remark 4.3. *The minimization algorithm used in Proposition 4.2 is an instance of a* Tychonov regularization scheme *(see Chapter 2 in* [Isa06] *for more information on this).*

The difficulty in Proposition 4.2 is that the operator

$$v \mapsto (-\Delta)^s v|_W$$

is a compact operator. As a consequence, its inverse cannot be bounded (as one would else be in a finite dimensional space). This indicates that the reconstruction properties in Proposition 4.2 are rather delicate and highly unstable.

Proof. The result follows from the direct method in the calculus of variations in combination with the global uniqueness theorem from Theorem 2. We leave the details as an exercise to the reader. □

Exercise 4.4. *Prove the constructive part of Proposition 4.2!*

Let us describe the strategy of the proof of Theorem 6 given the reconstruction algorithm from Proposition 4.2.

We argue in two main steps:

(1) Reconstruction of u from $f, \Lambda_q(f)$ by means of Proposition 4.2: Set $u = f + v$ with $v \in \widetilde{H}^s(\Omega)$.
   - Define $h = g - (-\Delta)^s f|_{W_2} \in H^{-s}(W_2)$.
   - Use Proposition 4.2 to determine $v = \lim v_\alpha$ with v_α as in Proposition 4.2 with W_2 replacing W.

– Define $u = f + v \in H^s(\mathbb{R}^n)$.

(2) Determine $q = -\frac{(-\Delta)^s u}{u}$.

We explain this in more detail, assuming the result of Proposition 4.2.

Proof of Theorem 6 using Proposition 4.2. We argue in two main steps.

Step 1: Recovery of u. Let $f \in \widetilde{H}^s(W_1) \setminus \{0\}$ and $\Lambda_q f|_{W_2} = (-\Delta)^s u|_{W_2}$ be given. We set $u = v + g$ with $v \in \widetilde{H}^s(\Omega)$. In particular, we obtain that

$$(-\Delta)^s v|_{W_2} = (-\Delta)^s u|_{W_2} - (-\Delta)^s f|_{W_2} = \Lambda_q f|_{W_2} - (-\Delta)^s f|_{W_2}.$$

By virtue of Proposition 4.2, the function v can then be globally and algorithmically reconstructed from the known data f and $\Lambda_q f|_{W_2} - (-\Delta)^s f|_{W_2}$. Hence, as f and $\Lambda_q(f)$ (and thus also $(-\Delta)^s v|_{W_2}$) are known, we have thus algorithmically reconstructed the function $u = f + v$.

Step 2: Reconstruction of q. As the function u has already been globally reconstructed in Step 1, we solve the fractional Schrödinger equation for q:

$$q(x) = -\frac{(-\Delta)^s u(x)}{u(x)}.$$

In particular, we have recovered q constructively and algorithmically, if we can show the well-definedness of the quotient $\frac{(-\Delta)^s u(x)}{u(x)}$. We split the proof of the well-definedness into two cases:

Step 2a: $q \in C^0(\overline{\Omega})$. If $q \in C^0(\overline{\Omega})$, we have for each $x_0 \in \overline{\Omega}$

$$q(x_0) = \lim_{x_k \to x_0} q(x_k) = -\frac{(-\Delta)^s u(x_k)}{u(x_k)}.$$

By the weak unique continuation property for fractional Schrödinger equations, we however always find a sequence $x_k \to x_0$ such that $u(x_k) \neq 0$ (else we would obtain a violation against the weak unique continuation property). This concludes the argument.

Step 2b: $q \in L^\infty(\Omega)$. If $q \in L^\infty(\Omega)$, it suffices to show that the quotient $\frac{(-\Delta)^s u(x)}{u(x)}$ is well-defined almost everywhere. To this end, it suffices to show that $u(x)$ cannot vanish on sets of positive Lebesgue measure. This is proved in [GRSU18] in the range $s \in [1/4, 1)$. □

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