Complex Systems Methods — 6. Interdependence between time series: Granger causality and transfer entropy

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Overview

1. Interdependence measures for time-series
   - Causality
   - Granger causality
2. Operationalisation
   - Vector autoregressive models
   - Transfer entropy and Granger causality
3. Problems
   - General Problems of observational causality concepts
   - Specific problem: State Dependence
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4. Estimating causal relationships
5. Summary

Why measuring interdependencies?

Consider two processes $X(t)$ and $Y(t)$. Possible questions:

- Is there any connection between the processes? Are they correlated?
- Is there a causal connection between the two processes? Are they coupled?
- Is one process driving the other?
- In particular interesting if no detailed model is available, e.g. in Neurosciences (EEG data) or Econometrics

Causality

- There is a difference between correlation and causation.
- Reichenbachs principle: Two processes A and B are statistically dependent (correlated) if either A causes B, B causes A, or both A and B have a common cause C.
- Causality can be formalized using the concept of an intervention (Pearl): A causes B, if we can change B by intervening at (manipulating) A.
- In models from physics: B is coupled to A.

Some Notation

- "World": a set V of $1 \leq N < \infty$ elements (agents, nodes) with state sets $\mathcal{X}_v$, $v \in V$.
- Given a probability vector p on $\mathcal{X}_V$ we get random variables X_V on V, X_A on $A \subseteq V$ and X_v on $v \in V$.
- World dynamics described as stationary stochastic process $X_V(t) = \{X_v(t)\},\ v \in V$
- discrete time

Definition — Wiener 1958, Granger 1964, Granger 1969

- past $\overline{X}_V(t-1) = (X_V(t-1), \ldots, X_V(t-\infty))$
- subprocess $X_{-j} = X_{V\setminus\{j\}}$
- $\sigma(X_A(t)|\overline{X}_A(t-1))$ denotes the standard deviation of the error predicting $X_A(t)$ using $\overline{X}_A(t-1)$.

Definition (Causality)

X_j causes X_i, if $\sigma(X_i(t)|\overline{\mathbf{X}}_V(t-1)) < \sigma(X_i(t)|\overline{\mathbf{X}}_{-j}(t-1))$, i.e. if the knowledge of the past values of X_j will improve the prediction of X_i.

Definition (Instantenous Causality)

X_j instantaneously causes X_i, if
$\sigma(X_i(t)|\overline{\mathbf{X}}_V(t-1), X_j(t)) < \sigma(X_i(t)|\overline{\mathbf{X}}(t-1))$, i.e. if the knowledge of the the actual value of X_j will improve the prediction of X_i.

Definition — Granger 1980

Axiom A: The past and the present may cause the future, but the future cannot cause the past

Axiom B: $\overline{\boldsymbol{X}}(t)$ contains no redundant information, so that if some variable $X_k(t')$ is functionally related to one or more other variables, in a deterministic fashion, then $X_k(t')$ should be excluded from $\boldsymbol{X}(t)$.
E.g. $x_j(t) = f(x_k(t-m))$, but also
$x_j(t) = f(x_j(t-1), x_j(t-2), \ldots, x_j(t-m))$, i.e. Granger excludes deterministic systems.

Definition

X_j causes X_i if $p(x_i(t)|\overline{\boldsymbol{x}}_V(t-1)) \neq p(x_i(t)|\overline{\boldsymbol{x}}_{-j}(t-1))$, i.e. X_j **non-causes** X_i if $X_i(t)$ is conditionally independent on X_j given $\overline{\boldsymbol{X}}_{-j}(t-1)$.

Operationalisation: Vector autoregressive models (VAR)

- A weakly stationary zero mean stochastic process has an autoregressive representation

$$\boldsymbol{x}_V(t) = \sum_{u=1}^{\infty} \boldsymbol{a}(u)\boldsymbol{x}(t-u) + \boldsymbol{\epsilon}(t)$$

- X_j is Granger non-causal to X_i with respect to X_V if $a_{ij}(u) = 0 \quad \forall \quad u$.
  X_j instantaneously non-causes X_i, if $\Sigma_{ij} = \langle \epsilon_i(t)\epsilon_j(t) \rangle = 0$.
- In the context of Graphical models: structural equations
- Problem: Only linear dependencies!

Transfer entropy — Information theoretic version of Granger casuality

- Schreiber 2000: Transfer entropy measures "directed information flow"; originally only bivariate

$$\begin{aligned} T_{j\to i} &= MI(X_i(t) : \overline{X}_j(t-1)|\overline{X}_i(t-1)) \\ &= H(X_i(t)|\overline{X}_i(t-1)) - H(X_i(t)|\overline{X}_i(t-1), \overline{X}_j(t-1)) \end{aligned}$$

- Palus 2001: Measuring conditional independence using conditional mutual information $\Rightarrow$ information theoretic formulation of the Granger causality — X_j Granger causes X_i if $T_{j\to i,V} > 0$.

$$\begin{aligned} T_{j\to i,V} &= MI(X_i(t) : \overline{X}_j(t-1)|\overline{\boldsymbol{X}}_{-j}(t-1)) \\ &= H(X_i(t)|\overline{\boldsymbol{X}}_{-j}(t-1)) - H(X_i(t)|\overline{\boldsymbol{X}}_V(t-1)) \end{aligned}$$

General Problems of observational causality concepts

- World description has to be causally complete in order to exclude common causes.
- Granger causality defined via conditional independence is purely observational, no interventions.
- $\Rightarrow$ if X_i and X_j are synchronized no causal interaction is detected
- But, this case is excluded by Grangers Axiom B!

Specific problem: State Dependence

Whether e.g. X_1 Granger causes X_2 depends on the representation of the rest of the world!

$$
\begin{aligned}
x_1(t) &= a_{11}x_1(t-1) + a_{12}x_2(t-1) + a_{13}x_3(t-1) + \epsilon_1(t) \\
x_2(t) &= a_{21}x_1(t-1) + a_{22}x_2(t-1) + a_{23}x_3(t-1) + \epsilon_2(t) \\
x_3(t) &= a_{31}x_1(t-1) + a_{32}x_2(t-1) + a_{33}x_3(t-1) + \epsilon_3(t)
\end{aligned}
$$

can be transformed into

$$
\begin{aligned}
x_1(t) &= (a_{11} - a_{13}\alpha)x_1(t-1) + (a_{12} - a_{13}\beta)x_2(t-1) + a_{13}\tilde{x}_3(t-1) + \epsilon_1(t) \\
x_2(t) &= (a_{21} - a_{23}\alpha)x_1(t-1) + (a_{22} - a_{23}\beta)x_2(t-1) + a_{23}\tilde{x}_3(t-1) + \epsilon_2(t) \\
\tilde{x}_3(t) &= (a_{31} - (a_{33} + a_{11})\alpha) - a_{13}\alpha^2)x_1(t-1) + \\
&\quad (a_{32} - (a_{33} + a_{12})\beta - a_{13}\beta^2)x_2(t-1) + \\
&\quad (a_{33} - a_{13}\alpha - a_{23}\beta)\tilde{x}_3(t-1) + \epsilon_3(t)
\end{aligned}
$$

using $\tilde{x}_3 = x_3 + \alpha x_1 + \beta x_2$ with $\alpha = a_{21}/a_{23} \Rightarrow X_2$ becomes independent on X_1 conditioned on $\tilde{X}_3$.

Specific problem: Deterministic Dynamics

- Deterministic dynamical system:

$$\boldsymbol{x}(t) = F(\boldsymbol{x}(t-1))$$

- Embedding theorem: The map $\boldsymbol{x}(t) \mapsto s(t) = h(\boldsymbol{x}(t)) \mapsto (s(t), s(t-1), \ldots, s(t-m+1))$ is an immersion with nowhere vanishing Jacobian, if $m > 2D_0$ with D_0 the box-counting dimension of the attractor
- $\Rightarrow$ state space can be reconstructed from any X_i
- KS-entropy

$$\begin{aligned} h_{KS} &= \lim_{\epsilon \to 0} h(\boldsymbol{X}(t)|\overline{\boldsymbol{X}}(t-1), \epsilon) \\ &= \lim_{\epsilon \to 0} h(X_i(t)|\overline{X}_i(t-1), \epsilon) \end{aligned}$$

- $\Rightarrow MI(X_i(t) : \overline{X}_j(t-1)|\overline{\boldsymbol{X}}_{-j}(t-1)) = 0$ if $h_{KS} = 0$
- $\Rightarrow$ No Granger causality in non-chaotic deterministic systems.
- But again, this situation is excluded by Axiom B!

Examplex: Granger causality in a VAR(2) process

$$
\begin{aligned}
x_1(t) &= a_{11}x_1(t-1) + a_{12}x_2(t-1) + \epsilon_1(t) \\
x_2(t) &= a_{21}x_1(t-1) + a_{22}x_2(t-1) + \epsilon_2(t)
\end{aligned}
$$

In which way implies $a_{12} > 0$ better predictability of X_1 knowing X_2?
Predicting $X_1(t)$ using only $\overline{X}_1(t-1)$

$$
\begin{aligned}
x_1(t) &= a_{11}x_1(t-1) + a_{12}a_{21}x_1(t-2) \\
&\quad + a_{12}a_{22}x_2(t-2) + a_{12}\epsilon_2(t-1) + \epsilon_1(t) \\
&= a_{11}x_1(t-1) + a_{12}a_{21}x_1(t-2) + a_{12}a_{22}a_{21}x_1(t-3) \\
&\quad + a_{12}a_{22}^2x_2(t-3) + a_{12}a_{22}\epsilon_2(t-2) + a_{12}\epsilon_2(t-1) + \epsilon_1(t)
\end{aligned}
$$

Special case $a_{22} = 0$

$$
x_1(t) = a_{11}x_1(t-1) + a_{12}a_{21}x_1(t-2) + a_{12}\epsilon_2(t-1) + \epsilon_1(t)
$$

Transfer entropy and effective noise level

- Granger causality: Improving predictability $\equiv$ Reducing noise level
- Stochastic dynamics for $X_i(t)$:

$$x_i(t) = f(\overline{x}_i(t-1), \xi_i(t)) \quad \langle \xi_i(t)^2 \rangle = 1$$

- Differential entropy $H(X) = -\int dx\ p(x) \log p(x)$ transforms for invertible function $y = f(x)$ according to

$$H(Y) = H(X) + \int dx\ p(x)\ \log |f'(x)|$$

because

$$p(x)dx = q(y)dy \quad \Rightarrow \quad q(y) = \left. \frac{p(x)}{df/dx} \right|_{x=f^{-1}(y)}$$

Applying this we get

$$h(x_i(t)|\overline{x}_i(t-1)) = H(\xi_i) + \langle \ln \left| \frac{\partial f}{\partial \xi_i} \right| \rangle .$$

Transfer entropy and effective noise level

- Using only the dynamics for $X_i(t)$ we got

$$h(x_i(t)|\overline{x}_i(t-1)) = H(\xi_i) + \langle \ln \left| \frac{\partial f}{\partial \xi_i} \right| \rangle \, .$$

- Stochastic dynamics for $X_i(t)$ and $X_j(t)$:

$$x_i(t) = g(\overline{x}_i(t-1), \overline{x}_j(t-1), \xi_{ij}(t)) \quad \langle \xi_{ij}(t)^2 \rangle = 1$$

- Same reasoning gives

$$h(x_i(t)|\overline{x}_i(t-1), \overline{x}_j(t-1)) = H(\xi_{ij}) + \langle \ln \left| \frac{\partial g}{\partial \xi_{ij}} \right| \rangle \, .$$

- Therefore

$$T_{j \to i} = H(\xi_i) - H(\xi_{ij}) + \langle \ln \left| \frac{\partial f}{\partial \xi_i} \right| \rangle - \langle \ln \left| \frac{\partial g}{\partial \xi_{ij}} \right| \rangle$$

Estimating "causal" relationships

- Linear: Fitting a VAR(m) model to the data, e.g. using least square estimation (e.g. ar-model in TISEAN) and then testing the coefficients a_{ij} against zero.
- Non-linear: Estimating the conditional mutual informations (transfer entropy) — Partitioning the data (if continous variables) and estimating the entropies $H(X_i(t)|\overline{\boldsymbol{X}}_{-j}(t-1), \epsilon)$ and $H(X_i(t)|\overline{\boldsymbol{X}}_V(t-1), \epsilon)$.
- Note that the result depends on the state space, e.g. on the embedding dimensions m_j, m_i in the Transfer entropy

$$T_{j \to i}(m_j, m_i, \epsilon)$$
$$= MI(X_i(t) : X_j(t-1), \dots, X_j(t-m_j+1)|X_i(t-1), \dots, X_i(t-m_i+1); \epsilon)$$

- The result might depend on ϵ. But, for stochastic systems the conditional mutual information should converge for $\epsilon \longrightarrow 0$ to the value for differential entropies!
- You have to correct for finite sample effects. Finite sample effects lead to overestimation.

Dependence on the resolution ϵ

T. Schreiber, Measuring Information Transfer, PRL 85(2000),461-464.

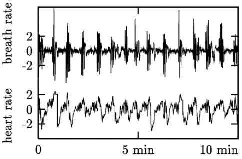


FIG. 3. Bivariate time series of the breath rate (upper) and instantaneous heart rate (lower) of a sleeping human. The data is sampled at 2 Hz. Both traces have been normalized to zero mean and unit variance.

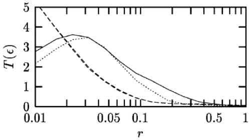
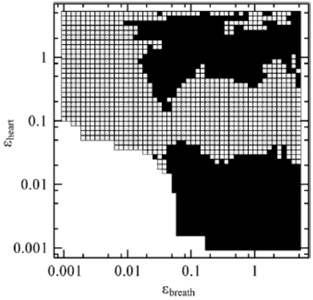


FIG. 4. Transfer entropies T(heart → breath) (solid line), T(breath → heart) (dotted line), and time delayed mutual information $M(\tau = 0.5$ s) (directions indistinguishable, dashed line) for the physiological time series shown in Fig. 3.

A. Kaiser and T. Schreiber, Information transfer in continuous processes,

Physica D 166(2002),43-62.



ϵ values with $T_{heart \rightarrow breath} > T_{breath \rightarrow heart}$ are marked by black squares.

Correcting for finite sample effects - effective transfer entropy

R. Marschinski and H. Kantz, Analysing the information flow between financial time series. An improved estimator for transfer entropy. Eur. Phys. J B 30(2002),275-281.

- Effective transfer entropy: Difference between the usual transfer entropy and the transfer entropy between $X_i(t)$ and a shuffled version of $X_j(t)$.

$$ET_{j\to i}(m_i, m_j) := T_{j\to i}(m_i, m_j) - T_{j,shuffled\to i}(m_i, m_j)$$

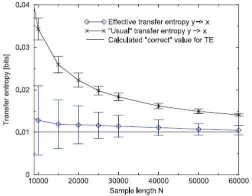


Fig. 6. Comparison of the behaviour of transfer entropy and effective transfer entropy for a varying sample size N: the information flow $y(t)$ to $x(t)$ (Eq. (11), with $\epsilon = 0.15$, $S = 3$ and $m = 4$) was measured for ten different realizations of the process, then average and standard deviation were calculated.

Application: DAX and Dow Jones

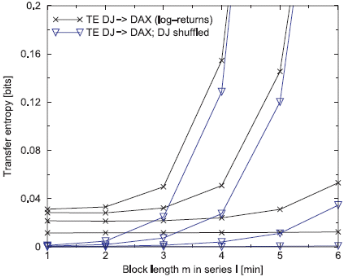


Fig. 2. Transfer entropy measuring the information flow from Dow Jones to DAX series, using various partitions of $S = 2$, 3, 4, 5 symbols (bottom to top). Upper lines have been calculated on the log-returns of DJ and DAX, for the lower ones (triangles) the log-returns of the DJ series have previously been shuffled.

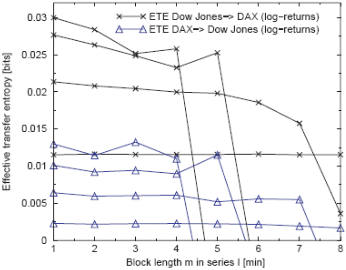


Fig. 3. Effective transfer entropy measuring the information flow between Dow Jones and DAX series, and *vice versa*, using four different partitions of $S = 2$, 3, 4, 5 symbols (bottom to top).

Summary

- Granger causality asks for interdependencies between stochastic processes
- It can be expressed using conditional mutual information (Transfer entropy)
- If we consider only linear interdependencies it can be studied with vector autoregressive(VAR)-models
- One has to be careful with causal interpretations because it is an purely observational measure.