Complex Systems Methods — 10. Self-Organized Criticality (SOC)

Eckehard Olbrich

e.olbrich@gmx.de
http://personal-homepages.mis.mpg.de/olbrich/complex_systems.html

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Overview

Introduction

"The aim of the science of self-organized criticality is to yield insight into the fundamental question of why nature is complex, not simple, as the laws of physics imply."
Per Bak in "How nature works", Springer, 1996)

- Term was introduced 1987 by Per Bak, Chao Tang and Kurt Wiesenfeld (PRL, 59, 381)
- SOC was intended to explain the ubiquity of "1/f"-noise (flicker noise) in nature
- "1/f" noise: power law for temporal or spatial correlations, self-similar, fractal structures
- Observed in light from quasars, intensity of sunspots, flow of rivers such as the Nile, stock exchange price indices

The main ideas

- General idea: In contrast to the critical point of equilibrium phase transitions no control parameters have to be tuned to reach the critical point. The system dynamics drives the system toward the critical point $\Rightarrow$ self-organized criticality, critical state as robust nonequlibrium "attractor", no fine-tuning of parameters necessary
- Criticality: local (in space/time) perturbation do not decay exponentially, but algebraically — long range/term correlations, power law correlations of the fluctuations
- Universality: Exponents independent on the details of the model
- Spatial extended system: critical state consists of a network of locally minimally stable states (*i.e. a local stability condition is needed*).

The BTW sandpile model - Definition in 1-d

The Bak-Tang-Wiesenfeld (BTW) model is a cellular automaton. Consider a one-dimensional sand pile of length N. The boundary conditions are so that sand can leave the system at the right hand side only. The numbers z_n represent height differences $z_n := h_n - h_{n+1}$. The dynamics is the following:

1. A unit of sand is added at a random position n:

$$\begin{aligned} z_n &\mapsto z_n + 1 \\ z_{n-1} &\mapsto z_{n-1} - 1 \end{aligned}$$

2. When the height difference becomes higher than a fixed critical value z_c, one unit of sand tumbles to the lower level, i.e.

$$\begin{aligned} z_n &\mapsto z_n - 2 \\ z_{n\pm1} &\mapsto z_{n\pm1} + 1 \qquad \text{for } z_n > z_c \end{aligned}$$

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3. Open boundary conditions

$$z_0 = 0 \qquad z_N \mapsto Z_N - 1 \qquad z_{N-1} \mapsto Z_{N-1} - 1 \qquad \text{for } z_N > z_c$$

The 1-d BTW sandpile

- If the sand is added randomly from an empty system, the pile will build up, eventually reaching the point were all the height differences z_n assume the critical value $z = z_c$. This is the *minimally stable state* of the system.
- Any additional sand simply falls from site to site and falls off at the end $n = N$ leaving the system in the minimally stable state.
- The state is only critical in the sense that any small perturbation can propagate infinitely through the system.
- The correlation functions are trivial.

The 2-d BTW sandpile

1. A unit of sand is added at a random position (x, y):

$$\begin{aligned} z(x-1,y) &\mapsto z(x-1,y)-1 \\ z(x,y-1) &\mapsto z(x,y-1)-1 \\ z(x,y) &\mapsto z(x,y)+2 \end{aligned}$$

2. When the height difference $z(x, y)$ becomes higher than a fixed critical value z_c

$$\begin{aligned} z(x,y) &\mapsto z(x,y)-4 \\ z_{x,y\pm1} &\mapsto z_{x,y\pm1}+1 \\ z_{x\pm1,y} &\mapsto z_{x\pm1,y}+1 \end{aligned}$$

The variable z has, however, no direct correspondence to the slope of a sandpile anymore.

Simplified BTW model - "height model"

1. Sand is added at a random position (x, y):

$$\begin{array}{ll} \text{Conservative:} & z(x,y) \mapsto z(x,y) + 4 \\ \text{Non-conservative:} & z(x,y) \mapsto z(x,y) + 1 \end{array}$$

2. When the height difference $z(x, y)$ becomes higher than a fixed critical value z_c

$$\begin{aligned} z(x,y) &\mapsto z(x,y) - 4 \\ z_{x,y\pm1} &\mapsto z_{x,y\pm1} + 1 \\ z_{x\pm1,y} &\mapsto z_{x\pm1,y} + 1 \end{aligned}$$

Criticality in the 2-d BTW model

- The homogeneous minimally stable state $z(x, y) = z_c$ is not critical, because the avalanches are amplifying, each node will render two other nodes unstable.
- BTW: "The network will become stable precisely at the point when the network of minimally stable clusters has been broken down to the level where the noise signal cannot be communicated through infinite distances."
- Adding sand randomly the "slope" will be built up to the point where stationarity is obtained, which is assumed by BTW to be a critical state
- No typical length and time scale $\Rightarrow$ power law distribution of avalanche times

Power law distributions of avalanche sizes and lifetimes at criticality for the 2-d BTW Sandpile

Figures from BTW, PRA 38(1988),364-374

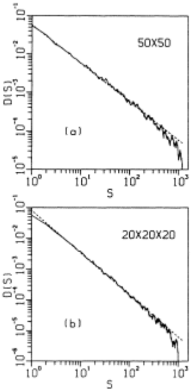

FIG. 3. Distribution of cluster sizes at criticality in two and three dimensions computed as described in the text. The data have been coarse grained. (a) 50×50 array, averaged over 200 samples. The dashed line is a straight line with slope -1.0; (b) $20\times 20\times 20$ array, averaged over 200 samples. The dashed straight line has a slope -1.37.

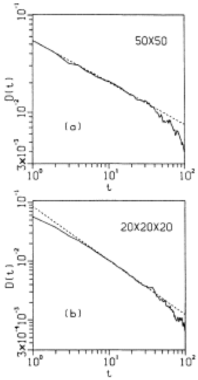

FIG. 4. Distribution of lifetimes corresponding to Fig. 3. (a) For the 50×50 array, the exponent $\alpha\approx 0.43$ yields a $1/f$ noise spectrum $f^{-1.57}$; (b) $20\times 20\times 20$ array, $\alpha\approx 0.92$, yielding an $f^{-1.08}$ spectrum.

Experiments: Sand and rice piles

- Sand piles: No critical behavior was found by Nagel (1992). The slop oscillates between the angle of repose θ_r and the maximum slope θ_m. The slope increases until the maximum slope is reached, then a landslide is is induced. The avalanche persists until the slope θ_r is reached.
- For small sand piles a power law was found by Held (1990) and later by Rosendahl et al. (1993) for avalanches containing from $s = 3$ to 80 grains distributed according to $P(s) \propto s^{-2.5}$. For larger sand piles a crossover to the oscillatory behavior was observed.
- The noncritical behavior of sand piles is assumed to be an effect of inertial effects. A growing avalanche gains kinetic energy and becomes increasingly harder to stop affecting finally the whole sandpile.
- In piles of rice of certain types (elongated grains, "Langkornreis") power law behavior could be found (Frette et al. 1996)

Earthquakes

- Gutenberg-Richter law: If M is the magnitude of an earthquake and N the of earthquakes of this magnitude in a particular region and time interval, it was found that

$$\log N = a - bM$$

- Magnitude of the earth quake is proportional to the released energy. i.e. $m \propto \log E$ and therefore

$$N \propto E^{-b}$$

- SOC was proposed as a mechanism to explain this law
- A continuous variable non-conservative version of the sandpile model was proposed as an earthquake model by Olami, Feder and Christensen (PRL 1992)
- There have been a (still ongoing?) debate about this explanation - see e.g. http://www.nature.com/nature/debates/earthquake/

The OFC model

- Olami, Feder and Christensen (1992) proposed a modification of the simplified BTW model which should mimic the dynamics of sliding tectonic sheets
- Local variable E_i energy related to local stress increases continuously

$$E_i(t + \Delta t) = E_i(t) + \nu$$

- When the local stress exceeds a threshold $E_i \geq E_c$, the next neighbors $E_{nn(i)}$ of i are updated according to

$$\begin{aligned} E_i(t + \Delta t) &= 0 \\ E_{nn(i)}(t + \Delta t) &= E_{nn(i)}(t) + \alpha E_i \end{aligned}$$

- In earthquakes energy is dissipated — non-conservative update, $\Delta E = (2d\alpha - 1)E_i$

Properties of the OFC model

- The model can be related to the more realistic Burridge-Knopoff spring-block model.
- Power law behavior for avalanche frequency vs. size was found with an exponent depending on α.
- The model is definitely non-critical for $\alpha = 0$. It is unclear, whether there is a transition at finite α.
- For periodic boundary conditions the critical behavior is lost and the system synchronizes and oscillates between E_c and 0

Evolution

- Evolution of species gradually at a slow, smooth pace, or via periods of hectic activity separated by long intervals of tranquility?
- Stephen J. Gould (1977) suggested the idea of "punctuated equilibrium" for the history of individual species scaled into geological time
- Species survive for long periods but then disappear within a relatively short span of years
- Often simultaneous extinction of many species, with greatly varying numbers of species in this "bursts" of extinction
- Idea: Avalanche-like nature of extinction: SOC

Bak-Sneppen model - extremal dynamics

- The Bak-Sneppen model is an example of extremal dynamics and arose from the modeling of the growth of random interfaces
- For the sake of simplicity again a lattice is considered, 1-d in the simplest case
- Local variable $b_i \in [0, 1)$ should represent a fitness barrier, i.e. how unlikely a change of fitness is
- It is assumed the it depends on the neighbors
- Update algorithm:
  1. Locate the site $i = j$ with the smallest b value $b_j \leq b_i \; \forall i$
  2. Perform the substitutions

     $$b_j = u_1 \quad b_{j+1} = u_2 \quad b_{j-1} = u_3$$

     where $u_i \in [0, 1)$ are random number drawn with uniform density
  3. Also the variant with random selection of neighbors exhibit criticality.

Bak-Sneppen model - extremal dynamics

- Distribution of barriers evolve toward a step function
- In the limit of infinite system size, the distribution is characterized by a single parameter b_c, with $p(b) = 0$ for $b < b_c$ and a uniform distribution $p(b) = 1/(1 - bc)$ for $b > b_c$
- Two point correlation function exhibits algebraic decay

$$G(i,j) = \langle b_i b_{i+\Delta} \rangle - \langle b_i^2 \rangle \propto (\Delta)^{-\eta} e^{-\Delta/\Delta_0}$$

with $\eta \approxeq 0.7$ and where $\Delta_0 \to \infty$ when $L \to \infty$ (Datta, Gilhøj and Jensen 1997)

- "Avalanche sizes depending on b_0 defined as t' with

$$b_{min}(t_{1,2}) > b_0 \qquad b_{min}(t) < b_0 \quad \text{for} \quad t_1 < t < t_2$$

were found for $b_0 = b_c$ to be distributed like $p(t) = 1/t$.

Forest fire model

- Introduced by Drossel and Schwabl (1992), not really related to forest fires, but applied with some success to the modeling of the spreading of measles of the isle of Bornholm and of the Faroe islands (Rhodes and Anderson 1996; Rhodes, Jensen, and Anderson 1997)
- Defined on a d-dimensional cubic lattice updated according to the following rules
  1. A site occupied by a *burning* tree becomes *empty*.
  2. A *green* tree becomes a burning tree if one or more of its nearest neighbors are burning trees.
  3. An empty site becomes a *green* tree with probability p (the growth rate).
  4. A *green* tree catches fire spontaneously with probability f (the lightning rate)
- Periodic boundary conditions are assumed. System spanning spiral-like fire fronts traverse the system with a period proportional to $1/p$.
- The system becomes critical only in the limit $p \to 0$ and $f/p \to 0$.

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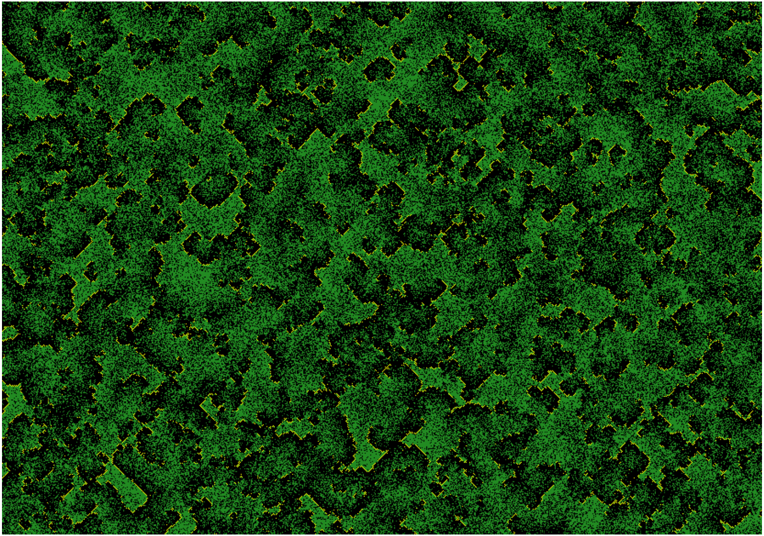
- Periodic boundary conditions are assumed. System spanning spiral-like fire fronts traverse the system with a period proportional to $1/p$.
- The system becomes critical only in the limit $p \to 0$ and $f/p \to 0$.
- The model is considered to be not self-organized critical, because the parameters have to be tuned.

Forest fire model



Game of Life

- Invented 1970 by the British mathematician John Conway
- 2-dimensional cellular automaton with a simple rule: two states (living and dead), 8 neighbors, if 2 or 3 neighbors are living, the cell survives, otherwise it dies. If it is dead and has 3 living neighbors it becomes living.
- Starting from a random initial condition it relaxes to a stationary state of constant or periodic patterns.
- There are moving patterns — best known are the "gliders".
- Perturbing the stationary state by a random perturbation produces transient activity - an avalanche
- Bak and others studied the duration and size of these avalanches and found power laws
- As in other models there was also here a controversy, whether the game of life is really critical or if there is a crossover to an exponential distribution for large system sizes.

What is SOC good for?

Jensen (1998) casts this question into four more specific ones:

1. Can we identify SOC as a well defined distinct phenomenon different from any other category of behavior?
2. Can we identify a certain construction that can be called a *theory* of self-organized critical systems?
3. Has SOC taught us anything about the world that we did not know prior to BTW's seminal paper 1987?
4. Is there any predictive power in SOC — that is, can we state the necessary and sufficient conditions a system must fulfill in order to exhibit SOC? And, if we are able to establish that a system belongs to the category of SOC systems, does that then actually help us to understand the behavior of the system?

The main ingredients of SOC (according to Jensen)

- **Interaction dominated** — the local dynamics is essentially affected by interactions
- **Time scale separation:** The time scale of the external driving force has to be much slower than the time scale of the response of the system. i.e. the internal relaxation process (avalanches); without this time scale separation the dynamics would be dominated by the external drive
- Existence of **thresholds** allows a large number of metastable static configurations
- Subset of this metastable states are the minimally stable states of BTW; these minimally stable states are the critical states
- We can expect SOC in **slowly driven, interaction-dominated threshold (SDIDT)** - systems

Criticality without tuning?

One of the main objection to SOC was, that also the SOC models produce criticality only in certain parameter regions. E.g exact criticality of the BTW model is only reached if the input rate goes to zero.

- It turned out that SOC is less general and universal than originally claimed by BTW. There are several examples of candidate systems, which not develop into a critical state, e.g. the real sandpile. Jensen tried to formulate some of the probably necessary (but clearly not sufficient) conditions by his notion of SDIDT systems.
- Some of the requirements to meet the *theoretical definition* of criticality are of *theoretical* nature, i.e. can not and need not to be fulfilled for real systems — think of the thermodynamical limit for equilibrium phase transitions
- Relation to evolution seems to be (for me) an open question

Power laws

Jensens proposal to characterize SOC as phenomenon in SDIDT systems is called by himself a *constructive* definition and contrasted with the *phenomenological* definition, which mainly relies on the identification of **power law** distributions in the fluctuations. The phenomenological approach has the problem that:

1. Not all systems exhibiting power laws in their fluctuations are SOC.
2. There are also trivial ways to produce power laws.
3. There are alternative theories to explain the ubiquity of power laws in evolved systems, e.g. highly optimized tolerance (HOT)

Conclusions

- SOC (alone) is certainly not "How nature works"
- But, SOC has been established as a subfield of nonequilibrium statistical mechanics
- SOC has called attention to thresholds, metastability and its relation to large fluctuations in the spatio-temporal behavior in a large class of many-body systems