NOTES ON THE MINIMAL LIPSCHITZ EXTENSION PROBLEM, TUG OF WAR WITH NOISE, AND THE INFINITY LAPLACIAN (WORK IN PROGRESS)

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These notes concern the minimal Lipschitz extension problem and tug-of-war games. These are two apparently very different topics. The first ostensibly belongs to the calculus of variations, as the goal is to minimize a certain energy functional and study the properties of minimizers. The second is a stochastic two-player game that could be filed under game theory: two players take turns controlling a random walker in the hopes of maximizing a pay-out function depending on its trajectory. We will see that these two problems are connected via an elliptic partial differential operator called the infinity Laplacian. Further, this story can be told using beautiful (and elementary) geometric and analytic arguments, which provide a nice setting for exploring topics like the maximum principle, Markov chains, convex analysis, and the theory of viscosity solutions.

0.1. **The Minimal Lipschitz Extension Problem.** Let φ be a norm in $\mathbb{R}^d$. The minimal Lipschitz extension problem can be stated as follows: given an open set $U \subseteq \mathbb{R}^d$ and a uniformly Lipschitz function $g : \partial U \to \mathbb{R}$, solve the following optimization problem

$$\min \left\{ \text{Lip}_\varphi(v; \overline{U}) \mid v : \overline{U} \to \mathbb{R} \quad \text{such that} \quad \forall x \in \partial U \quad v(x) = g(x) \right\}, \tag{1}$$

where $\text{Lip}_\varphi(v; \overline{U})$ is the Lipschitz seminorm defined by

$$\text{Lip}_\varphi(v; \overline{U}) = \inf \left\{ C > 0 \mid \forall x, y \in \overline{U} \quad v(x) - v(y) \leq C\varphi(x - y) \right\}.$$

A function v attaining the minimum in (1) is called a minimal Lipschitz extension of g in U. As we will see below, there may be more than one minimal Lipschitz extension, but there is at most one *absolutely minimizing Lipschitz extension*, defined next.

In the definition, we need to recall that a continuous function $u : U \to \mathbb{R}$ is called *locally Lipschitz continuous* if, for any compact set $K \subseteq U$,

$$\text{Lip}_\varphi(u; K) < \infty.$$

Exercise 4 below asks you to check that the definition does not depend on the choice of norm φ.

Let us also recall that if U and V are two open sets in $\mathbb{R}^d$, we say that V is *compactly contained in* U if V is bounded and $\overline{V} \subseteq U$, where $\overline{V}$ denotes the closure of V. For brevity, we write $V \subset\subset U$ to mean that V is compactly contained in U.

Definition 1. *Given any open set $U \subseteq \mathbb{R}^d$, a continuous function $u : U \to \mathbb{R}$ is called an* absolutely minimizing Lipschitz function *if u is locally Lipschitz continuous in U*

and, for any open subset $V \subset\subset U$,

$$Lip_\varphi(u;\overline{V}) = \min\left\{Lip_\varphi(v;\overline{V}) \mid v : \overline{V} \to \mathbb{R} \text{ such that } \forall x \in \partial V \ v(x) = u(x)\right\}.$$

By the end of these notes, we will prove the following theorem, asserting that there is at most one absolutely minimizing Lipschitz function with a given boundary value in any given bounded domain.

Theorem 1. *Let $U \subseteq \mathbb{R}^d$ be a bounded open set and fix a continuous function $g : \partial U \to \mathbb{R}$. Then there is a unique absolutely minimizing function $u : \overline{U} \to \mathbb{R}$ such that $u(x) = g(x)$ for each $x \in \partial U$. Further, if g is uniformly Lipschitz continuous in ∂U, then u attains the minimum in* (1).

Along the way, we will see another equivalent characterization of absolutely minimizing Lipschitz functions, involving a maximum principle. This will be used to prove the uniqueness assertion of the theorem, along with certain averaging operators that are intimately related to the tug-of-war games described next.

0.2. **Tug of War (with Noise) on $\mathbb{Z}^d$.** For our purposes, tug of twar with noise is a stochastic two-player game, involving a stochastic process indexed by p, the intensity of the noise. When $p = 1$, the process is simply the simple random walk (SRW). Recall that this is the Markov chain $(X_k)_{k\in\mathbb{N}_0}$ on $\mathbb{Z}^d$ such that, at each time, it randomly chooses its next position among its $2d$ nearest neighbors:

$$\mathbb{P}\{X_{n+1} = y \mid X_n = x\} = \begin{cases} \frac{1}{2d}, & \text{if } |y-x| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Even if we set aside probability for a moment, the SRW will be interesting for us since it leads to the solution of a "discrete PDE," namely, given a finite subset $A \subseteq \mathbb{Z}^d$ and a function $F : \mathbb{Z}^d \to \mathbb{R}$, study of the SRW leads to the analysis of the following linear equation

$$-\Delta_{\mathbb{Z}^d} u = 0 \quad \text{in } \mathrm{int}(A), \quad u = F \quad \text{on } \partial_{\mathrm{int}} A, \tag{2}$$

where $\mathrm{int}(A)$ denotes the set of points in A for which all nearest neighbors are also in A, $\partial_{\mathrm{int}} A = A \setminus \mathrm{int}(A)$, and $\Delta_{\mathbb{Z}^d}$ is the linear operator given by

$$(\Delta_{\mathbb{Z}^d} u)(x) = \frac{1}{2d} \sum_{y\in\mathbb{Z}^d : |y-x|=1} (u(y) - u(x)).$$

We will refer to $\Delta_{\mathbb{Z}^d}$ as the *discrete Laplacian* since it behaves like a discretized version of the partial differential operator Δ. Properties of this and related "discrete PDE" will be treated in detail in Section 1.

When $p \in [0, 1)$, the stochastic process is no longer a Markov chain *per se*. Instead, there are now two players, call them Max and Minnie, who get a chance to influence the trajectory of the random walk. The simplest version of the tug-of-war game with noise can be described informally as follows: fix a finite subset $A \subseteq \mathbb{Z}^d$ and a function $F : \partial_{\mathrm{int}} A \to \mathbb{R}$. At each time n, a biased coin is flipped, the probability of landing heads being p. If it is heads, the walker moves from position X_n to X_{n+1} as in the simple random walk, choosing one of its nearest neighbors uniformly at random. If the coin lands on tails, though, Max and Minnie get a chance to steer the walk. Another coin is flipped, this one unbiased. If this second coin is heads, it is Max's turn; otherwise, if it comes up tails, it is Minnie's. Either way, Max or Minnie get to move the walker from its current position X_n to any new position X_{n+1} as long as it is a nearest neighbor of X_n.

The game concludes the first time that the process X hits the boundary of A, i.e., $X_n \in \partial_{\text{int}} A$. Call this (random) time τ. At time τ, the rules of the game state that Minne has to pay Max the amount $F(X_\tau)$. Hence Max is inclined to influence X in such a way as to maximize the *(expected) payout*, that is, $\mathbb{E}[F(X_\tau)]$. Minnie, on the other hand, does her best to minimize the expected payout. Now you see why it is called tug-of-war (with noise): one player seeks to pull the walker towards the maximum, while the other player wants to drag it back towards the minimum.

Making the discussion above rigorous requires some notation, for which you can see Section 1, but the reader uninitiated (or uninterested) in probability can rest assured, as there is still something to see here even neglecting the details of the game. We will be interested in the connection between the tug-of-war and the following *nonlinear* equation, which turns out to describe the expected payout:

$$-p\Delta_{\mathbb{Z}^d} u - (1-p)\Delta^\infty_{\mathbb{Z}^d} u = 0 \quad \text{in int}(A), \quad u = F \quad \text{on } \partial_{\text{int}} A. \tag{3}$$

Clearly, when $p = 1$, this is the same as (3), whereas when $p < 1$, we add a term involving the *discrete infinity Laplacian* $\Delta^\infty_{\mathbb{Z}^d}$, which is given by

$$(\Delta^\infty_{\mathbb{Z}^d} u)(x) = \frac{1}{2} \max_{y \in \mathbb{Z}^d \,:\, |y-x|=1} \{u(y) - u(x)\} + \frac{1}{2} \min_{y \in \mathbb{Z}^d \,:\, |y-x|=1} \{u(y) - u(x)\}.$$

Notice that this is quite similar to $\Delta_{\mathbb{Z}^d}$ in so far as it outputs a weighted average of the differences $\{u(y) - u(x) \mid |y - x| = 1\}$. We will see below that (3) has unique solutions for any $p \in [0, 1]$ and we will show that the solution describes the expected payout of the game.

0.3. **Scaling Limit.** We started out discussing the minimal Lipschitz extension problem, which involves functions defined in the continuum but then went on to discuss tug-of-war games, involving functions in $\mathbb{Z}^d$. You may be forgiven for wondering about the connection between the two.

If you have seen a thing or two about finite-difference methods, you may already be anticipating what is coming next. Suppose we rescale by replacing $\mathbb{Z}^d$ by $\epsilon\mathbb{Z}^d$ so that the distance between nearest neighbors is now ϵ, and consider what happens when ϵ is small, but A remains a fixed size. At the level of the tug-of-war game, that means it takes many, many steps of the walker before it lands in $\partial_{\text{int}} A$. Alternatively, to account for the smaller step size, we should change our definition for the discrete infinity Laplacian by setting

$$(\Delta^\infty_{\epsilon\mathbb{Z}^d} u)(x) = \frac{1}{2} \max_{y \in \epsilon\mathbb{Z}^d \,:\, |y-x|=\epsilon} \{u(y) - u(x)\} + \frac{1}{2} \min_{y \in \epsilon\mathbb{Z}^d \,:\, |y-x|=\epsilon} \{u(y) - u(x)\},$$

We will prove the following result about the scaling limit of (3) as the step size $\epsilon \to 0$.

Theorem 2. *Fix a bounded open set $U \subseteq \mathbb{R}^d$ and a continuous function $g : \partial U \to \mathbb{R}$. Let $G : \overline{U} \to \mathbb{R}$ be a continuous extension of g. For any $\epsilon > 0$, define $U_\epsilon \subseteq \epsilon\mathbb{Z}^d$ and $G_\epsilon : U_\epsilon \to \mathbb{R}$ by the rule*

$$U_\epsilon = \{x \in \epsilon\mathbb{Z}^d \mid x \in U\}, \quad G_\epsilon(x) = G(x)$$

and let u^ϵ be the solution of the equation

$$-\epsilon^{-2}\Delta^\infty_{\epsilon\mathbb{Z}^d} u^\epsilon = 0 \quad \text{in } U_\epsilon, \quad u^\epsilon = G_\epsilon \quad \text{on } \partial_{int} U_\epsilon. \tag{4}$$

Then there is a continuous function $u : \overline{U} \to \mathbb{R}$ such that $u_\epsilon \to u$ uniformly in $\overline{U}$ as $\epsilon \to 0$ in the following sense:

$$\lim_{\delta \to 0} \sup \{|u_\epsilon(x) - u(y)| \mid \epsilon + |x - y| < \delta\} = 0. \tag{5}$$

Furthermore, u is an absolutely minimizing Lipschitz function in U with respect to the ℓ^1 (or Manhattan) norm.

A few remarks about the theorem are in order. First, the weird factor of ϵ^{-2} in (4) is unnecessary since the right-hand side is zero, but it will be meaningful later so it may be worth pondering its presence now.

Second, the strange notion of uniform convergence in (5) is necessary since u_ϵ and u are defined in different sets (indeed, I should have written $x \in U_\epsilon$ and $y \in U$, the set depends on ϵ).

Lastly, notice that we obtained an absolutely minimizing Lipschitz function in the limit! So there is, indeed, a connection between the two topics presented so far. Notice, though, that it isn't just any norm φ involved here, but the ℓ^1 "Manhattan" norm. This reflects the fact that our walk took place in $\mathbb{Z}^d$: if we had changed to a different lattice, we would have gotten a different norm.

Still, this all should seem fairly unsatisfying since we started with something that looked like a "discretized PDE" (or so I led you on), but all I have said so far is that the limit function is an absolutely minimizing Lipschitz function. As we will see, this is where the factor of ϵ^{-2} in (4) comes in handy.

Before getting too deep into the weeds, it is worth considering the more classical case when $p = 1$ instead of zero. In that case, the equation of interest is

$$-\epsilon^{-2}\Delta_{\epsilon\mathbb{Z}^d} u_\epsilon = 0 \quad \text{in } U_\epsilon, \quad u_\epsilon = G_\epsilon \quad \text{on } \partial_{\text{int}} U_\epsilon. \tag{6}$$

The behavior of u_ϵ is admittedly more delicate here: we have assume that ∂U is "nice" in a sense that can be made precise. The result then states that $u_\epsilon \to u$ in the same sense as (5), and $u : \overline{U} \to \mathbb{R}$ is a continuous function, which is smooth in U, such that

$$-\Delta u = 0 \quad \text{in } U, \quad u = g \quad \text{on } \partial U. \tag{7}$$

This is quite tidy: you could say that the so-called *Laplace equation* $-\Delta u = 0$ is the "limiting equation" of the discretized version (6).

Anticipating a little bit, we can rewrite the Laplace equation as $-H(Du, D^2u) = 0$, where H is the function $H(p, X) = \frac{1}{d}\sum_{i=1}^d X_{ii}$ taking a vector $p \in \mathbb{R}^d$ and a symmetric matrix $X \in \text{Sym}(d)$ and returning the average of the diagonal matrices of X.

In the case $p = 0$ (i.e., tug-of-war without noise), we obtain something similar, but with a twist. Analogous to H above, define two functions $G^*_{\ell^1}$ and $G^{\ell^1}_*$ on $\mathbb{R}^d \times \mathrm{Sym}(d)$ by

$$G^*_{\ell^1}(p, X) = \max_{i \in \{1,2,\ldots,d\} : |p_i| = |p|_\infty} X_{ii}$$

$$G^{\ell^1}_*(p, X) = \min_{i \in \{1,2,\ldots,d\} : |p_i| = |p|_\infty} X_{ii}.$$

Above $|p|_\infty$ denotes the ℓ^∞ norm of p, i.e., $|p|_\infty = \max\{|p_1|, \ldots, |p_d|\}$ if $p = (p_1, \ldots, p_d)$, whereas the ℓ^1 norm is $|p|_1 = \sum_{i=1}^d |p_i|$. The fact that ℓ^∞ appears in the definition even though the name involves ℓ^1 will require some explanation later on. The number X_{ii} appearing above are the ith diagonal entry of X. Thus, if $|p|_\infty = |p_i|$ for a unique i, then $G^*_{\ell^1}(p, X)$ and $G^*_{\ell^1}(p, X)$ both equal the corresponding diagonal entry of X, but, in general, they differ.

With this notation out of the way, we are prepared to state the result that gives an alternative characterization of the limit u.

Theorem 3. *Let u be the limiting function obtained in Theorem 2. Then u is the unique continuous function such that*

$$\begin{cases} -G^*_{\ell^1}(Du, D^2u) \leq 0 & \textit{in the viscosity sense in } U, \\ -G^{\ell^1}_*(Du, D^2u) \geq 0 & \textit{in the viscosity sense in } U, \\ \qquad u = g & \textit{on } \partial U. \end{cases} \tag{8}$$

In Section **??**, I will argue that the two inequalities in (8) are the "limiting equation" obtained from (4). We can call it the ℓ^1 *infinity Laplace equation*, and the two functions $G^{\ell^1}_*$ and $G^*_{\ell^1}$ together constitute the ℓ^1 *infinity Laplacian*. There is just one problem: emphatically unlike (7), we aren't dealing with an equation, but instead two inequalities. We will see below that this is unavoidable, and it necessitates some interesting analysis.

You may still be wondering: what does it mean for a partial differential inequality to hold *in the viscosity sense*? That is a question unto itself, as you could see by consulting a reference like the infamous "User's Guide" by Crandall, Ishii, and Lions. A key difficulty in all this is the function u will not even be C^1 in general, hence we need to reconsider what we mean when we talk about a solution of a PDE. Yet the two problems considered here (minimal Lipschitz extensions and tug-of-war games) will provide a nice setting to dip our feet into this somewhat esoteric terrain. So if you have never heard of a viscosity solution (of a PDE) or don't know the definition, rest assured. The problems presented above will provide a (hopefully pleasant) foretaste.

At last, of course, the analysis above could be repeated with an arbitrary $p \in [0, 1]$, and then one obtains, as you might guess, the two inequalities

$$-p\Delta u - (1-p)G^*_{\ell^1}(Du, D^2u) \leq 0 \quad \text{in the viscosity sense in } U,$$

$$-p\Delta u - (1-p)G^{\ell^1}_*(Du, D^2u) \geq 0 \quad \text{in the viscosity sense in } U.$$

Proving this requires more machinery than is developed here, but some of the discussion can be found in Section **??** below.

0.4. **Exercises.**

Exercise 1. *Prove that any Finsler norm is continuous.*

Exercise 2. *Let* $|\cdot|$ *be the Euclidean norm in* $\mathbb{R}^d$*, i.e.,* $|q|^2 = \sum_{i=1}^d q_i^2$ *if* $q = (q_1, \dots, q_d)$ *in the standard coordinates. Prove that if* φ *is a Finsler norm in* $\mathbb{R}^d$*, then there is a constant* $C \geq 1$ *such that*

$$C^{-1}|q| \leq \varphi(q) \leq C|q|. \tag{9}$$

(Hint: Use the previous exercise to deduce that $\min\{\varphi(q) \mid |q| = 1\} > 0$ *and* $\max\{\varphi(q) \mid |q| = 1\} < \infty$ *and then use positive one-homogeneity.)*

Exercise 3 (Requires knowledge of topology)**.** *Using the previous exercise, show that any norm on* $\mathbb{R}^d$ *induces the same topology. That is, if* φ *is a norm and we define the distance* $d_\varphi(x,y) = \varphi(x-y)$*, then the open sets in the metric topology of* d_φ *are the same as those in the standard Euclidean metric topology.*

Exercise 4. *Let* $C \subseteq \mathbb{R}^d$ *be a closed set and* $u : C \to \mathbb{R}$*. Using (9), prove that* $Lip_\varphi(u;C) < \infty$ *for some Finsler norm* φ *if and only if* $Lip_{|\cdot|}(u;C) < \infty$*. In particular, the definition of locally Lipschitz function given above does not depend on the choice of the (Finsler) norm* φ*.*

0.5. **Problems.**

Problem 1. *Let* $U \subseteq \mathbb{R}^d$ *be an open set and suppose that* $f : U \to \mathbb{R}^d$ *is smooth. Given* $x \in U \cap (\epsilon\mathbb{Z}^d)$*, define* $\Delta_{\epsilon\mathbb{Z}^d} f(x)$ *by*

$$\Delta_{\epsilon\mathbb{Z}^d} f(x) = \frac{1}{2d} \sum_{y \in \epsilon\mathbb{Z}^d \,:\, |y-x| = \epsilon} \left(f(y) - f(x)\right).$$

Show that, for any compact set $K \subseteq U$*,*

$$\lim_{\epsilon \to 0} \sup \left\{ \left| \frac{1}{\epsilon^2} \Delta_{\epsilon\mathbb{Z}^d} f(x) - \Delta f(x) \right| \;\middle|\; x \in K \cap (\epsilon\mathbb{Z}^d) \right\} = 0.$$

In this sense, the operator $\epsilon^{-2}\Delta_{\epsilon\mathbb{Z}^d}$ *approximates* Δ*.*

Problem 2. *Let* $U \subseteq \mathbb{R}^d$ *be an open set and suppose that* $f : U \to \mathbb{R}^d$ *is smooth. Given* $x \in U \cap (\epsilon\mathbb{Z}^d)$*, define* $\Delta^\infty_{\epsilon\mathbb{Z}^d} f(x)$ *by*

$$\Delta^\infty_{\epsilon\mathbb{Z}^d} f(x) = \frac{1}{2d} \max_{y \in \epsilon\mathbb{Z}^d \,:\, |y-x| = \epsilon} \left(f(y) - f(x)\right) + \frac{1}{2} \min_{y \in \epsilon\mathbb{Z}^d \,:\, |y-x| = \epsilon} \left(f(y) - f(x)\right).$$

Show that, for any compact set $K \subseteq U$*,*

$$\lim_{\epsilon \to 0} \sup \left\{ \frac{1}{\epsilon^2} \Delta^\infty_{\epsilon\mathbb{Z}^d} f(x) - G^*_{\ell^1}(Df(x), D^2 f(x)) \;\middle|\; x \in K \cap (\epsilon\mathbb{Z}^d) \right\} \leq 0,$$

$$\lim_{\epsilon \to 0} \inf \left\{ \frac{1}{\epsilon^2} \Delta^\infty_{\epsilon\mathbb{Z}^d} f(x) - G_*^{\ell^1}(Df(x), D^2 f(x)) \;\middle|\; x \in K \cap (\epsilon\mathbb{Z}^d) \right\} \geq 0.$$

In this sense, the pair of operators $(G^*_{\ell^1}, G_*^{\ell^1})$ *describe the limit of the operator* $\epsilon^{-2}\Delta_{\epsilon\mathbb{Z}^d}$ *as* $\epsilon \to 0$*.*

1. Tug of War with Noise

In this section, we discuss the stochastic two-player game called tug of war with noise. The study of this game will lead to a finite-difference equation, basically a discretized PDE involving a discrete analogue of the Laplacian and the infinity Laplacian.

Toward that end, recall from the introduction that we defined $\Delta_{\mathbb{Z}^d}$ and $\Delta^\infty_{\mathbb{Z}^d}$ for functions defined in subsets of $\mathbb{Z}^d$ by

$$\Delta_{\mathbb{Z}^d} u(x) = \frac{1}{2d} \sum_{e \in \mathbb{E}^d} \left(u(x+e) - u(x) \right), \tag{10}$$

$$\Delta^\infty_{\mathbb{Z}^d} u(x) = \frac{1}{2} \max_{e \in \mathbb{E}^d} \left(u(x+e) - u(x) \right) + \frac{1}{2} \min_{e \in \mathbb{E}^d} \left(u(x+e) - u(x) \right). \tag{11}$$

1.1. **Construction of the Game.** Let A be a finite subset of $\mathbb{Z}^d$ and $p \in [0,1]$. Recall that we denote by $\mathrm{int}(A)$ and $\partial_{\mathrm{int}} A$ the subsets of A given by

$$\partial_{\mathrm{int}} A = \{x \in A \mid \exists y \in \mathbb{Z}^d \setminus A \text{ such that } |y-x| = 1\}, \quad \mathrm{int}(A) = A \setminus \partial_{\mathrm{int}} A.$$

Fix a function $F : \partial_{\mathrm{int}} A \to \mathbb{R}$.

Let $\{\Theta_0, \Xi_0, Z_0, \Theta_1, \Xi_1, Z_1, \dots, \Theta_N, \Xi_N, Z_N, \dots\}$ be a family of independent random variables such that, for any $N \in \mathbb{N}_0$,

$$\mathbb{P}\{\Theta_N = 1\} = 1 - \mathbb{P}\{\Theta_N = 0\} = p,$$
$$\mathbb{P}\{\Xi_N = 1\} = \mathbb{P}\{\Xi_N = 0\} = \frac{1}{2},$$
$$\mathbb{P}\{Z_N = e\} = \frac{1}{2d} \quad \text{for each } e \in \mathbb{E}^d.$$

The sequence $\{\Theta_N\}_{N \in \mathbb{N}_0}$ models the biased coin that is flipped at each turn, with $\Theta_N = 1$ indicating the outcome is heads and $\Theta_N = 0$ indicating the outcome is tails. Similarly, $\{\Xi_N\}_{N \in \mathbb{N}_0}$ models the fair coin. The random vectors $\{Z_N\}_{N \in \mathbb{N}}$ model the random jumps occurring when the biased coin comes up heads.

We will model Max and Minnie's decisions using strategies defined in the following way. Define $\mathcal{A}$ by

$$\mathcal{A} = \{Q = (Q_j)_{j \in \mathbb{N}_0} \mid \forall j \in \mathbb{N}_0 \quad Q_j : (\mathbb{Z}^d)^{\{0,1,\dots,j\}} \to \mathbb{E}^d\}.$$

Given an $x \in A$ and two strategies $Q, S \in \mathcal{A}$, define the associated trajectory of the game $X^{x,Q,S} = (X_j^{x,Q,S})_{j \in \mathbb{N}_0}$ in the following way:

$$\begin{aligned} X_j^{x,Q,S} &= X_{j-1}^{x,Q,S} + \Theta_{j-1} Z_{j-1} + (1-\Theta_{j-1}) \Xi_{j-1} Q_{j-1}(X_0^{x,Q,S}, \dots, X_{j-1}^{x,Q,S}) \\ &\quad + (1-\Theta_{j-1})(1-\Xi_{j-1}) S_{j-1}(X_0^{x,Q,S}, \dots, X_{j-1}^{x,Q,S}), \\ X_0^{x,Q,S} &= x. \end{aligned} \tag{12}$$

We denote by $\tau = \tau^{x,Q,S}$ the exit time of the walk $X^{x,Q,S}$, that is,

$$\tau = \inf \left\{ j \geq 0 \mid X_j^{x,Q,S} \in \partial_{\mathrm{int}} A \right\}.$$

Recall that, at turn τ, Minnie has to pay Max the amount $F(X_{\tau}^{x,Q,S})$. Thus, Max would like to choose his strategy Q so as to maximize the payout F, while Minnie should choose S so as to minimize it. Since the amount Max gains is exactly equal to the amount Minnie loses, this is a zero-sum game.

To be slightly more precise, suppose that Max takes as his goal maximizing the *expected payout* $\mathbb{E}[F(X_{\tau}^{x,Q,S})]$, while Minnie decides to minimize it.[1] It then makes sense to consider the *lower* and *upper value functions* $u, v : A \to \mathbb{R}$ defined by

$$u(x) = \sup_{Q \in \mathcal{A}} \inf_{S \in \mathcal{A}} \mathbb{E}[F(X_{\tau}^{x,Q,S})],$$
$$v(x) = \inf_{S \in \mathcal{A}} \sup_{Q \in \mathcal{A}} \mathbb{E}[F(X_{\tau}^{x,Q,S})].$$

You can think of u as being the worst-case scenario for Max: without knowing Minnie's strategy S, it is the best he can hope to gain. Similarly, v is the best-case scenario for Minnie.

Note that $u(x) \leq v(x)$ for each $x \in A$ by definition, hence the words "lower" and "upper" in the name.

In the main result of this section, we prove that the upper and lower value functions coincide, that is, $u(x) = v(x)$ for each $x \in A$. (In game theory, one says that *the game has a value* in such a case.)

Theorem 4. *For any $x \in A$, we have $u(x) = v(x)$. Furthermore, u is the unique solution of the finite-difference equation*

$$\begin{cases} -p\Delta_{\mathbb{Z}^d} u(x) - (1-p)\Delta_{\mathbb{Z}^d}^{\infty} u(x) = 0 & \text{for each } x \in \operatorname{int}(A), \\ u(x) = F(x) & \text{for each } x \in \partial_{\mathrm{int}} A. \end{cases} \tag{13}$$

Since u and v are equal, we may as well refer to the *value function* and drop the qualifiers *upper* and *lower*. It turns out that the value function can be used to infer the optimal strategies that Max and Minnie should use. This is described in Theorem 5 below.

The proof of Theorem 4 proceeds in two steps. First, we prove that u and v are, respectively, super- and subsolution of (13).

Proposition 1. *For any $x \in \operatorname{int}(A)$,*

$$-p\Delta_{\mathbb{Z}^d} u(x) - (1-p)\Delta_{\mathbb{Z}^d}^{\infty} u(x) \geq 0, \tag{14}$$
$$-p\Delta_{\mathbb{Z}^d} v(x) - (1-p)\Delta_{\mathbb{Z}^d}^{\infty} v(x) \leq 0. \tag{15}$$

Next, we prove a *comparison principle*, which we invoke to establish that the inequality $u(x) \geq v(x)$ holds for any $x \in A$.

Proposition 2. *If $u_1, u_2 : A \to \mathbb{R}$ satisfy, for any $x \in \operatorname{int}(A)$,*

$$-p\Delta_{\mathbb{Z}^d} u_1(x) - (1-p)\Delta_{\mathbb{Z}^d}^{\infty} u_1(x) \geq 0 \geq -p\Delta_{\mathbb{Z}^d} u_2(x) - (1-p)\Delta_{\mathbb{Z}^d}^{\infty} u_2(x),$$

then

$$\max\{u_2(x) - u_1(x) \mid x \in A\} = \max\{u_2(x) - u_1(x) \mid x \in \partial_{\mathrm{int}} A\}.$$

[1] This would make sense if Max and Minnie planned to play the game many times repeatedly.

Remark 1. *Due to the finite-difference inequality satisfied by u_1, we say that u_1 is a* supersolution *of the equation $-p\Delta_{\mathbb{Z}^d}w - (1-p)\Delta^{\infty}_{\mathbb{Z}^d}w = 0$. Symmetrically, we say that u_2 is a* subsolution.

Since $u(x) = v(x) = F(x)$ for $x \in \partial_{\mathrm{int}}A$, we combine Propositions 1 and 2 to deduce that $u(x) \geq v(x)$ for any $x \in A$. Since we know $u \leq v$ by definition, this prove that $u \equiv v$. Further, Proposition 2 implies that solutions of (13) are unique (see Exercise 5 below). Thus, Theorem 4 is proved as soon as we prove the two propositions.

Exercise 5. *Fix $F, G : \partial_{\mathrm{int}}A \to \mathbb{R}$. Assuming that $u_F, u_G : A \to \mathbb{R}$ satisfy*

$$\begin{aligned} -p\Delta_{\mathbb{Z}^d}u_F(x) - (1-p)\Delta^{\infty}_{\mathbb{Z}^d}u_F(x) = 0 \quad &\text{for each } x \in \mathrm{int}(A), \\ -p\Delta_{\mathbb{Z}^d}u_G(x) - (1-p)\Delta^{\infty}_{\mathbb{Z}^d}u_G(x) = 0 \quad &\text{for each } x \in \mathrm{int}(A), \\ u_F(x) = F(x) \text{ and } u_G(x) = G(x) \quad &\text{for each } x \in \partial_{\mathrm{int}}A, \end{aligned}$$

use Proposition 2 to establish that

$$\max\{|u_F(x) - u_G(x)| \mid x \in A\} \leq \max\{|G(x) - F(x)| \mid x \in \partial_{\mathrm{int}}A\}.$$

Conclude from this that there is at most one function u satisfying (13).

1.2. **Proof of Proposition 1.** Here we prove that u and v are super- and subsolution of (13). This will follow from a relatively elementary Markov chain-style argument, the crux of which is treated in the next problem.

We will need some additional notation. Given $x \in \mathbb{Z}^d$ and $Q \in \mathcal{A}$, let $Q^{(1)} = (Q^{(1)}_j)_{j\in\mathbb{N}_0}$ be the strategy in $\mathcal{A}$ defined by

$$\begin{aligned} &Q^{(1)}_j : (\mathbb{Z}^d)^{\{0,1,\ldots,j\}} \to \mathbb{E}^d, \\ &Q^{(1)}_j(y_0, y_1, \ldots, y_j) = Q_{j+1}(x, y_0, y_1, \ldots, y_j). \end{aligned} \tag{16}$$

In words, $Q^{(1)}$ is the strategy we get from Q by fixing the initial position of the walker at x and shifting to the next turn. The significance of this is detailed in the next problem.

Problem 1. *Fix $x \in \mathbb{Z}^d$, and let $Q, S \in \mathcal{A}$.*

(i) Fix $e \in \mathbb{E}^d$. Prove that the law of the shifted walk $(X^{x,Q,S}_{j+1})_{j\in\mathbb{N}_0}$ conditioned on the event $\{\Theta_0 = 1, Z_0 = e\}$ equals the law of $(X^{x+e,Q^{(1)},S^{(1)}}_j)_{j\in\mathbb{N}_0}$. More precisely, for any $M \in \mathbb{N}$ and any $y_0, \ldots, y_M \in \mathbb{Z}^d$, we have

$$\begin{aligned} &\mathbb{P}\left\{X^{x,Q,S}_1 = y_0, \ldots, X^{x,Q,S}_{M+1} = y_M \mid \Theta_0 = 1, Z_0 = e\right\} \\ &\quad = \mathbb{P}\left\{X^{x+e,Q^{(1)},S^{(1)}}_0 = y_0, \ldots, X^{x+e,Q^{(1)},S^{(1)}}_M = y_M\right\}. \end{aligned}$$

(Hint: Use equation (12).*)*

(ii) Prove that the law of the shifted walk $(X^{x,Q,S}_{j+1})_{j\in\mathbb{N}_0}$ conditioned on the event $\{\Theta_0 = 0, \Xi_0 = 1\}$ equals the law of $(X^{x+Q_0(x),Q^{(1)},S^{(1)}}_j)_{j\in\mathbb{N}_0}$. More precisely, for any

$M \in \mathbb{N}$ and any $y_0, \dots, y_M \in \mathbb{Z}^d$, we have

$$\mathbb{P}\left\{X_1^{x,Q,S} = y_0, \dots, X_{M+1}^{x,Q,S} = y_M \mid \Theta_0 = 0, \Xi_0 = 1\right\}$$
$$= \mathbb{P}\left\{X_0^{x+Q_0(x),Q^{(1)},S^{(1)}} = y_0, \dots, X_M^{x+Q_0(x),Q^{(1)},S^{(1)}} = y_M\right\}.$$

(iii) Prove that the law of the shifted walk $(X_{j+1}^{x,Q,S})_{j\in\mathbb{N}_0}$ conditioned on the event $\{\Theta_0 = 0, \Xi_0 = 0\}$ equals the law of $(X_j^{x+S_0(x),Q^{(1)},S^{(1)}})_{j\in\mathbb{N}_0}$. More precisely, for any $M \in \mathbb{N}$ and any $y_0, \dots, y_M \in \mathbb{Z}^d$, we have

$$\mathbb{P}\left\{X_1^{x,Q,S} = y_0, \dots, X_{M+1}^{x,Q,S} = y_M \mid \Theta_0 = 0, \Xi_0 = 1\right\}$$
$$= \mathbb{P}\left\{X_0^{x+S_0(x),Q^{(1)},S^{(1)}} = y_0, \dots, X_M^{x+S_0(x),Q^{(1)},S^{(1)}} = y_M\right\}.$$

In addition to the previous problem, we will also use the next result, which concerns the existence of ϵ-optimal strategies for Max. (A similar result also applies to Minnie.) To state it, it will be convenient to introduce the following notation. Given $Q \in \mathcal{A}$, let $u_Q : A \to \mathbb{R}$ be the function

$$u_Q(x) = \inf_{S\in\mathcal{A}} \mathbb{E}[F(X_\tau^{x,Q,S})].$$

Note that, with this notation, $u(x) = \sup_{Q\in\mathcal{A}} u_Q(x)$. The next proposition asserts that, for any $\epsilon > 0$, it is possible to choose a $Q_\epsilon \in \mathcal{A}$ such that $u_{Q_\epsilon} \geq u - \epsilon$ pointwise in A.

Proposition 3. *For any $\epsilon > 0$, there is a strategy $Q_\epsilon \in \mathcal{A}$ such that*

$$u_{Q_\epsilon}(x) \geq u(x) - \epsilon \quad \text{for each } x \in A.$$

Proof. By definition, for any $y \in A$, we can choose $Q_y \in \mathcal{A}$ such that

$$u_{Q_y}(y) \geq u(y) - \epsilon.$$

Fix a $y_* \in A$ for convenience. Define $Q = (Q_j)_{j\in\mathbb{N}_0}$ by the following rule:

$$Q_j(y_0, y_1, \dots, y_j) = \begin{cases} (Q_{y_0})_j(y_0, y_1, \dots, y_j), & \text{if } y_0 \in A, \\ (Q_{y_*})(y_0, y_1, \dots, y_j), & \text{otherwise.} \end{cases}$$

If $x \in A$, then $u_Q(x) = u_{Q_x}(x) \geq u(x) - \epsilon$, as desired. □

Actually, it will be useful to know that the previous result still holds true if we replace Q by $Q^{(1)}$. The next exercise asks you to check this.

Exercise 6. *Fix $x \in A$. Prove that, for any $\epsilon > 0$, there is a strategy Q_ϵ such that*

$$u(x + Q_{\epsilon 0}(x)) = \max_{e\in\mathbb{E}^d} u(x + e),$$
$$u_{Q_\epsilon^{(1)}}(y) \geq u(y) - \epsilon \quad \text{for each } y \in A.$$

(Recall that $Q_\epsilon^{(1)}$ is defined by (16).)

Finally, here is the proof of Proposition 1.

Proof of Proposition 1. We will only prove that $-p\Delta_{\mathbb{Z}^d}u - (1-p)\Delta^\infty_{\mathbb{Z}^d}u \geq 0$ holds pointwise in $\operatorname{int}(A)$. The corresponding inequality for v follows by symmetrical arguments.

Fix $x \in \operatorname{int}(A)$ and $Q, S \in \mathcal{A}$. By Problem 1,

$$\begin{aligned}\mathbb{E}[F(X_\tau^{x,Q,S})] &= \frac{p}{2d}\sum_{e\in\mathbb{E}^d}\mathbb{E}[F(X_\tau^{x,Q,S}) \mid \Theta_0 = 1, Z_0 = e] \\ &\quad + \frac{1-p}{2}\mathbb{E}[F(X_\tau^{x,Q,S}) \mid \Theta_0 = 0, \Xi_0 = 1] \\ &\quad + \frac{1-p}{2}\mathbb{E}[F(X_\tau^{x,Q,S}) \mid \Theta_0 = 0, \Xi_0 = 0] \\ &= \frac{p}{2d}\sum_{e\in\mathbb{E}^d}\mathbb{E}[F(X_\tau^{x+e,Q^{(1)},S^{(1)}})] \\ &\quad + \frac{1-p}{2}\mathbb{E}[F(X_\tau^{x+Q_0(x),Q^{(1)},S^{(1)}})] \\ &\quad + \frac{1-p}{2}\mathbb{E}[F(X_\tau^{x+S_0(x),Q^{(1)},S^{(1)}})]\end{aligned}$$

Let $u_Q : A \to \mathbb{R}$ be the function $u_Q(y) = \inf_{S\in\mathcal{A}}\mathbb{E}[F(X_\tau^{y,Q,S})]$. Taking the infimum over S, we find

$$\begin{aligned}u_Q(x) \geq \frac{p}{2d}\sum_{e\in\mathbb{E}^d} u_{Q^{(1)}}(x+e) + \frac{1-p}{2}u_{Q^{(1)}}(x+Q_0(x)) \\ + \frac{1-p}{2}\min_{e\in\mathbb{E}^d} u_{Q^{(1)}}(x+e).\end{aligned}$$

At this stage, we choose $\epsilon > 0$ and invoke Exercise 6, which allows us to fix $Q \in \mathcal{A}$ such that

$$u(x+Q_0(x)) = \max_{e\in\mathbb{E}^d} u(x+e) \quad \text{and} \quad u_{Q^{(1)}}(y) \geq u(y) - \epsilon \quad \text{for each } y \in A.$$

With this choice of Q, we find

$$\begin{aligned}u(x) \geq u_Q(x) \geq \frac{p}{2d}\sum_{e\in\mathbb{E}^d}(u(x+e)-\epsilon) + \frac{1-p}{2}\max_{e\in\mathbb{E}^d}(u(x+e)-\epsilon) \\ + \frac{1-p}{2}\min_{e\in\mathbb{E}^d}(u(x+e)-\epsilon).\end{aligned}$$

Since $\frac{p}{2d}\sum_{e\in\mathbb{E}^d}(1) + \frac{1-p}{2}(1) + \frac{1-p}{2}(1) = 1$, we conclude after sending $\epsilon \to 0$ that

$$u(x) \geq \frac{p}{2d}\sum_{e\in\mathbb{E}^d} u(x+e) + \frac{1-p}{2}\max_{e\in\mathbb{E}^d} u(x+e) + \frac{1-p}{2}\min_{e\in\mathbb{E}^d} u(x+e).$$

After rearranging, this becomes $-p\Delta_{\mathbb{Z}^d}u(x) - (1-p)\Delta^\infty_{\mathbb{Z}^d}u(x) \geq 0$ (see (11)). □

1.3. **Proof of Proposition 2.** We give two different proofs of the comparison principle, an easier one that applies when $p > 0$ and a slight variation when $p = 0$.

In both proofs, we will use the topology of $\mathbb{Z}^d$. Toward that end, say that an N-tuple $(x_0, \dots, x_N) \in \mathbb{Z}^{dN}$ is a *path of length* N if

$$|x_{i+1} - x_i| = 1 \quad \text{for each } i \in \{0, 1, \dots, N-1\}.$$

We say that a set $A \subseteq \mathbb{Z}^d$ is *path connected* if, for any $x, y \in A$, there is an $N \in \mathbb{N}_0$ and a path $(x_0, \dots, x_N)$ such that $x_0 = x$ and $x_N = y$. If A is not path connected, then it can be written as a union $A = \cup_i A_i$, where each set A_i (the so-called path components of A) are path connected and $A_i \cap A_j = \emptyset$ if $i \neq j$.

With the topological preliminaries out of the way, here is the proof of the proposition.

Proof of Proposition 2 when $p > 0$. Let $M = \max\{u_2(x) - u_1(x) \mid x \in A\}$ and $\tilde{A} = \{x \in A \mid u_2(x) - u_1(x) = M\}$. If $x \in \tilde{A} \cap \text{int}(A)$, then

$$\begin{aligned} u_1(x) + M = u_2(x) &\leq \frac{p}{2d} \sum_{e \in \mathbb{E}^d} u_2(x+e) + \frac{1-p}{2} \max_{e \in \mathbb{E}^d} u_2(x+e) \\ &\quad + \frac{1-p}{2} \min_{e \in \mathbb{E}^d} u_2(x+e) \\ &\leq \frac{p}{2d} \sum_{e \in \mathbb{E}^d} (u_1(x+e) + M) + \frac{1-p}{2} \max_{e \in \mathbb{E}^d} (u_1(x+e) + M) \\ &\quad + \frac{1-p}{2} \min_{e \in \mathbb{E}^d} (u_1(x+e) + M) \\ &\leq u_1(x) + M. \end{aligned}$$

It follows that all inequalities invoked above are actually equalities. In particular,

$$0 = \frac{1}{2d} \sum_{e \in \mathbb{E}^d} (u_2(x+e) - u_1(x+e) + M),$$

which implies, since each summand is nonpositive, that $u_2(x+e) = u_1(x+e) + M$ for each $e \in \mathbb{E}^d$.

We proved that if $x \in \tilde{A} \cap \text{int}(A)$, then $\{x + e \mid e \in \mathbb{E}^d\} \subseteq \tilde{A}$. From this, we deduce that if $\tilde{A}$ intersects some path component P of A, then $P \subseteq \tilde{A}$. At the same time, any path component of A necessarily intersects $\partial_{\text{int}} A$. Therefore, there is a point $x \in \tilde{A} \cap \partial_{\text{int}} A$, and this implies $M = \max\{u_2(x) - u_1(x) \mid x \in \partial_{\text{int}} A\}$. □

Proof of Proposition 2 when $p = 0$. Since the last proof applies otherwise, we assume that $p = 0$. As in the last proof, let $M = \max\{u_2(x) - u_1(x) \mid x \in A\}$ and $\tilde{A} = \{x \in A \mid u_2(x) - u_1(x) = M\}$. It will be convenient to let $K = \max\{u_2(x) \mid x \in \tilde{A}\}$ and $\tilde{A}_* = \{x \in \tilde{A} \mid u_2(x) = K\}$.

If $x \in \tilde{A}_* \cap \operatorname{int}(A)$, then

$$\begin{aligned} u_1(x) + M &= u_2(x) \\ &\le \frac{1}{2}\max_{e\in\mathbb{E}^d} u_2(x+e) + \frac{1}{2}\min_{e\in\mathbb{E}^d} u_2(x+e) \\ &\le \frac{1}{2}\max_{e\in\mathbb{E}^d}(u_1(x+e)+M) + \frac{1}{2}\min_{e\in\mathbb{E}^d}(u_1(x+e)+M) \\ &\le u_1(x) + M. \end{aligned}$$

If $p = 1$, we conclude as above. Otherwise, if $p < 1$, we instead deduce that

$$\max_{e\in\mathbb{E}^d} u_2(x+e) = \max_{e\in\mathbb{E}^d}(u_1(x+e)+M), \tag{17}$$

$$\min_{e\in\mathbb{E}^d} u_2(x+e) = \min_{e\in\mathbb{E}^d}(u_1(x+e)+M). \tag{18}$$

Since $u_2 \le u_1 + M$ pointwise in A, this implies that the maximum and minimum are both attained at points e such that $x + e \in \tilde{A}$. At the same time, u_2 is at most K in $\tilde{A}$, hence from the identity

$$K = \frac{1}{2}\max_{e\in\mathbb{E}^d} u_2(x+e) + \frac{1}{2}\min_{e\in\mathbb{E}^d} u_2(x+e)$$

we deduce that

$$K = \frac{1}{2}\max_{e\in\mathbb{E}^d} u_2(x+e) + \frac{1}{2}\min_{e\in\mathbb{E}^d} u_2(x+e)$$

hence $u_2(x+e) = K$ for each $e \in \mathbb{E}^d$. Since (17) and (18) both hold, we also know that $u_1(x+e) = K - M$ for each $e \in \mathbb{E}^d$.

We conclude that if $x \in \tilde{A}_* \cap \operatorname{int}(A)$, then $\{x + e \mid e \in \mathbb{E}^d\} \subseteq \tilde{A}_*$. We deduce from this, as before, that $\tilde{A}_*$ intersects $\partial_{\text{int}}A$, and, thus, $M = \max\{u_2(x) - u_1(x) \mid x \in \partial_{\text{int}}A\}$. □

1.4. **Optimal Strategies.** In this section, we use the value function u to determine optimal strategies for Max and Minnie. In fact, sub- and supersolutions of (13) can always be used to construct suboptimal strategies, as the next proposition shows.

To be precise, we will need the following construction. Given a function $w : A \to \mathbb{R}$, fix a function $E_w^Q : \operatorname{int}(A) \to \mathbb{E}^d$ such that

$$w(x + E_w^Q(x)) = \max\{w(x+e) \mid e \in \mathbb{E}^d\}.$$

Define a strategy $Q_w = (Q_{wj})_{j\in\mathbb{N}} \in \mathcal{A}$ by setting

$$Q_{wj}(y_0, \ldots, y_j) = \begin{cases} E_w^Q(y_j), & \text{if } y_j \in \operatorname{int}(A), \\ (1, 0, 0, \ldots, 0), & \text{otherwise.} \end{cases}$$

Similarly, fix a function $E_w^S : \operatorname{int}(A) \to \mathbb{E}^d$ such that

$$w(x + E_w^S(x)) = \min\{w(x+e) \mid e \in \mathbb{E}^d\}.$$

Let $S_w \in \mathcal{A}$ be defined analogously to Q_w.

The next result shows that sub- and supersolutions of (13) can be used to control the expected payout.

Proposition 4. *If $u_1, u_2 : A \to \mathbb{R}$ satisfy, for any $x \in \operatorname{int}(A)$,*

$$-p\Delta_{\mathbb{Z}^d} u_1(x) - (1-p)\Delta_{\mathbb{Z}^d}^{\infty} u_1(x) \geq 0 \geq -p\Delta_{\mathbb{Z}^d} u_2(x) - (1-p)\Delta_{\mathbb{Z}^d}^{\infty} u_2(x)$$

and $u_1(x) = u_2(x) = F(x)$ for each $x \in \partial_{\text{int}} A$, then, for any $Q, S \in \mathcal{A}$,

$$\mathbb{E}[F(X_\tau^{x,Q_{u_2},S})] \geq u_2(x), \quad \mathbb{E}[F(X_\tau^{x,Q,S_{u_1}})] \leq u_1(x).$$

Notice that the proposition says that if Max uses u_2 to determine his strategy, then his payout is at least $u_2(x)$, *no matter which strategy Minnie chooses.* Symmetrically, if Minnie uses u_1, then she pays out no more than $u_1(x)$, irrespective of Max's strategy.

Proof. Fix $x \in A$ and suppose that $S \in \mathcal{A}$. Given any $j \in \mathbb{N}_0$, we compute, as in the proof of Proposition 1,

$$\begin{aligned}
\mathbb{E}[u_2(X_{j+1}^{x,Q,S})] &= \frac{p}{2d} \sum_{e \in \mathbb{E}^d} \mathbb{E}[u_2(X_j^{x,Q,S} + e)] + \frac{1-p}{2} \mathbb{E}[u_2(X_j^{x,Q,S} + E_{u_2}^{Q}(X_j^{x,Q,S}))] \\
&\quad + \frac{1-p}{2} \mathbb{E}[u_2(X_j^{x,Q,S} + S_j)] \\
&\geq \frac{p}{2d} \sum_{e \in \mathbb{E}^d} \mathbb{E}[u_2(X_j^{x,Q,S} + e)] + \frac{1-p}{2} \mathbb{E}[\max_{e \in \mathbb{E}^d} u_2(X_j^{x,Q,S} + e)] \\
&\quad + \frac{1-p}{2} \mathbb{E}[\min_{e \in \mathbb{E}^d} u_2(X_j^{x,Q,S} + e)] \\
&\geq \mathbb{E}[u_2(X_j^{x,Q,S})].
\end{aligned}$$

This proves that the function $j \mapsto \mathbb{E}[u_2(X_j^{x,Q_{u_2},S})]$ is nondecreasing. Sending $j \to \infty$ (and invoking Lebesgue's dominated convergence theorem), we find

$$u_2(x) = \mathbb{E}[u_2(x)] = \mathbb{E}[u_2(X_0^{x,Q_{u_2},S})] \leq \lim_{j \to \infty} \mathbb{E}[u_2(X_j^{x,Q_{u_2},S})] = \mathbb{E}[F(X_\tau^{x,Q_{u_2},S})].$$

The proof involving u_1 is entirely analogous. □

In view of Theorem 4, the implication of Proposition 4 is fundamental. We know that the value function u is both a sub- and supersolution. Hence if Max uses Q_u as his strategy, he is guaranteed a payout of at least $u(x)$, while if Minnie uses S_u as her strategy, she will owe Max no more than $u(x)$ at the end of the game. Further, by definition of the upper and lower value functions, Max and Minnie can't expect to do any better. Therefore, Q_u and S_u are (loosely speaking) the optimal strategies for Max and Minnie, and, if they both play the game optimally, the expected payout of the game started at x is exactly $u(x)$.

For the sake of precision, the optimality of Q_u and S_u is summarized in the next theorem.

Theorem 5. *If $u : A \to \mathbb{R}$ is the value function of the game, then, for any $x \in A$ and any $Q, S \in \mathcal{A}$,*

$$u(x) \leq \mathbb{E}[F(X_\tau^{x,Q_u,S})], \quad \mathbb{E}[F(X_\tau^{x,Q,S_u})] \leq u(x).$$

In particular, if Max plays strategy Q_u and Minne plays strategy S_u, then the expected payout equals $u(x)$.

1.5. **Special Case: $p = 0$.** When $p = 0$, the equation $-\Delta^\infty_{\mathbb{Z}^d} u = 0$ is equivalent to a discrete version of the Lipschitz extension problem. To see how this works, let us first define the discrete Lipschitz seminorm $\mathrm{Lip}_{\mathbb{Z}^d}$ by

$$\mathrm{Lip}_{\mathbb{Z}^d}(u; K) = \inf\{C > 0 \mid \forall x, y \in K \quad u(x) - u(y) \le C\|x - y\|_1\},$$

where $\|\cdot\|_1$ is the ℓ^1 norm (i.e., if $q = (q_1, \dots, q_d)$, then $\|q\|_1 = \sum_{i=1}^d |q_i|$). This is a natural generalization of the seminorm Lip_φ defined for functions in $\mathbb{R}^d$ in the introduction.

As in $\mathbb{R}^d$, we can define absolutely minimizing Lipschitz functions.

Definition 2. *A function $u : A \to \mathbb{R}$ defined in some subset $A \subseteq \mathbb{Z}^d$ is called an* absolutely minimizing Lipschitz function *if, for any finite set $K \subseteq A$ and any function $v : K \to \mathbb{R}$ such that $v(x) = u(x)$ for each $x \in \partial_{\mathrm{int}} K$, we have*

$$Lip_{\mathbb{Z}^d}(u; K) \le Lip_{\mathbb{Z}^d}(v; K).$$

The next proposition asserts that any absolutely minimizing Lipschitz function is a solution of the equation $-\Delta^\infty_{\mathbb{Z}^d} u = 0$.

Proposition 5. *Let $A \subseteq \mathbb{Z}^d$. If $u : A \to \mathbb{R}$ is an absolutely minimizing Lipschitz function, then $-\Delta^\infty_{\mathbb{Z}^d} u(x) = 0$ for each $x \in \mathrm{int}(A)$.*

In the course of the proof, it will be convenient to use the observations of the next two exercises.

Exercise 7. *Given $x \in \mathbb{Z}^d$, let $K = \{y \in \mathbb{Z}^d \mid |y - x| \le 1\}$. Prove that if $u : K \to \mathbb{R}$, then*

$$Lip_{\mathbb{Z}^d}(u; K) = \max\left\{|u(x + e) - u(x)| \mid e \in \mathbb{E}^d\right\}.$$

Exercise 8. *Fix $\{a(e) \mid e \in \mathbb{E}^d\} \subseteq \mathbb{R}$ and $a(0) \in \mathbb{R}$. Show that if, for any $t \in \mathbb{R}$,*

$$\max\{|a(e) - a(0) + t| \mid e \in \mathbb{E}^d\} \ge \max\{|a(e) - a(0)| \mid e \in \mathbb{E}^d\},$$

then $a(0) = \frac{1}{2}\max_{e\in\mathbb{E}^d} a(e) + \frac{1}{2}\min_{e\in\mathbb{E}^d} a(e)$. Further, prove that the converse also holds.

Proof of Proposition 5. Fix $x \in \mathrm{int}(A)$ and let $K = \{y \in \mathbb{Z}^d \mid |y - x| \le 1\}$. Since $x \in \mathrm{int}(A)$, we know that $K \subseteq A$.

For $t \in \mathbb{R}$, consider the function $v_t : K \to \mathbb{R}$ such that $v_t(x) = u(x) + t$ and $v_t(y) = u(y)$ if $y \in K \setminus \{x\}$. Since u is an absolutely minimizing Lipschitz function,

$$\mathrm{Lip}_{\mathbb{Z}^d}(u; K) \le \mathrm{Lip}_{\mathbb{Z}^d}(v_t; K),$$

which, due to the definition of K and Exercise 7, implies that

$$\max\left\{|u(x + e) - u(x)| \mid e \in \mathbb{E}^d\right\} \le \max\{|u(x + e) - t - u(x)| \mid e \in \mathbb{E}^d\}.$$

Thus, by Exercise 8, $u(x) = \frac{1}{2}\max_{e\in\mathbb{E}^d} u(x+e) + \frac{1}{2}\min_{e\in\mathbb{E}^d} u(x+e)$, which is equivalent to $-\Delta^\infty_{\mathbb{Z}^d} u(x) = 0$. □

It is worth pointing out that the converse of Proposition 5 also holds.

Proposition 6. *Let $A \subseteq \mathbb{Z}^d$. If $u : A \to \mathbb{R}$ satisfies $-\Delta^{\infty}_{\mathbb{Z}^d} u(x) = 0$ for each $x \in \text{int}(A)$, then u is an absolutely minimizing Lipschitz function.*

The proof of this result will be taken up in the problems at the end of this section.

It is worthwhile to note that Exercise 7 can be extended to certain other subsets of $\mathbb{Z}^d$. Toward that end, some additional definitions will be useful.

Definition 3. *An N-tuple $(x_0, x_1, \dots, x_N) \in \mathbb{Z}^{dN}$ is a* path beginning at x_0 and terminating at x_N *if*

$$|x_{i+1} - x_i| = 1 \quad \textit{for each} \quad i \in \{0, 1, \dots, N-1\}.$$

The length *of the path $(x_0, x_1, \dots, x_N)$ is defined to be N.*

A path $(x_0, x_1, \dots, x_N)$ is a geodesic of length N *if $N = \|x_N - x_0\|_1$, where $\|\cdot\|_1$ is the ℓ^1 norm.*

The next problems determines the sets for which $\text{Lip}_{\mathbb{Z}^d}$ can be determined by maximizing $|u(x+e) - u(x)|$.

Problem 2. *Prove that if $(x_0, x_1, \dots, x_N)$ is a geodesic of length N, then N is the minimal length of all paths beginning at x_0 and terminating at x_N.*

Problem 3. *Say that a set $A \subseteq \mathbb{Z}^d$ is* geodesically connected *if, for any $x, y \in A$, there exists a geodesic $(x_0, x_1, \dots, x_N)$ such that $x_0 = x$, $x_N = y$, and $x_0, \dots, x_N \in A$. Prove that if A is geodesically connected and $u : A \to \mathbb{R}$, then*

$$Lip_{\mathbb{Z}^d}(u; A) = \max\left\{|u(x+e) - u(x)| \ \mid \ x \in \text{int}(A), \ e \in \mathbb{E}^d\right\}.$$

It is worth emphasizing that the identity in the previous problem has a continuum analogue. In particular, if $U \subseteq \mathbb{R}^d$ is an open convex set, then, for any C^1 function $f : \overline{U} \to \mathbb{R}$,

$$\text{Lip}_{\varphi}(f; \overline{U}) = \sup\left\{\varphi^*(Df(x)) \ \mid \ x \in \overline{U}\right\}.$$

See Problems 14 and 15 for this identity and more.

Problem 4. *Say that a set $A \subseteq \mathbb{Z}^d$ is* path connected *if, for any $x, y \in A$, there is a path $(x_0, x_1, \dots, x_N)$ such that $x_0, x_1, \dots, x_N \in A$, $x_0 = x$, and $x_N = y$. Prove that if A is a finite, path connected set and $u : A \to \mathbb{R}$, then there is a constant $C(A) > 0$ depending on A such that*

$$Lip_{\mathbb{Z}^d}(u; A) \leq C(A) \max\{|u(x+e) - u(x)| \ \mid \ x \in \text{int}(A), \ e \in \mathbb{E}^d\}.$$

Problem 5. *Find an example of a set $A \subseteq \mathbb{Z}^d$ and a function $u : A \to \mathbb{R}$ such that*

$$Lip_{\mathbb{Z}^d}(u; A) > \max\left\{|u(x+e) - u(x)| \ \mid \ x \in \text{int}(A), \ e \in \mathbb{E}^d\right\}.$$

1.6. Further Problems.

Problem 6. *When $p = 1$, we simply obtain the equation $-\Delta_{\mathbb{Z}^d} u = 0$. Prove that a function $u : A \to \mathbb{R}$ satisfies $-\Delta_{\mathbb{Z}^d} u(x) = 0$ for each $x \in \operatorname{int}(A)$ if and only if*

$$\sum_{x\in\operatorname{int}(A)} \sum_{e\in\mathbb{E}^d} (u(x+e) - u(x))^2$$
$$= \min \left\{ \sum_{x\in\operatorname{int}(A)} \sum_{e\in\mathbb{E}^d} (v(x+e) - v(x))^2 \;\middle|\; v : A \to \mathbb{R} \;:\; \forall x \in \partial_{\text{int}} A \quad v(x) = u(x) \right\}.$$

2. Minimal Lipschitz Extensions

In this section, we study the minimal Lipschitz extension problem (1). As in the introduction, we consider the specific class of solution called *absolutely minimizing Lipschitz functions*. The main goal will be to prove that these functions are characterized by another property, called *comparison with cones*.

Before defining the comparison-with-cones property, we enlarge the class of functions φ we consider in the Lipschitz seminorm $\operatorname{Lip}_\varphi$. We will see that it is not necessary to assume that the "norm" is symmetric, hence it is natural to consider so-called *Finsler norms*.

2.1. Finsler Norms.

It turns out that the results in these notes do not rely on the symmetry assumption inherent in the definition of a norm. Toward that end, let us consider not only norms but also, more generally, *Finsler norms*.

Definition 4. *A function $\varphi : \mathbb{R}^d \to [0, \infty)$ is said to be a* Finsler norm *if it satisfies the following three properties:*

(i) *(positive definiteness) $\varphi(q) = 0$ if and only if $q = 0$.*

(ii) *(positive 1-homogeneity) For any $q \in \mathbb{R}^d$ and any $\lambda \geq 0$,*
$$\varphi(\lambda q) = \lambda\varphi(q).$$

(iii) *(subadditivity) For any $q_1, q_2 \in \mathbb{R}^d$,*
$$\varphi(q_1 + q_2) \leq \varphi(q_1) + \varphi(q_2).$$

For the remainder of this section, we fix a Finsler norm φ. As in the introduction, we define the φ-Lipschitz seminorm of a function $g : C \to \mathbb{R}$ in a closed set C by

$$\operatorname{Lip}_\varphi(g; C) = \inf \{M > 0 \mid \forall x, y \in C \quad g(x) \leq g(y) + M\varphi(x - y)\}. \tag{19}$$

Exercise 9. *Prove that a Finsler norm φ is a norm if and only if, in addition to conditions (i), (ii), and (iii) above, it also satisfies*

$$\textit{(iv) (symmetry)} \quad \varphi(-q) = \varphi(q) \quad \textit{for each} \quad q \in \mathbb{R}^d.$$

Exercise 10. *Fix a finite subset $\{v_1, \ldots, v_N\} \subseteq \mathbb{R}^d$ and define φ by*

$$\varphi(q) = \max \{\langle v_1, q\rangle, \ldots, \langle v_N, q\rangle\}.$$

Assume that $\varphi(\mathbb{R}^d) \subseteq [0,\infty)$ and φ is positive definite: $\varphi(q) \geq 0$ if and only if $q = 0$. Prove that, in this case, φ is a Finsler norm. (See Problem 8 below for a geometric condition on $\{v_1, \dots, v_N\}$ that is equivalent to the positive-definiteness of φ.)

Exercise 11. *Let $g : C \to \mathbb{R}$ be a function defined in a closed set $C \subseteq \mathbb{R}$. Prove that*

$$Lip_\varphi(g; C) = \sup\left\{ \frac{g(x) - g(y)}{\varphi(x-y)} \mid x, y \in C,\ x \neq y \right\}.$$

(Recall that $Lip_\varphi(g; C)$ is defined by (19).)

2.2. **Comparison with Cones.** The goal of this section will be to prove that a function is an absolutely minimizing Lipschitz function if and only if it satisfies the two comparison-with-cones properties, namely, comparison with cones from above and comparison with cones from below. These are defined next.

In the next definition, we assume that some Finsler norm φ and some open set $U \subseteq \mathbb{R}^d$ are given.

Definition 5. *A continuous function $u : U \to \mathbb{R}$ satisfies* comparison with φ-cones from above (with respect to φ) *if, for any open $V \subset\subset U$,*[2] *any $v \in \mathbb{R}^d \setminus V$, and any $\lambda > 0$,*

$$\max\left\{u(x) - \lambda\varphi(x-q) \mid x \in \overline{V}\right\} = \max\left\{u(x) - \lambda\varphi(x-q) \mid x \in \partial V\right\}.$$

Similarly, a continuous function $v : U \to \mathbb{R}$ satisfies comparison with φ-cones from below (with respect to φ) *if, for any $V \subset\subset U$, any $v \in \mathbb{R}^d \setminus V$, and any $\lambda > 0$,*

$$\max\left\{v(x) + \lambda\varphi(q-x) \mid x \in \overline{V}\right\} = \min\left\{v(x) + \lambda\varphi(q-x) \mid x \in \partial V\right\}.$$

It will be useful to keep in mind another, more geometric formulation of the definition above. This is tackled in the next exercise.

Exercise 12. *Suppose that $u : U \to \mathbb{R}$ satisfies comparison with cones from above, and assume that $V \subset\subset U$, $q \in \mathbb{R}^d \setminus \{0\}$, $\lambda > 0$, and $M \in \mathbb{R}$ are chosen in such a way that*

$$u(x) \leq M + \lambda\varphi(x-q) \quad \textit{for each} \quad x \in \partial V.$$

Prove that $u(x) \leq M + \lambda\varphi(x-v)$ must hold for each $x \in \overline{V}$. Also, prove that the converse also holds: any function with this property necessarily satisfies comparison with cones from above.

Exercise 13. *Suppose that $v : U \to \mathbb{R}$ satisfies comparison with cones from below, and assume that $V \subset\subset U$, $q \in \mathbb{R}^d \setminus V$, $\lambda > 0$, and $m \in \mathbb{R}$ are chosen in such a way that*

$$v(x) \geq m - \lambda\varphi(q-x) \quad \textit{for each} \quad x \in \partial V.$$

Prove that $v(x) \geq m - \lambda\varphi(q-x)$ must hold for each $x \in \overline{V}$. Also, prove that the converse also holds: any functino with this property necessarily satisfies comparison with cones from below..

[2]Recall that the notation $V \subset\subset U$ means that V is bounded and $\overline{V} \subseteq U$.

We will see that quite a lot can be said about functions satisfying the comparison-with-cones properties, and doing so requires only relatively elementary (though clever) arguments. The next theorem, which is the main result of this section, states that a function is an absolutely minimizing Lipschitz extension if and only if it satisfies both comparison-with-cones properties.

Theorem 6. *Let φ be a Finsler norm in $\mathbb{R}^d$ and fix an open set $U \subseteq \mathbb{R}^d$. Given a continuous function $u : U \to \mathbb{R}$, the following are equivalent:*

(i) *u is a φ-absolutely minimizing Lipschitz function in U.*
(ii) *u satisfies comparison with φ-cones from above and below in U.*

The theorem above will be useful later, in Section **??**, when we prove that the minimal Lipschitz extension problem has a unique absolute minimizer. For now, to appreciate the connection between the comparison-with-cones properties and Lipschitz functions, consider the following preliminary result.

Proposition 7. *Let $U \subseteq \mathbb{R}^d$ be an open set. If $u : \overline{U} \to \mathbb{R}$ satisfies comparison with cones from above in U, then, for any open set $V \subset\subset U$,*

$$Lip_{\varphi}(u; \overline{V}) = \sup \left\{ \frac{u(y) - u(x)}{\varphi(y - x)} \mid x \in V,\ y \in \partial V \right\}.$$

The proof given below is what is referred to as a "maximum principle" argument (also called a "comparison argument" or "barrier argument").

Assume that u satisfies comparison with φ-cones from above in U. If we think of difference quotients $\frac{u(y)-u(x)}{\varphi(y-x)}$ as "slopes," then the previous proposition shows that the maximal slope of u can always be computed using one point on the boundary and one in the interior (and with the orientation such that the slope goes "down" into the domain).

Proof. Let $M = \sup \left\{ \frac{u(x)-u(y)}{\varphi(x-y)} \mid x \in U,\ y \in \partial U \right\}$. If $M = \infty$, there is nothing to prove as then also $\mathrm{Lip}_{\varphi}(u; \overline{V}) = \infty$. Hence assume that $M < \infty$ in what follows.

By definition, $M \leq \mathrm{Lip}_{\varphi}(u; V)$ holds. Therefore, we only need to prove that $M \geq \mathrm{Lip}_{\varphi}(u; \overline{V})$.

Suppose $x \in V$. By definition of M,

$$u(y) \leq u(x) + M\varphi(y - x) \quad \text{for each} \quad y \in \partial V.$$

Of course, $u(x) \leq u(x) + M\varphi(x - x)$ also holds. Therefore, the inequality $u \leq u(x) + M\varphi(\cdot - x)$ holds pointwise in $V \setminus \{x\}$. Since u satisfies comparison with cones from above,

$$u(x') \leq u(x) + M\varphi(x' - x) \quad \text{for each} \quad x' \in \overline{V}.$$

Since x was arbitrary, we deduce that

$$u(x') \leq u(x) + M\varphi(x' - x) \quad \text{for each} \quad x, x' \in \overline{V},$$

and, therefore, $\mathrm{Lip}_{\varphi}(u; \overline{V}) \leq M$ by definition. ∎

It should be noted that the previous proposition already implies that the comparison-with-cones-from-above property rules out functions with certain shapes. That is explored in the next exercise.

Exercise 14. *Let $U \subseteq \mathbb{R}^d$ be open and assume that $u : U \to \mathbb{R}$ satisfies comparison with cones from above. Prove that if there is a $V \subset\subset U$ and an $m \in \mathbb{R}$ such that*

$$u(x) = m \quad \text{for each} \quad x \in \partial V, \quad u(x) \geq m \quad \text{for each} \quad x \in \overline{V},$$

then $u(x) = m$ for each $x \in \overline{V}$.

Problem 7. *Say that a function $u : U \to \mathbb{R}$ is* locally constant *if, for any $x \in U$, there is an open set $V \subset U$ such that $x \in V$ and the restriction of u to V is constant. Prove that if U is connected, u satisfies comparison with cones from above, and u has a global maximum in U, then u is locally constant. (In particular, u is constant by connectedness of U.)*

Exercise 15. *Prove the corresponding version of Proposition 7 with "comparison with cones from above" replaced by "comparison with cones from below." (Hint: "Slopes" should go "up" as you go from the boundary into the domain.)*

2.3. **McShane-Whitney Extensions.** In the introduction, we claimed that the minimal Lipschitz extension problem need not have a unique minimizer in general. This motivated the definition of absolute minimizer (or absolutely minimizing Lipschitz function). In this subsection, we define the McShane-Whitney extensions, which are useful in miriad ways, not the least being the proof of nonuniqueness of minimizers.

In what follows, given a closed set $C \subseteq \mathbb{R}^d$ and a function $u : C \to \mathbb{R}$, we say that u is *uniformly Lipschitz continuous* in C if

$$\text{Lip}_\varphi(u; C) < \infty.$$

Note that, by definition, this implies that, for each $x, y \in C$,

$$u(y) - \text{Lip}_\varphi(u; C)\varphi(y - x) \leq u(x) \leq u(y) + \text{Lip}_\varphi(u; C)\varphi(x - y).$$

These inequalities motivate the following definition.

Definition 6. *Given an open set $U \subseteq \mathbb{R}^d$ and a uniformly Lipschitz continuous function $u : \overline{U} \to \mathbb{R}$, the* McShane-Whitney extensions *of u are the functions $u^+ : \overline{U} \to \mathbb{R}$ and $u_- : \overline{U} \to \mathbb{R}$ given by*

$$u^+(x) = \inf_{y \in \partial U} \left\{ u(y) + Lip_\varphi(u; \partial U)\varphi(x - y) \right\},$$
$$u_-(x) = \sup_{y \in \partial U} \left\{ u(y) - Lip_\varphi(u; \partial U)\varphi(y - x) \right\}.$$

We claim that u^+ and u_- are always minimal Lipschitz extensions, i.e., minimizers of the variational problem (1). To see this, we begin with the following observation.

Exercise 16. *Prove that if $U \subseteq \mathbb{R}^d$ is an open set and $u : \overline{U} \to \mathbb{R}$ is uniformly Lipschitz continuous, then*

$$u^+(x) = u(x) = u_-(x) \quad \textit{for each} \quad x \in \partial U,$$
$$Lip_\varphi(u^+;\overline{U}) = Lip_\varphi(u;\partial U) = Lip_\varphi(u_-;\overline{U}).$$

The result of the previous exercise shows that u^+ and u_- are minimal Lipschitz extensions, as established by the next proposition.

Proposition 8. *Given an open set $U \subseteq \mathbb{R}^d$ and a uniformly Lipschitz function $u : \overline{U} \to \mathbb{R}$, if $v : \overline{U} \to \mathbb{R}$ satisfies $v(x) = u(x)$ for each $x \in \partial U$, then*

$$Lip_\varphi(u_-;\overline{U}) = Lip_\varphi(u^+;\overline{U}) \leq Lip_\varphi(v;\overline{U}).$$

In particular, u^+ and u_- are both minimizers of the minimal Lipschitz extension problem (1)*.*

Proof. It suffices to observe that, since $v(x) = u(x)$ for each $x \in \partial U$, we have

$$\mathrm{Lip}_\varphi(u;\partial U) = \mathrm{Lip}_\varphi(v;\partial U) \leq \mathrm{Lip}_\varphi(v;\overline{U}).$$

Thus, the desired conclusion follows from the previous exercise. □

Exercise 17. *Let $U \subseteq \mathbb{R}^d$ be an open set and assume that $u : \overline{U} \to \mathbb{R}$ is uniformly Lipschitz continuous. Show that $v : \overline{U} \to \mathbb{R}$ is a minimal Lipschitz extension of u if and only if $v(x) = u(x)$ for each $x \in \partial U$ and*

$$Lip_\varphi(v;\overline{U}) = Lip_\varphi(v;\partial U).$$

Problem 10 below asks you to prove that there is a function u such that $u^+ \neq u_-$.

2.4. **One-Sided Absolute Minimizers.** We defined the notion of an absolutely miniming Lipschitz function in Section 0.1. In what follows, it will be convenient to generalize the definition slightly. We will consider absolutely *subminimizing* and absolutely *superminimizing* Lipschitz functions. These are one-sided notions of (absolute) minimality.

Before we state the definitions, some additional notation will be helpful. Given a function $u : C \to \mathbb{R}$ defined in some set $E \subseteq \mathbb{R}^d$, define $\mathcal{F}_u^+(E)$ and $\mathcal{F}_u^-(E)$ by

$$\mathcal{F}_u^+(E) = \{v : E \to \mathbb{R} \mid \forall x \in E \quad v(x) \geq u(x), \quad \forall x \in \partial E \quad v(x) = u(x)\},$$
$$\mathcal{F}_u^-(E) = \{v : E \to \mathbb{R} \mid \forall x \in E \quad v(x) \leq u(x), \quad \forall x \in \partial E \quad v(x) = u(x)\}.$$

Definition 7. *Given an open set $U \subseteq \mathbb{R}^d$, a continuous function $u : U \to \mathbb{R}$ is said to be an* absolutely subminimizing Lipschitz function *(respectively,* absolutely superminimizing Lipschitz function*) in U if it is locally Lipschitz continuous in U and, for any $V \subset\subset U$,*

$$Lip_\varphi(u;\overline{V}) = \min\left\{Lip_\varphi(v;\overline{V}) \mid v \in \mathcal{F}_u^-(\overline{V})\right\},$$
$$\left(\textit{respectively,} \quad Lip_\varphi(u;\overline{V}) = \min\left\{Lip_\varphi(v;\overline{V}) \mid v \in \mathcal{F}_u^+(\overline{V})\right\}\right).$$

We will see below that, if nothing else, these one-sided notions will help us to organize our thoughts.

One might be tempted to conclude at this stage that a function is an absolutely minimizing Lipschitz function if and only if it is both absolutely subminimizing and absolutely superminimizing. This is, indeed, true, as we can see using the McShane-Whitney extensions.

Exercise 18. *Prove that a function $u : \overline{U} \to \mathbb{R}$ is absolutely minimizing if and only if it is both absolutely subminimizing and absolutely superminimizing. (Hint: Use the characterization established in Exercise 17.)*

2.5. **Cone Comparison implies Minimizing.** In this subsection, we establish that the cone comparison properties imply the one-sided absolute minimization properties. This is a particularly simple application of Proposition 7.

Proposition 9. *Let $U \subseteq \mathbb{R}^d$ be an open set. If $u : U \to \mathbb{R}$ satisfies comparison with cones from above and u is locally Lipschitz continuous in U, then u is absolutely subminimizing.*

Proof. Fix an open set $V \subset\subset U$ and assume that $v \in \mathcal{F}_u^-(\overline{V})$, i.e., $v : \overline{V} \to \mathbb{R}$ satisfies $v(x) \leq u(x)$ for each $x \in \overline{V}$ with equality for $x \in \partial V$. To see that u is absolutely subminimizing, we need to show that $\text{Lip}_\varphi(u; \overline{V}) \leq \text{Lip}_\varphi(v; \overline{V})$.

Toward that end, since $v \leq u$, we can invoke Proposition 7 to find

$$\text{Lip}_\varphi(u; \overline{V}) \leq \sup \left\{ \frac{v(y) - v(x)}{\varphi(y - x)} \mid y \in \partial V, \ x \in \overline{V} \right\} \leq \text{Lip}_\varphi(v; \overline{V}).$$

□

By completely analogous arguments, we deduce that a locally Lipschitz function that satisfies comparison with cones from below is absolutely superminimizing.

Proposition 10. *Let $U \subseteq \mathbb{R}^d$ be an open set. If $u : \overline{U} \to \mathbb{R}$ satisfies comparison with φ-cones from below in U and u is locally Lipschitz continuous in U, then u is φ-absolutely subminimizing.*

Exercise 19. *Prove Proposition 10 by first proving a version of Proposition 7 with "comparison with cones from above" replaced by "comparison with cones from below" and then mimicking the proof of Proposition 9.*

There is a small wrinkle in the results above. We assumed that u was locally Lipschitz continuous in U, as, indeed, is required in the definition of absolutely sub- and superminimizing (Definition 7). The next result shows that this is no restriction: if u satisfies comparison with φ-cones from above or below in U, then u is locally Lipschitz in U.

In the result, we will use the notion of the *oscillation* of a function. Toward that end, if $u : A \to \mathbb{R}$ for some $A \subseteq \mathbb{R}^d$, then the oscillation $\text{osc}(u; A)$ is defined by

$$\text{osc}(u; A) = \sup\{|u(x) - u(y)| \mid x, y \in A\}.$$

Proposition 11. *Let $U \subseteq \mathbb{R}^d$ be an open set. If $u : U \to \mathbb{R}$ is a continuous function satisfying comparison with φ-cones from above in U, then, for any $V \subset\subset V' \subset\subset U$, there is a constant $C(V, V') < \infty$ such that*

$$Lip_\varphi(u; \overline{V}) \leq C(V, V')\, osc(u; V').$$

In particular, $Lip_\varphi(u; \overline{V}) \leq 2C(V, V') \max\{|u(x)| \mid x \in \overline{V}'\}$, and u is locally Lipschitz continuous in U.

Proof. Fix $V \subset\subset V' \subset\subset U$. Define $C(V, V') > 0$ by

$$C(V, V') = \max\left\{ \frac{1}{\varphi(x-y)} \mid x \in V,\ y \in \partial V' \right\}.$$

Note that $C(V, V') < \infty$ since $V \subset\subset V'$.

Fix $x_0 \in V$. For any $x \in \partial V'$, we compute

$$u(x) - u(x_0) \leq \mathrm{osc}(u; V') \leq C(V, V')\mathrm{osc}(u; V')\varphi(x - x_0).$$

Thus, by comparison with cones from above,

$$\begin{aligned} &\max\left\{ u(x) - C(V, V')\mathrm{osc}(u; V')\varphi(x - x_0) \mid x \in \overline{V \setminus \{x_0\}} \right\} \\ &\quad = \max\left\{ u(x) - C(V, V')\mathrm{osc}(u; V')\varphi(x - x_0) \mid x \in \partial(V \setminus \{x_0\}) \right\} = u(x_0). \end{aligned}$$

In particular, for any $x \in V'$,

$$u(x) \leq u(x_0) + C(V, V')\mathrm{osc}(u; V')\varphi(x - x_0).$$

Since x_0 was an arbitrary point in V, this implies

$$\mathrm{Lip}_\varphi(u; V) \leq C(V, V')\mathrm{osc}(u; V').$$

Further, it is straightforward to check that $\mathrm{osc}(u; V') \leq 2 \max\{|u(x)| \mid x \in \overline{V'}\}$ so we can also say that $\mathrm{Lip}_\varphi(u; V) \leq 2C(V, V') \max\{|u(x)| \mid x \in \overline{V'}\}$.

Finally, if $K \subseteq U$ is a compact set contained in U, then there are open sets $V, V' \subseteq \mathbb{R}^d$ such that $V \subset\subset V' \subset\subset U$ such that $K \subseteq V$, and then $\mathrm{Lip}_\varphi(u; K) \leq \mathrm{Lip}_\varphi(u; \overline{V}) < \infty$ by our computations above. This proves that u is locally Lipschitz continuous in U according to the definition. □

Remark 2 (improvement of regularity)**.** *The previous result is an example of* improvement of regularity *in the theory of elliptic PDE. We see that if u is a bounded function satisfying comparison with cones from above in U, then, for any $V \subset\subset U$, we have a bound*

$$Lip_\varphi(u; \overline{V}) \leq C(V) \sup\{|u(x)| \mid x \in U\}.$$

Hence the map $u \mapsto u \restriction_{\overline{V}}$ improves *regularity from "bounded and continuous in U" to "bounded and uniformly Lipschitz continuous in V."*

Combining Propositions 10 and 11, we conclude that if u satisfies comparison with φ-cones from above in U, then u is a φ-absolutely subminimizing Lipschitz function in U.

2.6. **Minimizing implies Cone Comparison.** Finally, we prove that absolutely sub- and superminimizing functions satisfy comparison with cones from above and below, respectively.

Proposition 12. *Let $U \subseteq \mathbb{R}^d$ be an open set. If $u : U \to \mathbb{R}$ is φ-absolutely subminimizing in U, then u satisfies comparison with φ-cones from above in U.*

Proof. Fix an open set $V \subset\subset U$, a $q \in \mathbb{R}^d \setminus V$, and a $\lambda > 0$. We need to prove that

$$\max\left\{u(x) - \lambda\varphi(x - v) \mid x \in \overline{U}\right\} = \max\left\{u(x) - \lambda\varphi(x - v) \mid x \in \partial U\right\}.$$

Let $M = \max\{u(x) - \lambda\varphi(x - v) \mid x \in \partial U\}$. What we wish to prove is equivalent to establishing that

$$u(x) \leq M + \lambda\varphi(x - v) \quad \text{for each} \quad x \in \overline{V}.$$

Toward this end, let $W = \{x \in V \mid u(x) > M + \lambda\varphi(x - v)\}$. We need to show that W is empty. Notice that W is open and $W \subset\subset U$. Let $w(x) = M + \lambda\varphi(x - v)$ and notice that $w \in \mathcal{F}_u^-(\overline{W})$. Thus, since u is absolutely subminimizing,

$$\text{Lip}_\varphi(u; \overline{W}) \leq \text{Lip}_\varphi(w; \overline{W}). \tag{20}$$

We claim that (20) implies that W is empty. Indeed, if W were not empty, then we could let $x_0 \in W$. Since $v \notin W$ and $x_0 \in W$, we can find $\mu \in [0, 1)$ such that

$$v + \mu(x_0 - v) \in \partial W.$$

Since φ is a Finsler norm and $\mu \geq 0$,

$$\begin{aligned}
\varphi(x_0 - v) &= \mu\varphi(x_0 - v) + (1 - \mu)\varphi(x_0 - v) \\
&= \varphi(\mu(x_0 - v)) + \varphi((1 - \mu)(x_0 - v)) \\
&= \varphi(\mu(x_0 - v)) + \varphi(\{x_0 - v\} - \{\mu(x_0 - v\}).
\end{aligned}$$

In particular, in terms of w,

$$w(x_0) = w(v + \mu(x_0 - v)) + \lambda\varphi(\{x_0 - v\} - \{\mu(x_0 - v)\})$$

and, thus,

$$\text{Lip}_\varphi(u; \overline{W}) \geq \frac{u(x_0) - u(v + \mu(x_0 - v))}{\varphi(\{x_0 - v\} - \{\mu(x_0 - v)\})} > \frac{w(x_0) - w(v + \mu(x_0 - v))}{\varphi(\{x_0 - v\} - \{\mu(x_0 - v)\})} = \lambda.$$

Combining this with (20), we deduce that $\text{Lip}_\varphi(w; \overline{W}) > \lambda$, but the definition of w readily implies $\text{Lip}_\varphi(w; \overline{W}) \leq \lambda$. This contradiction forces us to conclude that W is empty. □

2.7. **Further Exercises.**

Exercise 20. *Fix an open set $U \subseteq \mathbb{R}^d \setminus \{0\}$. Prove that the Finsler norm φ is itself a φ-absolutely minimizing function in U. (Multiple proofs are possible.)*

2.8. **Problems.**

Problem 8. *Fix a finite set* $\{v_1, \ldots, v_N\} \subseteq \mathbb{R}^d$. *Define* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *by*

$$\varphi(q) = \max\{\langle v_1, q\rangle, \ldots, \langle v_N, q\rangle\}.$$

(i) Prove that $\varphi(\mathbb{R}^d) \subseteq [0, \infty)$ *if and only if the convex hull of* $\{v_1, \ldots, v_N\}$ *contains the origin.*

(ii) Assume that the convex hull of $\{v_1, \ldots, v_N\}$ *contains the origin. Prove that* φ *is positive definite (i.e.,* $\varphi(q) = 0$ *if and only if* $q = 0$*) if and only if the convex hull of* $\{v_1, \ldots, v_N\}$ *contains the origin in its interior.*

Problem 9. *Using the comparison-with-cones properties, prove that if* $u : U \to \mathbb{R}$ *is a* φ*-absolutely minimizing Lipschitz function in* U *and* $V \subset\subset U$, *then*

$$Lip_\varphi(u; \overline{V}) = \sup\left\{\frac{u(y_1) - u(y_2)}{\varphi(y_1 - y_2)} \mid y_1, y_2 \in \partial V\right\} = Lip_\varphi(u; \partial V).$$

(Hint: Mimic the proof of Proposition 7.) Compare to Exercise 17.

Problem 10. *Provide an example of a bounded open set* $U \subseteq \mathbb{R}^d$ *and a uniformly Lipschitz function* $u : \overline{U} \to \mathbb{R}$ *such that* $u^+ \neq u_-$.

2.9. **Supplementary Material.** In the next exercises, we will use the so-called *dual norm* φ^* associated with a given Finsler norm φ. In what follows, we denote by $\langle \cdot, \cdot\rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ the Euclidean inner product, hence $\langle v, w\rangle = \sum_{i=1}^d v_i w_i$ if $v = (v_1, \ldots, v_d)$ and $w = (w_1, \ldots, w_d)$.

Definition 8. *Given a Finsler norm* φ^*, *the* dual norm φ^* *of* φ *is the function* φ^* *given by*

$$\varphi^*(p) = \sup\left\{\frac{\langle q, p\rangle}{\varphi(q)} \mid q \in \mathbb{R}^d \setminus \{0\}\right\}.$$

Exercise 21. *Prove that if* φ *is a Finsler norm, then so is* φ^*.

Exercise 22. *Prove that if* φ *is the* ℓ^2 *norm,* $\varphi(q) = \left(\sum_{i=1}^d q_i^2\right)^{\frac{1}{2}}$, *then* $\varphi^* = \varphi$.

Exercise 23. *Prove that* $\langle p, q\rangle \leq \varphi(q)\varphi^*(p)$ *for any* $p, q \in \mathbb{R}^d$.

Exercise 24. *Prove that if* φ *and* ψ *are two Finsler norms and if* $\langle p, q\rangle \leq \varphi(q)\psi(p)$ *for any* $p, q \in \mathbb{R}^d$, *then* $\varphi^* \leq \psi$.

Problem 11. *Prove that if* $\varphi^* = \varphi$, *then* φ *equals the* ℓ^2 *norm.*

Problem 12. *Let* φ *be a Finsler norm and* $\gamma : [0, T] \to \mathbb{R}^d$ *be a smooth curve. Prove that, for any smooth function* f *defined in a neighborhood of* $\{\gamma(s) \mid s \in [0, T]\}$,

$$f(\gamma(T)) - f(\gamma(0)) \leq \max\{\varphi^*(Df(\dot\gamma(s))) \mid s \in [0, T]\} \int_0^T \varphi(\dot\gamma(s))\, ds.$$

Deduce from the above that if f *is defined and smooth in an open set* $U \subseteq \mathbb{R}^d$ *containing the line segment between two points* $x, y \in \mathbb{R}^d$, *then*

$$f(x) - f(y) \leq \sup\{\varphi^*(Df(x')) \mid x' \in U\}\, \varphi(x - y).$$

Problem 13. *Prove that if $\gamma : [0,T] \to \mathbb{R}^d$ is a C^1 function, then*

$$\varphi(\gamma(T) - \gamma(0)) \leq \int_0^T \varphi(\dot{\gamma}(t))\, dt.$$

Problem 14. *Prove that if $U \subseteq \mathbb{R}^d$ is an open convex set and if $f : \overline{U} \to \mathbb{R}$ is C^1, then*

$$Lip_\varphi(f; \overline{U}) = \sup\left\{\varphi^*(Df(x')) \mid x' \in U\right\}.$$

Problem 15 (requires measure theory)**.** *Prove that if $U \subseteq \mathbb{R}^d$ is an open convex set and $u : \overline{U} \to \mathbb{R}$ is any function such that $Lip_\varphi(u; \overline{U}) < \infty$, then*

$$Lip_\varphi(u; \overline{U}) = \|\varphi^*(Du)\|_{L^\infty(U)},$$

where $\|\cdot\|_{L^\infty(U)}$ denotes the L^∞-norm with respect to the Lebesgue measure in U.

The previous problem should be compared to Problem 3 above.

3. Uniqueness of Absolutely Minimizing Lipschitz Extensions and the (Finsler) Infinity Laplacian

In this section, there are three goals. First, we prove that absolutely minimizing Lipschitz extensions are unique. Second, we prove that they exist. Finally, we uncover a PDE (or, actually, two partial differential inequalities) that they solve.

To start with, let us be slightly more precise about what we mean by an absolutely minimizing Lipschitz extension. The definition does not make reference to our original motivation, the variational problem (1), but we will prove that it gives a solution to the problem in Corollary 1.

As in the last section, we fix a Finsler norm φ without further comment.

Definition 9. *Given an open set $U \subseteq \mathbb{R}^d$ and a continuous function $g : \partial U \to \mathbb{R}$, we say that a function $u : \overline{U} \to \mathbb{R}$ is an* absolutely minimizing Lipschitz extension of g *if $u(x) = g(x)$ for each $x \in \partial U$ and the restriction of u to U is an absolutely minimizing Lipschitz function.*

The next theorem asserts that, in a bounded domain U, there is an absolutely minimizing Lipschitz extension u_g corresponding to any boundary condition g.

Theorem 7. *If $U \subseteq \mathbb{R}^d$ is a bounded open set and $g : \partial U \to \mathbb{R}$ is continuous, then there is a unique absolutely minimizing Lipschitz extension $u_g : \overline{U} \to \mathbb{R}$ of g.*

In the next corollary, we prove that if g is uniformly Lipschitz on the boundary ∂U, then u_g is a minimal Lipschitz extension in the sense of (1).

Corollary 1. *If $U \subseteq \mathbb{R}^d$ is a bounded open set and $g : \partial U \to \mathbb{R}$ is uniformly Lipschitz continuous on ∂U, then*

$$Lip_\varphi(u; \overline{U}) = Lip_\varphi(g; \partial U).$$

In particular, u attains the minimum in (1).

It will take some time to describe the PDE (or partial differential inequalities) satisfied by absolutely minimizing Lipschitz functions, and, indeed, even the sense in which the PDE is solved will require some explanation. Nonetheless, at this stage, let us at least state the result.

Recall that, as in the introduction, we write $\mathrm{Sym}(d)$ to denote the space of all symmetric, $d \times d$ matrices.

Theorem 8. *There are functions $G^*_\varphi, G^\varphi_* : \mathbb{R}^d \times Sym(d) \to \mathbb{R}$ such that if $u : U \to \mathbb{R}$ is an absolutely minimizing Lipschitz function (with respect to φ) in some open set $U \subseteq \mathbb{R}^d$, then*

$$-G^\varphi_*(Du, D^2u) \geq 0 \quad \textit{in the viscosity sense in } U, \tag{21}$$

$$-G^*_\varphi(Du, D^2u) \leq 0 \quad \textit{in the viscosity sense in } U. \tag{22}$$

Of course, now a natural question might occur to you: do the partial differential inequalities (21) and (22) also characterize absolutely minimizing Lipschitz functions? I claim that the answer is yes, but now we are getting ahead of ourselves, and the proof goes beyond the scope of these notes.

3.1. **Convexity Properties.** In preparation for the proof of Theorem 7, we will give a fundamental property of absolutely minimizing Lipschitz functions.

Toward that end, some additional notation will be convenient. Define the "forward" and "backward" open balls $B^\varphi_r(x)$ and $B^r_\varphi(x)$ for $x \in \mathbb{R}^d$ and $r \geq 0$ by

$$B^\varphi_r(x) = \{y \in \mathbb{R}^d \mid \varphi(y - x) < r\}, \quad B^r_\varphi(x) = \{y \in \mathbb{R}^d \mid \varphi(x - y) < r\}.$$

Similarly, let $\bar{B}^\varphi_r(x)$ and $\bar{B}^r_\varphi(x)$ be the closed balls, defined analogously to $B^\varphi_r(x)$ and $B^r_\varphi(x)$, but with "$< r$" replaced by "$\leq r$." We need *both* notions of balls since φ is only assumed to be a Finsler norm (hence is not necessarily symmetric).

Given a function u and an x in the domain of u, define the function M^φ_x by

$$M^\varphi_x(r) = \max\left\{u(y) \mid y \in \bar{B}^\varphi_r(x)\right\},$$

at least for those r for which the above supremum makes sense. The next result shows that if u is an absolutely minimizing Lipschitz function (with respect to φ), then M^φ_x is a convex function in its interval of definition.

Proposition 13. *Let $U \subseteq \mathbb{R}$ and fix a continuous function $u : U \to \mathbb{R}$. Fix $x \in U$ and let $R(x) = \sup\{r > 0 \mid \bar{B}^\varphi_r(x) \subseteq U\}$. If u is φ-absolutely subminimizing, then M^φ_x is a convex function in $[0, R(x))$.*

Proof. Fix $r_1, r_2 \in [0, R)$ and $\lambda \in [0, 1]$. By definition, we know that

$$u(y) \leq M^\varphi_x(r_1) \quad \text{if } y \in \bar{B}^\varphi_r(x),$$
$$u(y) \leq M^\varphi_x(r_2) \quad \text{if } y \in \bar{B}^\varphi_r(x).$$

At the same time, observe that we can fix $A \in \mathbb{R}$ and $B > 0$ such that the function $f : \mathbb{R}^d \to \mathbb{R}$ given by

$$f(y) = A + B\varphi(y - x)$$

satisfies $f(y) = M_x^\varphi(r_1)$ if $y \in \partial B_{r_1}^\varphi(x)$ and $f(y) = M_x^\varphi(r_2)$ if $y \in \partial B_{r_2}^\varphi(x)$.

Since u is φ-absolutely subminimizing, Theorem **??** says that u satisfies comparison with cones from above. Therefore,

$$\max\{u(y) - f(y) \mid y \in \bar{B}_{r_2}^\varphi \setminus B_{r_1}^\varphi(x)\} \leq 0.$$

Finally, notice that if $\varphi(y-x) = (1-\lambda)r_1 + \lambda r_2$, then we can fix y_1 and y_2 such that $(1-\lambda)y_1 + \lambda y_2 = y$ and $\varphi(y_i - x) = r_i$ for $i \in \{1, 2\}$. Therefore, by convexity of φ,

$$\begin{aligned} u(y) \leq f(y) = A + B\varphi(y-x) &\leq (1-\lambda)(A + B\varphi(y_1 - x)) + \lambda(A + B\varphi(y_2 - x)) \\ &= (1-\lambda)M_x^\varphi(r_1) + \lambda M_x^\varphi(r_2). \end{aligned}$$

This proves that

$$\max\left\{u(y) \mid y \in \partial B_{(1-\lambda)r_1 + \lambda r_2}(x)\right\} \leq (1-\lambda)M_x^\varphi(r_1) + \lambda M_x^\varphi(r_2).$$

Therefore, since the maximum occurs on the boundary (Exercise **??**),

$$M_x^\varphi((1-\lambda)r_1 + \lambda r_2) \leq (1-\lambda)M_x^\varphi(r_1) + \lambda M_x^\varphi(r_2).$$

This proves M_x^φ is convex in $[0, R(x))$. □

Symmetrically, if we define m_x^φ by

$$m_x^\varphi(r) = \min\left\{v(y) \mid y \in \bar{B}_\varphi^r(x)\right\},$$

then m_x^φ is a *concave* function whenever v is φ-absolutely superminimizing.

Proposition 14. *Let $U \subseteq \mathbb{R}^d$ be an open set and fix $x \in U$. If $v : U \to \mathbb{R}$ is φ-absolutely superminimizing and $R(x) = \sup\{r > 0 \mid \bar{B}_\varphi^r(x) \subseteq U\}$, then the function $m_x^\varphi : [0, R(x)) \to \mathbb{R}$ is concave.*

Exercise 25. *Prove the previous proposition (e.g., by mimicking that of Proposition 13).*

3.2. **Finite-Differences, Revisited.** In this subsection, I recount the elementary uniqueness argument of Armstrong and Smart. Finite-difference operators like those dealt with in Section 1 reappear at this stage of the discussion.

The crux of the matter is the following observation. Suppose that u is a φ-absolutely subminimizing Lipschitz function in some open set $U \subseteq \mathbb{R}^d$. Given $r > 0$, let U_r be the subdomain

$$U_r = \{x \in U \mid \bar{B}_r^\varphi(x) \subseteq U \quad \text{and} \quad \bar{B}_\varphi^r(x) \subseteq U\}.$$

and define the function u_r^φ by

$$u_r^\varphi(x) = \max\left\{u(y) \mid y \in \bar{B}_r^\varphi(x) \cap \overline{U}\right\}. \tag{23}$$

The next result asserts that u_r^φ is a subsolution of a finite-difference equation that will look familiar if perused Section 1.

Lemma 1. *If $u : U \to \mathbb{R}$ is a φ-absolutely subminimizing Lipschitz function in U, then, for each $x \in U_{2r}$, u_r^φ satisfies*

(24)
$$-\frac{1}{2}\sup\left\{u_r^\varphi(y) - u_r^\varphi(x) \mid x \in \bar{B}_r^\varphi(x)\right\} - \frac{1}{2}\inf\left\{u_r^\varphi(y) - u_r^\varphi(x) \mid y \in \bar{B}_\varphi^r(x)\right\} \leq 0.$$

It is worth observing at this stage that u_r^φ relates to the function M_r^φ defined in the last section via the formula

$$u_r^\varphi(x) = M_x^\varphi(r).$$

It turns out that the convexity property of $M_x^\varphi(r)$ readily implies the conclusion of the lemma.

Proof. Fix $x \in U_r$. Since $R \mapsto M_x^\varphi(R)$ is convex by Proposition 13, we know that

$$\begin{aligned} u_r^\varphi(x) &= M_x^\varphi(r) \\ &\leq \frac{1}{2} M_x^\varphi(0) + \frac{1}{2} M_x^\varphi(2r) \\ &= \frac{1}{2} u(x) + \frac{1}{2} \max\left\{u(y) \mid y \in \bar{B}_{2r}^\varphi(x)\right\}. \end{aligned}$$

Notice that, by definition of u_r^φ and the triangle inequality satisfied by φ (see Definition 4),

$$\max\left\{u(y) \mid y \in \bar{B}_{2r}^\varphi(x)\right\} = \max\left\{u_r^\varphi(y) \mid y \in \bar{B}_r^\varphi(x)\right\}.$$

At the same time, if $y \in \bar{B}_\varphi^r(x)$, that is, if $\varphi(x-y) \leq r$, then $u(x) \leq u_r^\varphi(y)$ by definition, hence

$$u(x) \leq \min\left\{u_r(y) \mid y \in \bar{B}_\varphi^r(x)\right\}.$$

This leads us to conclude that

$$u_r^\varphi(x) \leq \frac{1}{2} \max\left\{u_r^\varphi(y) \mid y \in \bar{B}_r^\varphi(x)\right\} + \frac{1}{2} \min\left\{u_r^\varphi(y) \mid y \in \bar{B}_\varphi^r(x)\right\}.$$

□

A completely analogous property is satisfied by φ-absolutely superminimizing Lipschitz functions.

Lemma 2. *If $v : U \to \mathbb{R}$ is a φ-absolutely superminimizing Lipschitz function in U, then the function $v_\varphi^r : U_r \to \mathbb{R}$ given by*

$$v_\varphi^r(x) = \min\left\{v(y) \mid y \in \bar{B}_\varphi^r(x) \cap \overline{U}\right\} \tag{25}$$

satisfies, for each $x \in U_{2r}$,

$$-\frac{1}{2} \sup\left\{v_\varphi^r(y) - v_\varphi^r(x) \mid x \in \bar{B}_r^\varphi(x)\right\} - \frac{1}{2} \inf\left\{u_\varphi^r(y) - u_\varphi^r(x) \mid y \in \bar{B}_\varphi^r(x)\right\} \geq 0. \tag{26}$$

Exercise 26. *Using Proposition 14, prove Lemma 2.*

3.3. **Comparison Principle.** At this stage, the goal is to use the finite-difference inequalities (24) and (26) to prove a comparison principle for φ-absolutely minimizing Lipschitz functions analogous to Proposition 2 in Section 1. Precisely, we prove the following result:

Theorem 9. *Let $U \subseteq \mathbb{R}^d$ be a bounded open set. If u is a φ-absolutely subminimizing function in U and v is a φ-absolutely superminimizing function in U, both of which extend to continuous functions in the closure $\overline{U}$, then*

$$\max\left\{u(x) - v(x) \mid x \in \overline{U}\right\} = \max\left\{u(x) - v(x) \mid x \in \partial U\right\}.$$

The proof use the functions u_r^φ and v_φ^r defined above. First, we prove a result analogous to Proposition 2. Before we state the result, we need to define the notion of *upper* and *lower semicontinuous functions*.

Recall that, given a closed set $C \subseteq \mathbb{R}^d$, a function $u : C \to \mathbb{R}$ is called *upper semicontinuous* if, for any $\alpha \in \mathbb{R}$,

$$\{x \in C \mid u(x) \geq \alpha\} \quad \text{is a closed subset of} \quad C.$$

Similarly, a function $v : C \to \mathbb{R}$ is called *lower semicontinuous* if, for any $\lambda > 0$,

$$\{x \in C \mid v(x) \leq \alpha\} \quad \text{is a closed subset of} \quad C.$$

For our purposes, we only need to know that (i) the functions u_r^φ and v_φ^r defined above are, respectively, upper and lower semicontinuous and (ii) upper semicontinuous functions attain their maximum in any compact set (and lower semicontinuous functions attain their minimum in any compact set).

Exercise 27. *(i) Let $U \subseteq \mathbb{R}^d$ be an open set and assume that $u : \overline{U} \to \mathbb{R}$ is any function. Show that the function u_r^φ defined by the formula* (23) *is an upper semicontinuous function in $\overline{U_r}$.*

(ii) Let $U \subseteq \mathbb{R}^d$ be an open set and assume that $v : \overline{U} \to \mathbb{R}$ is any function. Show that the function v_φ^r defined by the formula (25) *is a lower semicontinuous function in $\overline{U}_r$.*

Exercise 28. *(i) Let $C \subseteq \mathbb{R}^d$ be a closed set and assume that $u : C \to \mathbb{R}$ is upper semicontinuous. Prove that if $K \subseteq C$ is compact, then there is an $x \in K$ such that $u(x) = \sup\{u(y) \mid y \in K\}$.*

(ii) Let $C \subseteq \mathbb{R}^d$ be a closed set and assume that $v : C \to \mathbb{R}$ is lower semicontinuous. Prove that if $K \subseteq C$ is compact, then there is an $x \in K$ such that $v(x) = \inf\{v(y) \mid y \in K\}$.

Remark 3. *Indeed, in the previous exercise, C did not need to be a closed subset of $\mathbb{R}^d$. Our definition of upper and lower semicontinuous extends to any topological space, and then one can show that an upper (resp. lower) semicontinuous function attains its maximum (resp. minimum) in any compact set.*

Proposition 15. *Let $U \subseteq \mathbb{R}^d$ be a bounded open set and fix $r > 0$. Suppose that $u_r : \overline{U} \to \mathbb{R}$ is upper semicontinuous in $\overline{U}$ and $v^r : \overline{U} \to \mathbb{R}$ is lower semicontinuous in $\overline{U}$. If, for each $x \in U_r$, the following inequalities hold*

$$-\frac{1}{2}\sup\left\{u_r(y) - u_r(x) \mid y \in \bar{B}_r^\varphi(x)\right\} - \frac{1}{2}\inf\left\{u_r(y) - u_r(x) \mid y \in \bar{B}_\varphi^r(x)\right\} \leq 0,$$
$$-\frac{1}{2}\sup\left\{v^r(y) - v^r(x) \mid y \in \bar{B}_r^\varphi(x)\right\} - \frac{1}{2}\inf\left\{v^r(y) - v^r(x) \mid y \in \bar{B}_\varphi^r(x)\right\} \geq 0,$$

then

$$\max\left\{u_r(x) - v^r(x) \mid x \in \overline{U}\right\} = \max\left\{u_r(x) - v^r(x) \mid \overline{U} \setminus U_r\right\}.$$

Problem 16. *Prove Proposition 15 by following the same strategy as in the proof of Proposition 2.*

Finally, we know that $u_r^\varphi \to u$ and $v_\varphi^r \to v$ as $r \to 0$, hence we can use Proposition 15 to conclude the proof of Theorem 9.

Proof of Theorem 9. Define u_r^φ and v_φ^r as in (23) and (25). By Proposition 15,

$$\max\left\{u_r^\varphi(x) - v_\varphi^r(x) \mid x \in \overline{U}\right\} = \max\left\{u_r^\varphi(x) - v_\varphi^r(x) \mid x \in \overline{U} \setminus U_{2r}\right\}.$$

Since u and v are continuous in $\overline{U}$, we know that

$$\lim_{r\to 0^+} \sup\left\{|u_r^\varphi(x) - u(x)| + |v_\varphi^r(x) - v(x)| \mid x \in \overline{U}\right\} = 0.$$

This leads us to conclude that

$$\begin{aligned}\max\left\{u(x) - v(x) \mid x \in \overline{U}\right\} &= \lim_{r\to 0^+} \max\left\{u_r^\varphi(x) - v_\varphi^r(x) \mid x \in \overline{U} \setminus U_{2r}\right\} \\ &= \max\left\{u(x) - v(x) \mid x \in \partial U\right\}.\end{aligned}$$

□

3.4. **Existence.** Now that we know that absolutely minimizing Lipschitz functions are uniquely determined by their boundary values, it remains to prove that such a function exists for any given choice of boundary value.

Proposition 16. *Let $U \subseteq \mathbb{R}^d$ be a bounded open set. Given any continuous function $g : \partial U \to \mathbb{R}$, there is continuous function $u : \overline{U} \to \mathbb{R}$ such that $u(x) = g(x)$ for $x \in \partial U$ and u is a φ-absolutely minimizing Lipschitz function in U.*

We will prove the proposition using the comparison-with-cones properties and Perron's method.

3.5. **The Finsler Infinity Laplacian.** The finite-difference equations can be used to prove that φ-absolutely sub- and superminimizing Lipschitz functions are sub- and supersolutions, respectively, of a certain partial differential inclusion. Specifically, this equation takes the form

$$-\langle D^2 u \cdot \partial\varphi^*(Du), \partial\varphi^*(Du)\rangle \ni 0 \quad \text{in } U. \tag{27}$$

Above $\partial\varphi^*$ is the subdifferential of the dual norm φ^* (defined in Definition 8 above) and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product.

To make this precise, let us define the subdifferential $\partial\psi$ of a Finsler norm ψ.

Definition 10. *Given a Finsler norm $\psi : \mathbb{R}^d \to [0,\infty)$, the subdifferential $\partial\psi$ is the set-valued function in $\mathbb{R}^d$ given by*

$$\partial\psi(q) = \left\{p \in \mathbb{R}^d \mid \psi^*(p) \le 1,\ \langle p, q\rangle = \psi(q)\right\}.$$

On the one hand, if $q = 0$, then the subdifferential $\partial\psi(0)$ simply equals the closed unit ball $\{p \in \mathbb{R}^d \mid \psi^*(p) \leq 1\}$.

On the other hand, recall that, for any Finsler norm ψ, $\langle p, q\rangle \leq \psi(q)\psi^*(p)$ for any $p, q \in \mathbb{R}^d$. Thus, if $q \neq 0$, then the subdifferential $\partial\psi(q)$ consists precisely of those vectors p for which equality holds in this inequality, normalized so that $\psi^*(p) = 1$.

The subdifferential $\partial\psi(q)$ is always nonempty, as you are asked to prove in Problem 17 below.

Exercise 29. *Prove that if ψ is a Finsler norm in $\mathbb{R}^d$ and $q \in \mathbb{R}^d$, then $\partial\psi(q)$ is a closed, convex subset of $\mathbb{R}^d$.*

Exercise 30. *Prove that if ψ is a Finsler norm in $\mathbb{R}^d$, then ψ is not differentiable at zero. (Hint: First, consider the case of the Euclidean norm.)*

Exercise 31. *Prove that if ψ is a Finsler norm in $\mathbb{R}^d$ and ψ is differentiable at some point $q \in \mathbb{R}^d \setminus \{0\}$, then*

$$\partial\psi(q) = \{D\psi(q)\}.$$

(The converse is also true: if $\partial\psi(q) = \{p\}$ for some $p \in \mathbb{R}^d$, then ψ is differentiable at q and $D\psi(q) = p$.)

Problem 17. *Prove that if ψ is a Finsler norm in $\mathbb{R}^d$ and $q \in \mathbb{R}^d$, then $\partial\psi(q)$ is nonempty.*

To appreciate the relevance of (27), it is useful to consider that $u_r^\varphi \approx u_\varphi^r \approx u$ for small r, hence, for any $x \in U$,

$$-\frac{1}{2}\max\left\{u(y) - u(x) \mid y \in \bar{B}_r^\varphi(x)\right\} - \frac{1}{2}\min\left\{u(y) - u(x) \mid y \in \bar{B}_\varphi^r(x)\right\} \approx 0,$$

at least for small $r > 0$. Now we would like to understand what happens in this expression in the limit $r \to 0^+$. Toward that end, we may as well assume that u is smooth in a neighborhood of x (the best case scenario if we want to see a partial differential equation of some sort).

Lemma 3. *If f is a smooth function defined in a neighborhood of some point $x \in \mathbb{R}^d$ and $(x_r)_{r>0} \subseteq \mathbb{R}^d$ satisfies $\lim_{r\to 0^+} x_r = x$, then*

$$\frac{1}{2}G_\varphi^*(Df(x), D^2f(x)) \geq$$
$$\limsup_{r\to 0^+}\left(\frac{1}{2r^2}\max\left\{f(y) - f(x_r) \mid y \in \bar{B}_r^\varphi(x_r)\right\} + \frac{1}{2r^2}\min\left\{f(y) - f(x_r) \mid y \in \bar{B}_\varphi^r(x_r)\right\}\right),$$
$$\frac{1}{2}G_*^\varphi(Df(x), D^2f(x)) \leq$$
$$\liminf_{r\to 0^+}\left\{\frac{1}{2r^2}\max\left\{f(y) - f(x_r) \mid y \in \bar{B}_r^\varphi(x_r)\right\} + \frac{1}{2r^2}\min\left\{f(y) - f(x_r) \mid y \in \bar{B}_\varphi^r(x_r)\right\}\right),$$

where $G_*^\varphi, G_\varphi^* : \mathbb{R}^d \times Sym(d) \to \mathbb{R}$ are the functions defined by

$$G_\varphi^*(p, X) = \max\left\{\langle Xq, q\rangle \mid q \in \partial\varphi^*(p)\right\},$$
$$G_*^\varphi(p, X) = \min\left\{\langle Xq, q\rangle \mid q \in \partial\varphi^*(p)\right\}.$$

Proof. The factors of r^2 in the denominator suggest that we are interested in the behavior of f very close to x. Therefore, it makes sense to try a blow-up argument. Toward that end, consider the functions $(F_r)_{r>0}$ defined by $F_r(\xi) = r^{-1}(f(x_r + r\xi) - f(x_r))$.

For any $r > 0$ small enough, let ξ_r^+ and ξ_r^- be points such that $\varphi(\xi_r^+), \varphi(\xi_r^-) \leq 1$ and

$$\begin{aligned} f(x_r + r\xi_r^+) &= \max\left\{ f(y) \mid y \in \bar{B}_r^\varphi(x_r) \right\}, \\ f(x_r - r\xi_r^-) &= \min\left\{ f(y) \mid y \in \bar{B}_\varphi^r(x_r) \right\}. \end{aligned}$$

This means that

$$\begin{aligned} F_r(\xi_r^+) &= \max\left\{ F_r(\xi) \mid \varphi(\xi) \leq 1 \right\}, \\ F_r(-\xi_r^-) &= \min\left\{ F_r(-\xi) \mid \varphi(\xi) \leq 1 \right\}. \end{aligned}$$

Since f is smooth close to x, we know that

$$\limsup_{r \to 0} \left\{ \left| F_r(\xi) - \langle Df(x_r), \xi \rangle - \frac{r}{2} \langle Df(x_r)\xi, \xi \rangle \right| \mid \xi \in \bar{B}_1^\varphi(0) \cup \bar{B}_\varphi^1(0) \right\} = 0. \tag{28}$$

By compactness, we can fix a subsequence $(r_j)_{j \in \mathbb{N}} \subseteq (0, \infty)$ and points $\xi^+, \xi^- \in \mathbb{R}^d$ such that $r_j \to 0$ as $j \to \infty$ and

$$\lim_{j \to \infty} \xi_{r_j}^+ = \xi^+, \quad \lim_{j \to \infty} \xi_{r_j}^- = \xi^-.$$

By the choice of ξ_r^+ and ξ_r^-, we know that

$$\begin{aligned} \langle Df(x), \xi^+ \rangle &= \max\left\{ \langle Df(x), \xi \rangle \mid \varphi(\xi) \leq 1 \right\}, \\ \langle Df(x), -\xi^- \rangle &= \min\left\{ \langle Df(x), (-\xi) \rangle \mid \varphi(\xi) \leq 1 \right\} \\ &= \min\left\{ \langle (-Df(x)), \xi \rangle \mid \varphi(\xi) \leq 1 \right\} \\ &= -\max\left\{ \langle Df(x), \xi \rangle \mid \varphi(\xi) \leq 1 \right\}. \end{aligned}$$

Thus, $\{\xi^+, \xi^-\} \subseteq \partial\varphi^*(Df(x))$ by Definition 10.

Finally, by (28),

$$\begin{aligned} &\limsup_{r \to 0^+} \left(\frac{1}{2r} \max\left\{ F_r(\xi) \mid \varphi(\xi) \leq 1 \right\} + \frac{1}{2r} \min\left\{ F_r(-\xi) \mid \varphi(\xi) \leq 1 \right\} \right) \\ &\quad \leq \lim_{r \to 0^+} \left(\frac{1}{2r} \langle Df(x_r), \xi_r^+ \rangle + \frac{1}{4} \langle D^2 f(x_r)\xi_r^+, \xi_r^+ \rangle - \frac{1}{2r} \langle Df(x_r), \xi_r^+ \rangle \right. \\ &\qquad \left. + \frac{1}{4} \langle D^2 f(x_r)\xi_r^+, \xi_r^+ \rangle \right) \\ &= \frac{1}{2} \langle D^2 f(x)\xi^+, \xi^+ \rangle \leq \frac{1}{2} G_\varphi^*(Df(x), D^2 f(x)). \end{aligned}$$

Similarly,

$$\begin{aligned}
&\liminf_{r\to 0^+}\left(\frac{1}{2r}\max\{F_r(\xi)\mid \varphi(\xi)\leq 1\}+\frac{1}{2r}\min\{F_r(-\xi)\mid \varphi(\xi)\leq 1\}\right)\\
&\quad\geq \lim_{r\to 0^+}\left(\frac{1}{2r}\langle Df(x_r),\xi_r^-\rangle+\frac{1}{4}\langle D^2f(x_r)\xi_r^-,\xi_r^-\rangle-\frac{1}{2r}\langle Df(x_r),\xi_r^-\rangle\right.\\
&\quad\quad\left.+\frac{1}{4}\langle D^2f(x_r),\xi_r^-,\xi_r^-\rangle\right)\\
&\quad=\frac{1}{2}\langle D^2f(x)\xi^-,\xi^-\rangle\geq\frac{1}{2}G_*^{\varphi}(Df(x),D^2f(x)).
\end{aligned}$$

In view of the definition of $(F_r)_{r>0}$, this completes the proof. □

In the next section, we will show that if u is a φ-absolutely minimizing Lipschitz function in some open set $U\subseteq\mathbb{R}^d$, then

$$-G^*_{\varphi}(Du,D^2u)\leq 0\leq -G_*^{\varphi}(Du,D^2u)\quad\text{in the viscosity sense in } U.$$

Along the way, we will need to define what "in the viscosity sense" means.

3.6. **Viscosity Solutions.** In the previous section, we saw that the two operators $(G_*^{\varphi},G^*_{\varphi})$ emerge as the limit of the finite-difference inequalities (24) and (26) when $r\to 0^+$. This was the observation of Lemma 3, but there we had to work with smooth functions.

In general, a φ-absolutely minimizing Lipschitz function will not be smooth, or even C^1. Hence we need a notion of solution of a PDE (or partial differential inclusion) applicable to functions that are merely continuous. This is the purpose of the theory of viscosity solutions.

Definition 11. *An upper semicontinuous function $u:U\to\mathbb{R}$ defined in an open set $U\subseteq\mathbb{R}^d$ is said to satisfy the partial differential inequality $-G^*_{\varphi}(Du,D^2u)\leq 0$ in the viscosity sense in U if, for any $x\in\mathbb{R}^d$ and any smooth function f defined in a neighborhood of U for which the difference $u-f$ has a local maximum at x,*

$$-G_*^{\varphi}(Df(x),D^2f(x))\leq 0.$$

Similarly, a lower semicontinuous function $v:U\to\mathbb{R}$ is said to satisfy the partial differential inequality $-G_*^{\varphi}(Dv,D^2v)\geq 0$ in the viscosity sense in U if, for any $x\in\mathbb{R}^d$ and any smooth function f defined in a neighborhood of U for which the difference $u-f$ has a local minimum at x,

$$-G_*^{\varphi}(Df(x),D^2f(x))\geq 0.$$

A function u satisfying $-G^*_{\varphi}(Du,D^2u)\leq 0$ in the viscosity sense is called a *viscosity subsolution* of the equation $-G^*_{\varphi}(Du,D^2u)=0$, while a function v satisfying $-G_*^{\varphi}(Dv,D^2v)\geq 0$ in the viscosity sense is called a *viscosity supersolution* of the equation $-G_*^{\varphi}(Dv,D^2v)=0$.

In effect, as we will see in the next theorem, the definition of viscosity sub- and supersolutions proceeds by passing the derivatives onto a smooth test function using the maximum principle. This will become more-or-less apparent in the proof of the next theorem.

Theorem 10. *If u is a φ-absolutely subminimizing Lipschitz function in some open set $U \subseteq \mathbb{R}^d$, then $-G^*_\varphi(Du, D^2u) \leq 0$ in the viscosity sense in U. Similarly, if v is a φ-absolutely superminimizing Lipschitz function in U, then $-G^\varphi_*(Dv, D^2v) \geq 0$ in the viscosity sense in U.*

Proof. We only give the proof for a subminimizer u since the proof for a superminimizer v is analogous.

Suppose that $x \in U$ and f is a smooth function defined in a neighborhood of x for which the difference $u - f$ has a local maximum at x. In particular, we can fix an open ball $B \subset\subset U$ centered at x such that f is defined in the closure $\bar{B}$ and

$$u(x) - f(x) = \max\left\{u(y) - f(y) \mid y \in \bar{B}\right\}.$$

We need to show that $-G^*_\varphi(Df(x), D^2f(x)) \leq 0$. Since adding a constant does not change the derivatives of f, there is no loss of generality in assuming that the maximum is zero, that is, $u(x) = f(x)$ and $u(y) \leq f(y)$ for $y \in \bar{B}$.

It is convenient to make x a *strict* maximum of $u - f$. Toward that end, define f_ϵ by

$$f_\epsilon(y) = f(y) + \frac{\epsilon}{2}|y - x|^2,$$

where $|\cdot|$ is the Euclidean norm. Notice that $u(x) = f_\epsilon(x)$ and $u(y) < f_\epsilon(y)$ if $y \in \bar{B} \setminus \{x\}$.

Consider the approximation u^φ_r defined by (23). Since u^φ_r is upper semicontinuous (Exercises 27 and 28), we can fix a point $x_r \in \bar{B}$ for which

$$u^\varphi_r(x_r) - f_\epsilon(x_r) = \max\left\{u^\varphi_r(y) - f_\epsilon(y) \mid y \in \bar{B}\right\}.$$

Fix a subsequence $(r_j)_{j\in\mathbb{N}} \subseteq (0, \infty)$ and a point $y_0 \in \bar{B}$ such that $\lim_{j\to\infty} r_j = 0$ and $\lim_{j\to\infty} x_{r_j} = y_0$. Since $u^\varphi_r \to u$ as $r \to 0^+$, we know that

$$u(y_0) - f_\epsilon(y_0) = \lim_{j\to\infty} \max\left\{u^\varphi_{r_j}(y) - f_\epsilon(y) \mid y \in \bar{B}\right\} = \max\left\{u(y) - f_\epsilon(y) \mid y \in \bar{B}\right\}.$$

It follows that $y_0 = x$. Since the subsequence was chosen arbitrarily, we deduce from this that $x_r \to x$ as $r \to 0^+$.

Finally, since $x \in B$ and $x_r \to x$ as $r \to 0^+$, there is an $r_* > 0$ such that $\bar{B}^\varphi_r(x_r) \cup \bar{B}^r_\varphi(x_r) \subseteq \bar{B}$ if $r \in (0, r_*)$. Thus, since $u^\varphi_r - f_\epsilon$ is maximized at x_r, for any $r \in (0, r_*)$, we can write

$$\begin{aligned}
&-\frac{1}{2}\sup\left\{u^\varphi_r(y) - u^\varphi_r(x_r) \mid y \in \bar{B}^\varphi_r(x_r)\right\} - \frac{1}{2}\inf\left\{u^\varphi_r(y) - u^\varphi_r(x_r) \mid y \in \bar{B}^r_\varphi(x_r)\right\} \\
&\quad \geq -\frac{1}{2}\max\left\{f_\epsilon(y) - f_\epsilon(x_r) \mid y \in \bar{B}^\varphi_r(x_r)\right\} - \frac{1}{2}\min\left\{f_\epsilon(y) - f_\epsilon(x_r) \mid y \in \bar{B}^r_\varphi(x_r)\right\}.
\end{aligned}$$

Therefore, by Lemmas 1 and 3, in the limit $r \to 0^+$, we find

$$0 \geq -\frac{1}{2}G^*_\varphi(Df_\epsilon(x), D^2f_\epsilon(x)).$$

Finally, we know that $Df_\epsilon(x) = Df(x)$ and $D^2 f_\epsilon(x) = D^2 f(x) + \epsilon \text{Id}$. Thus, since the function G^*_φ is upper semicontinuous, in the limit $\epsilon \to 0^+$, we find

$$0 \geq -G^*_\varphi(Df(x), D^2 f(x)).$$

□

It is worth reflecting on what we have proved thus far. At first, I claimed that the PDE, really a partial differential inclusion, should be

$$-\langle D^2 u \cdot \partial\varphi^*(Du), \partial\varphi^*(Du)\rangle \ni 0 \quad \text{in } U. \tag{29}$$

Above, since $\partial\varphi^*$ is set-valued (Definition 10), the right-hand side is a set. Notice, however, that if $p \in \mathbb{R}^d$ and $X \in \text{Sym}(d)$ are such that

$$\{\langle Xq, q\rangle \mid q \in \partial\varphi^*(p)\} = \langle X\partial\varphi^*(p), \partial\varphi^*(p)\rangle \ni 0,$$

then, since $\partial\varphi^*(p)$ is a connected set by Exercise 29,

$$\max\{\langle Xq, q\rangle \mid q \in \partial\varphi^*(p)\} \geq 0 \geq \min\{\langle Xq, q\rangle \mid q \in \partial\varphi^*(p)\}.$$

Thus, a natural way to interpret the partial differential inclusion (29) is to simply ask that the partial differential inequalities $-G^*_\varphi(Du, D^2 u) \leq 0 \leq -G^\varphi_*(Du, D^2 u)$ hold in U. We just proved that if u is a φ-absolutely minimizing Lipschitz function, then these inequalities do hold (in the viscosity sense).

3.7. **Further Exercises.**

3.8. **Problems.**

Problem 18. *Prove that if h is a smooth function defined in some open set $U \subseteq \mathbb{R}^d$ such that*

$$-G^*_\varphi(Dh(x), D^2 h(x)) \leq 0 \quad \text{for each } x \in U,$$

then h satisfies $-G^*_\varphi(Dh, D^2 h) \leq 0$ in the viscosity sense in U. (Hint: If $h - f$ has a local maximum at some point $x \in U$, use the second derivative test to deduce that $-G^*_\varphi(Dh(x), D^2 h(x)) \leq 0$.)

Problem 19. *Say that a continuous function $u : U \to \mathbb{R}$ satisfies $\varphi^*(Du) \leq M$ in the viscosity sense if for any $x \in U$ and any smooth function f defined in a neighborhood of x for which $u - f$ has a local maximum, we have $\varphi^*(Df(x)) \leq M$. Prove that if u extends to a continuous function in $\bar{U}$ and $Lip_\varphi(u; \overline{U}) \leq M$, then $\varphi^*(Du) \leq M$ in the viscosity sense. (The converse is also true, but it is not so easy to prove.)*

4. Scaling Limit of the Value Function

In this section, we prove Theorems 2 and 3.

4.1. **Rescaled Value Function.** Let $U \subseteq \mathbb{R}^d$ be a bounded open set and $g : \partial U \to \mathbb{R}$ be continuous. Fix a continuous function $G : \overline{U} \to \mathbb{R}$ such that $G(x) = g(x)$ for each $x \in \partial U$.

It will be convenient to assume for a while that G is uniformly Lipschitz in $\overline{U}$:

$$\mathrm{Lip}_{\ell^1}(G; \overline{U}) < \infty.$$

In what follows, let $U_\epsilon = U \cap (\epsilon \mathbb{Z}^d)$. Since U is bounded, the set U_ϵ is finite. Therefore, by Proposition **??**, there is a unique function $u_\epsilon : \epsilon^{-1} U_\epsilon \to \mathbb{R}$ such that

$$-\Delta^\infty_{\mathbb{Z}^d} u_\epsilon(x) = 0 \quad \text{for each } x \in \mathrm{int}(\epsilon^{-1} U_\epsilon),$$
$$u_\epsilon(x) = \epsilon^{-1} G(\epsilon x) \quad \text{for each } x \in \partial_{\mathrm{int}}(\epsilon^{-1} U_\epsilon).$$

Furthermore, the next proposition shows that

$$\max\{u_\epsilon(x) \mid x \in \epsilon^{-1} U_\epsilon\} \leq \epsilon^{-1} \max\left\{|G(x)| \mid x \in \overline{U}\right\},$$
$$\mathrm{Lip}_{\mathbb{Z}^d}(u_\epsilon; \epsilon^{-1} U_\epsilon) \leq \mathrm{Lip}_{\ell^1}(G; \overline{U}).$$

Proposition 17. *Let $A \subseteq \mathbb{Z}^d$ be a finite set. If $u : A \to \mathbb{R}$ satisfies $-\Delta^\infty_{\mathbb{Z}^d} u(x) = 0$ for each $x \in \mathrm{int}(A)$, then*

$$\max\{u(x) \mid x \in A\} = \max\{u(x) \mid x \in \partial_{\mathrm{int}} A\},$$
$$Lip_{\mathbb{Z}^d}(u; A) = Lip_{\mathbb{Z}^d}(u; \partial_{\mathrm{int}} A).$$

Proof. First, notice that the constant function $v(x) = 0$ satisfies $-\Delta^\infty_{\mathbb{Z}^d} v(x) = 0$ for each $x \in \mathrm{int}(A)$. Therefore, by the comparison principle (Proposition 2), the maximum of u in A is achieved on $\partial_{\mathrm{int}} A$. Similarly, replacing u by $-u$, we see that the minimum is achieved on the boundary as well.

Next, notice that if $y \in \mathbb{Z}^d \setminus \mathrm{int}(A)$ and $\lambda > 0$, then the function $v(x) = \lambda \|x - y\|_1$ satisfies $-\Delta^\infty_{\mathbb{Z}^d} v(x) = 0$ for each $x \in \mathrm{int}(A)$. Thus, setting $\lambda = \mathrm{Lip}_{\mathbb{Z}^d}(u; \partial_{\mathrm{int}} A)$ and using the comparison principle (Proposition 2) as in Proposition 7 (see also Problem 9), we find

$$\mathrm{Lip}_{\mathbb{Z}^d}(u; A) = \mathrm{Lip}_{\mathbb{Z}^d}(u; \partial_{\mathrm{int}} A).$$

□

Define the rescaled functions $(\tilde{u}_\epsilon)_{\epsilon > 0}$ by

$$\tilde{u}_\epsilon(x) = \epsilon u(\epsilon^{-1} x) \quad \text{for each } x \in U_\epsilon.$$

What we just proved can be rephrased as

$$\max\{|\tilde{u}_\epsilon(x)| \mid x \in U_\epsilon\} \leq \max\left\{|G(x)| \mid x \in \overline{U}\right\},$$
$$\sup\left\{\frac{\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)}{\|x - y\|_1} \mid x, y \in U_\epsilon\right\} \leq \mathrm{Lip}_{\ell^1}(G; \overline{U}).$$

By mimicking the proof of the Arzelà-Ascoli Theorem, we can prove the following compactness result:

Proposition 18. *For any sequence $(\epsilon_j)_{j\in\mathbb{N}} \subseteq (0,\infty)$ such that $\epsilon_j \to 0$ as $j \to \infty$, there is a subsequence $(j_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $j_k \to \infty$ as $k \to \infty$ and a continuous function $\tilde{u} : \overline{U} \to \mathbb{R}$ such that*

$$\lim_{\delta\to 0^+} \sup\left\{ |\tilde{u}_{\epsilon_{j_k}}(y) - \tilde{u}(x)| \ \Big| \ \frac{1}{k} + |x-y| < \delta \right\} = 0. \tag{30}$$

Furthermore, $\tilde{u}(x) = G(x) = g(x)$ for each $x \in \partial U$ and

$$\max\{|\tilde{u}(x)| \mid x \in \overline{U}\} \leq \max\left\{|G(x)| \mid x \in \overline{U}\right\},$$
$$Lip_{\ell^1}(\tilde{u};\overline{U}) \leq Lip_{\ell^1}(G;\overline{U}).$$

Problem 20. *Prove Proposition 18 by mimicking the proof of the Arzelà-Ascoli Theorem.*

Next, we prove that the limit $\tilde{u}$ is uniquely determined as the ℓ^1-absolutely minimizing Lipschitz function that coincides with G on ∂U.

Proposition 19. *If $\tilde{u}$ is a subsequential limit of $(\tilde{u}_\epsilon)_{\epsilon>0}$ as in* (30)*, then $\tilde{u}$ is the ℓ^1-absolutely minimizing Lipschitz function that equals g on ∂U.*

Proof. Assume that $(\epsilon_j)_{j\in\mathbb{N}} \subseteq (0,\infty)$ satisfies $\lim_{j\to\infty}\epsilon_j = 0$ and (30). We begin by proving that $\tilde{u}$ satisfies comparison with φ-cones from above and below in U. By Theorem **??**, this implies that $\tilde{u}$ is a φ-absolutely minimizing Lipschitz function in U.

Fix $V \subset\subset U$, $q \in \mathbb{R}^d \setminus V$, and $\lambda > 0$. We want to show that

$$\max\left\{\tilde{u}(x) - \lambda\|x-q\|_1 \mid x \in \overline{V}\right\} = \max\left\{\tilde{u}(x) - \lambda\|x-q\|_1 \mid x \in \partial V\right\},$$
$$\min\left\{\tilde{u}(x) - \lambda\|x-q\|_1 \mid x \in \overline{V}\right\} = \min\left\{\tilde{u}(x) - \lambda\|x-q\|_1 \mid x \in \partial V\right\}.$$

To avoid needless repetition, we only prove the identity involving maxima.

Let $V_\epsilon = V \cap (\epsilon\mathbb{Z}^d)$. Since $q \notin V$, the function $v(x) = \lambda\|x-q\|_1$ satisfies

$$-\frac{1}{2}\max_{e\in\mathbb{E}^d}\left(v(x+\epsilon e) - v(x)\right) - \frac{1}{2}\min_{e\in\mathbb{E}^d}\left(v(x+\epsilon e) - v(x)\right) = 0 \quad \text{for each } x \in \text{int}(V_\epsilon).$$

Thus, by the comparison principle (Proposition 2),

$$\max\left\{\tilde{u}_{\epsilon_j}(x) - \lambda\|x-q\|_1 \mid x \in V_{\epsilon_j}\right\} = \max\left\{\tilde{u}_{\epsilon_j}(x) - \lambda\|x-q\|_1 \mid x \in \partial_{\text{int}}V_{\epsilon_j}\right\}.$$

Since $\tilde{u}_{\epsilon_j} \to \tilde{u}$ in the sense of (30), we conclude that

$$\max\left\{\tilde{u}(x) - \lambda\|x-q\|_1 \mid x \in \overline{V}\right\} = \max\left\{\tilde{u}(x) - \lambda\|x-q\|_1 \mid x \in \partial V\right\}.$$

Finally, it only remains to prove that $\tilde{u}(x) = G(x) = g(x)$ for $x \in \partial U$. Indeed, let us fix an $x \in \partial U$ and observe that there are points $(x_j)_{j\in\mathbb{N}} \subseteq \overline{U}$ such that $x_j \in \partial_{\text{int}}U_{\epsilon_j}$ for each $j \in \mathbb{N}$ and $\lim_{j\to\infty} x_j = x$. Using (30), we find

$$\tilde{u}(x) = \lim_{j\to\infty}\tilde{u}_{\epsilon_j}(x_j) = \lim_{j\to\infty}G(x_j) = G(x) = g(x).$$

□

We have already seen in Section **??** that if u is ℓ^1-absolutely minimizing Lipschitz function, then $-G^*_{\ell^1}(Du, D^2u) \le 0 \le -G^{\varphi}_*(Du, D^2u)$. Here we give a direct proof involving the discrete infinity Laplacian $-\Delta^\infty_{\mathbb{Z}^d}$. We will use the following lemma, which is analogous to Lemma 3.

Lemma 4. *Fix $x \in \mathbb{R}^d$ and let f be a smooth function defined in a neighborhood of x. If $(\epsilon_j)_{j\in\mathbb{N}} \subseteq (0,\infty)$ and $(x_j)_{j\in\mathbb{N}} \subseteq \mathbb{R}^d$ satisfy $\lim_{j\to\infty} \epsilon_j = 0$ and $\lim_{j\to\infty} x_j = x$, then*

$$\begin{aligned}
&\frac{1}{2} G^*_{\ell^1}(Df(x), D^2 f(x)) \\
&\qquad \ge \limsup_{j\to\infty} \left\{ \frac{1}{2\epsilon_j^2} \max_{e\in\mathbb{E}^d} \left(f(x_j+\epsilon_j e) - f(x_j)\right) + \frac{1}{2\epsilon_j^2} \min_{e\in\mathbb{E}^d} \left(f(x_j+\epsilon_j e) - f(x_j)\right) \right\}, \\
&\frac{1}{2} G^{\ell^1}_*(Df(x), D^2 f(x)) \\
&\qquad \le \liminf_{j\to\infty} \left\{ \frac{1}{2\epsilon_j^2} \max_{e\in\mathbb{E}^d} \left(f(x_j+\epsilon_j e) - f(x_j)\right) + \frac{1}{2\epsilon_j^2} \min_{e\in\mathbb{E}^d} \left(f(x_j+\epsilon_j e) - f(x_j)\right) \right\}.
\end{aligned}$$

Problem 21. *Prove Lemma 4 by mimicking the proof of Lemma 3.*

Proposition 20. *If $\tilde{u}$ is the limit of $(\tilde{u}_\epsilon)_{\epsilon>0}$*