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Deep Narrow Boltzmann Machines are Universal Approximators

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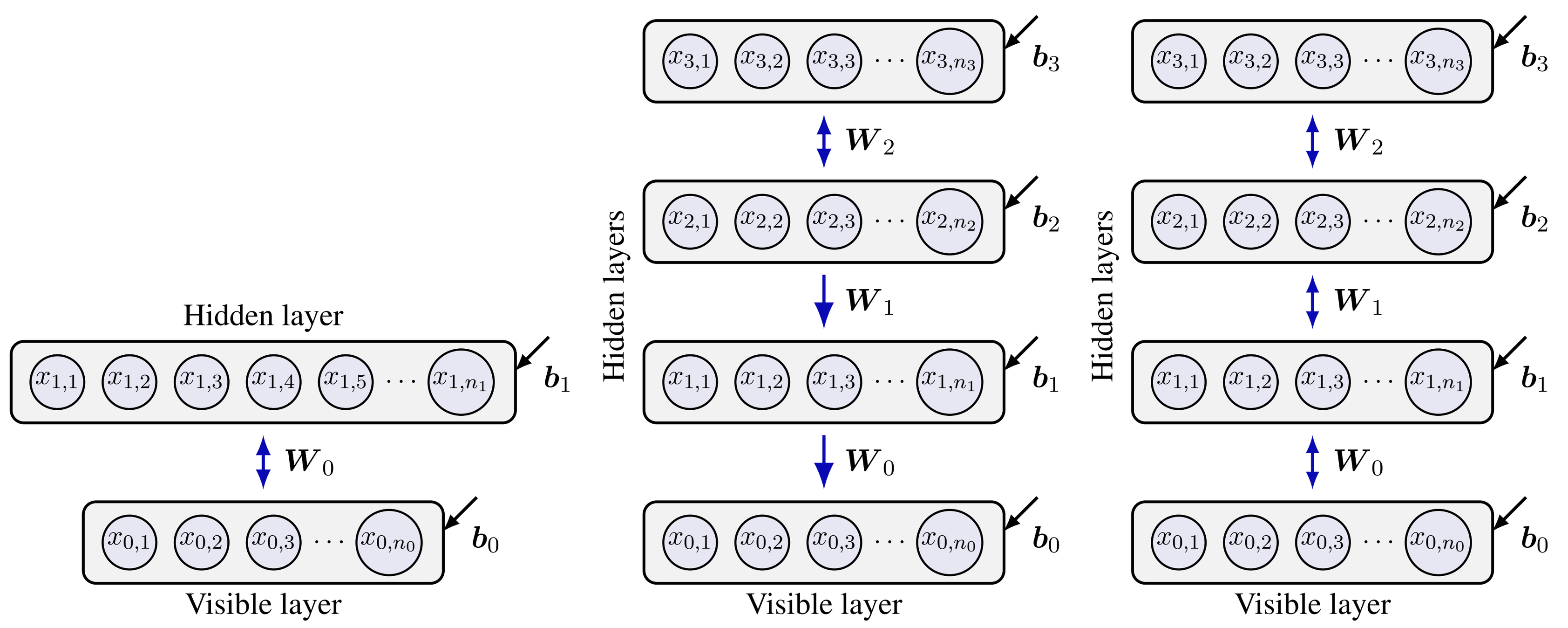
Introduction

- It is an interesting question how the representational power of **deep** artificial neural networks compares with that of **shallow** neural networks.
- Furthermore, it is interesting how the representational power of layered networks compares in the cases of **undirected** and **directed** connections.
- A **basic question** in this respect is whether a given network type can reach any degree of representation accuracy, when endowed with sufficiently many units.
- **Universal approximation** has been verified for many types of neural networks, but has remained an **open problem** for **deep narrow Boltzmann machines**.



RBM	DBN	DBM
(shallow undirected)	(deep directed)	(deep undirected)
✓ [1,5,13,8]	✓ [12,6,8,9]	✓ [this work]

Figure. Restricted Boltzmann machine (RBM), deep belief network (DBN), and deep Boltzmann machine (DBM).

Definition. A deep Boltzmann machine with n_0 visible units and L hidden layers of $n_1, \dots, n_L$ units is a model of probability distributions of the form

$$p_{\mathbf{W},\mathbf{b}}(\mathbf{x}_0) = \sum_{\mathbf{x}_1,\dots,\mathbf{x}_L} \frac{1}{Z(\mathbf{W},\mathbf{b})} \exp\Big(\sum_{l=0}^{L-1} \mathbf{x}_l^\top \mathbf{W}_l \mathbf{x}_{l+1} + \sum_{l=0}^{L} \mathbf{x}_l^\top \mathbf{b}_l\Big).$$

The model is **narrow**, when all layers have about the same number of units.

Overview

- At an intuitive level, undirected networks are expected to be more powerful than directed networks, since "they allow information to flow both ways."
- This intuition is not straightforward to verify. Feedforward networks can be naturally studied in a sequential way, but undirected networks are more subtle.
- We develop a method to study undirected architectures in a sequential way.

Sequential Analysis

- Express the visible probability distribution of a DBM in terms of the distributions of two smaller DBMs.

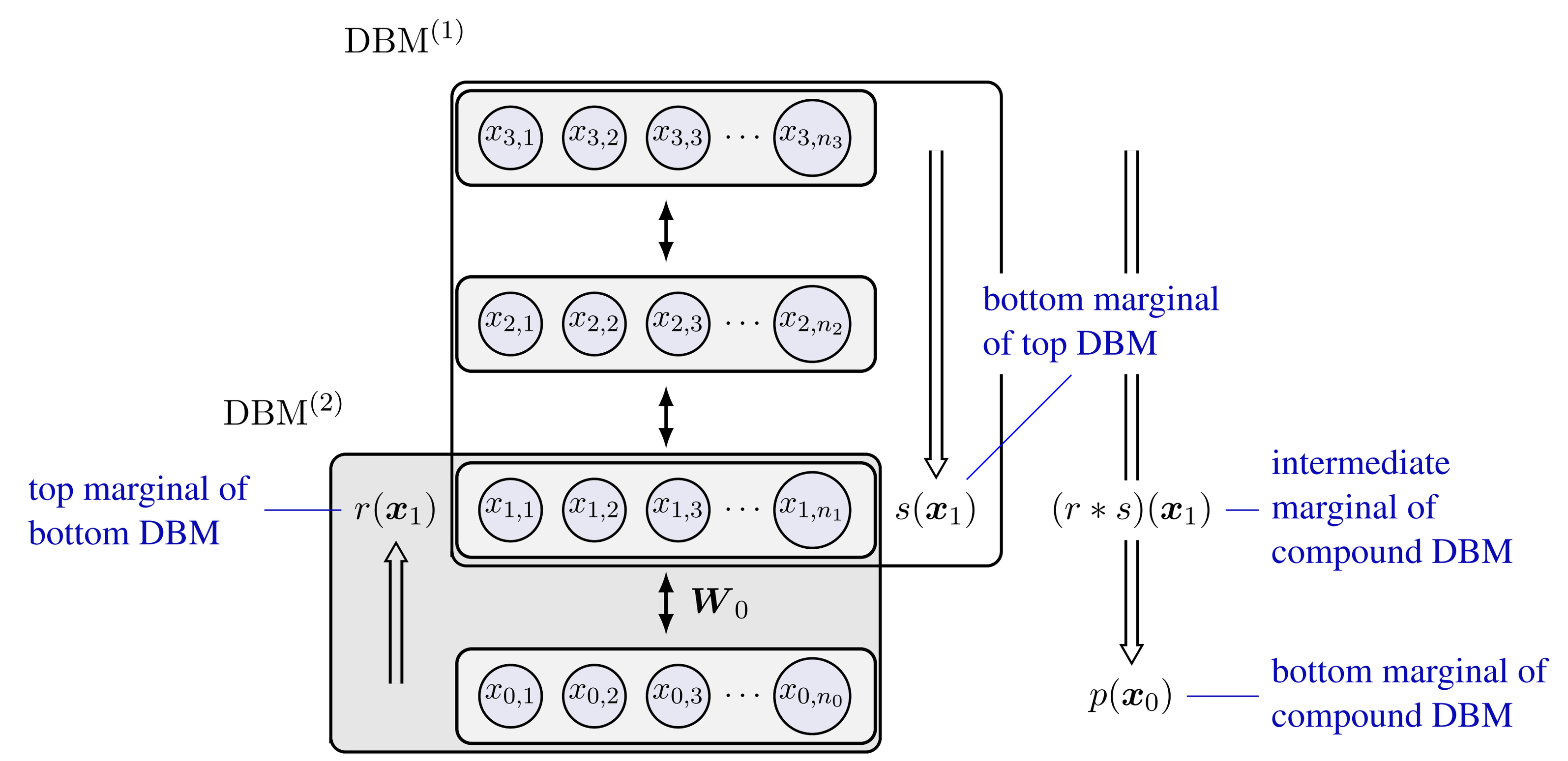


Figure. Composition of two DBMs to form a compound DBM.
Here $(r * s)$ denotes the renormalized entrywise product of r and s.

- The bottom marginal $p(\mathbf{x}_0)$ is the feedforward pass of the intermediate marginal $(r * s)(\mathbf{x}_1)$ by the feedforward map
$$q_{\mathbf{W}_0,\mathbf{b}_0}(\mathbf{x}_0|\mathbf{x}_1) = \frac{1}{Z(\mathbf{W}_0\mathbf{x}_1 + \mathbf{b}_0)} \exp(\mathbf{x}_0^\top \mathbf{W}_0 \mathbf{x}_1 + \mathbf{x}_0^\top \mathbf{b}_0).$$
- Problem: **shared parameters** of intermediate marginal and feedforward map.
- Solution: restrict attention to special marginals s from the top DBM to obtain **independent parameters** for the feedforward maps.
- In this way, with each additional layer we can transform the visible distribution by an independent feedforward map.

Universal Approximation

Theorem. A deep and narrow Boltzmann machine with a visible layer of n units and L hidden layers of n units each **is a universal approximator** of probability distributions on the states of the visible layer, provided L is large enough.

- Sufficient condition:
$$L \geq \frac{2^{n'}}{2(n' - \log_2(n') - 1)},$$
for any $n' = 2^k + k + 1 \geq n$, $k \in \mathbb{N}$.
- Necessary condition:
$$L \geq \frac{2^n - (n+1)}{n(n+1)}.$$
- For universal approximation, the first hidden layer must have at least as many units as the visible layer (minus one).
- Similar results for discriminative and multinomial models.

Conclusions

- We investigated the compositional structure of DBMs and presented a trick to separate the activities on the upper part of the network from those on the lower part of the network.
- Within certain parameter regions, deep Boltzmann machines can be studied as feedforward networks.
- We showed that deep narrow Boltzmann machines are universal approximators, and provided upper and lower bounds on the sufficient depth and width.
- In a specific sense, deep narrow Boltzmann machines are at least as powerful as narrow sigmoid belief networks and restricted Boltzmann machines.
- The methods appear valuable for studying the effects of training undirected networks sequentially, from layer to layer.

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