

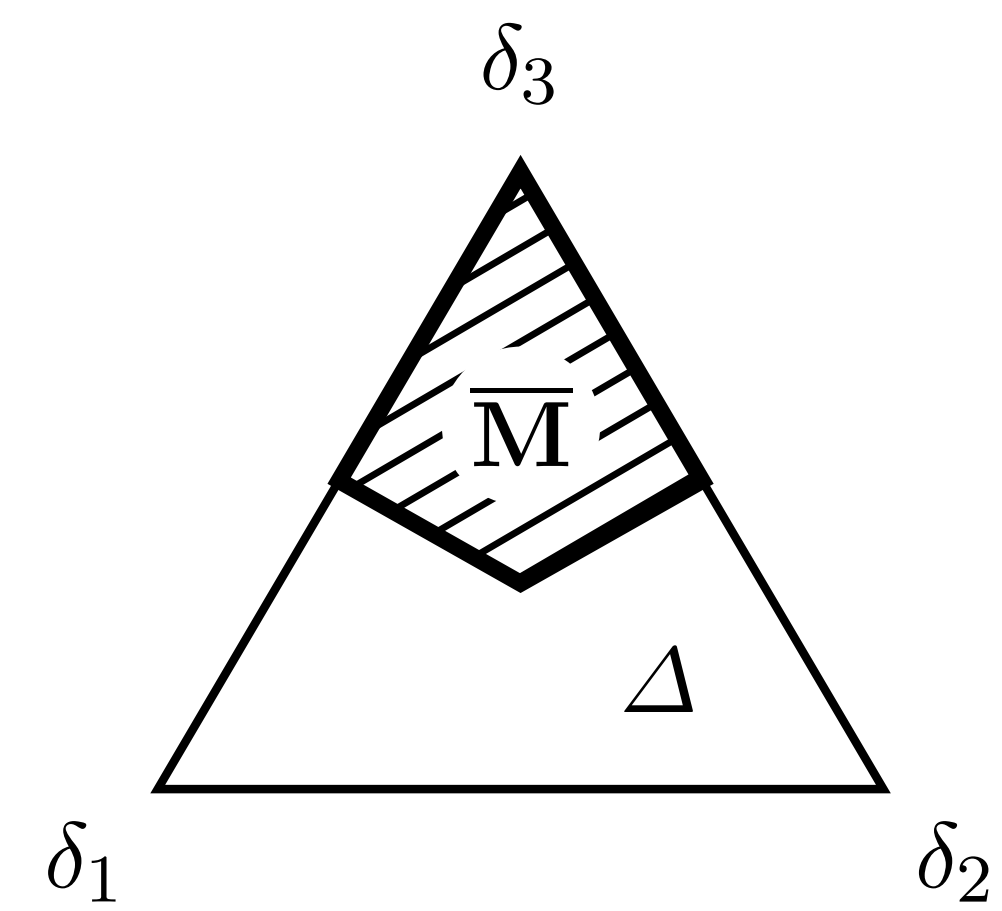
## INTRODUCTION

- A mode of a probability vector is a local maximum with respect to some vicinity structure on the set of elementary events. The mode inequalities cut out a polytope from the simplex of probability vectors.
- Statistical models are usually constrained in the patterns of modes that they can realize. The set of patterns of modes, or strong modes, realizable by a statistical model gives a combinatorial characteristic of the model.
- The mode characteristic of a statistical model can be used to describe a portion of its complement and represents a coarse form of implicit semialgebraic description.
- We study the vertices, facets, and volume of mode poset probability polytopes, depending on the sets of (strong) modes and the vicinity structures.

## SETTINGS

Finite set of elementary events:	$V$
Probability simplex:	$\Delta(V)$
Vicinity structure:	$G = (V, E)$ , a simple graph
Set of (strong) modes:	$\mathcal{C} \subseteq V$ , an independent set in $G$

## THE POLYTOPE OF MODES



- A point  $x \in V$  is a **mode** of a probability distribution  $p \in \Delta(V)$  if

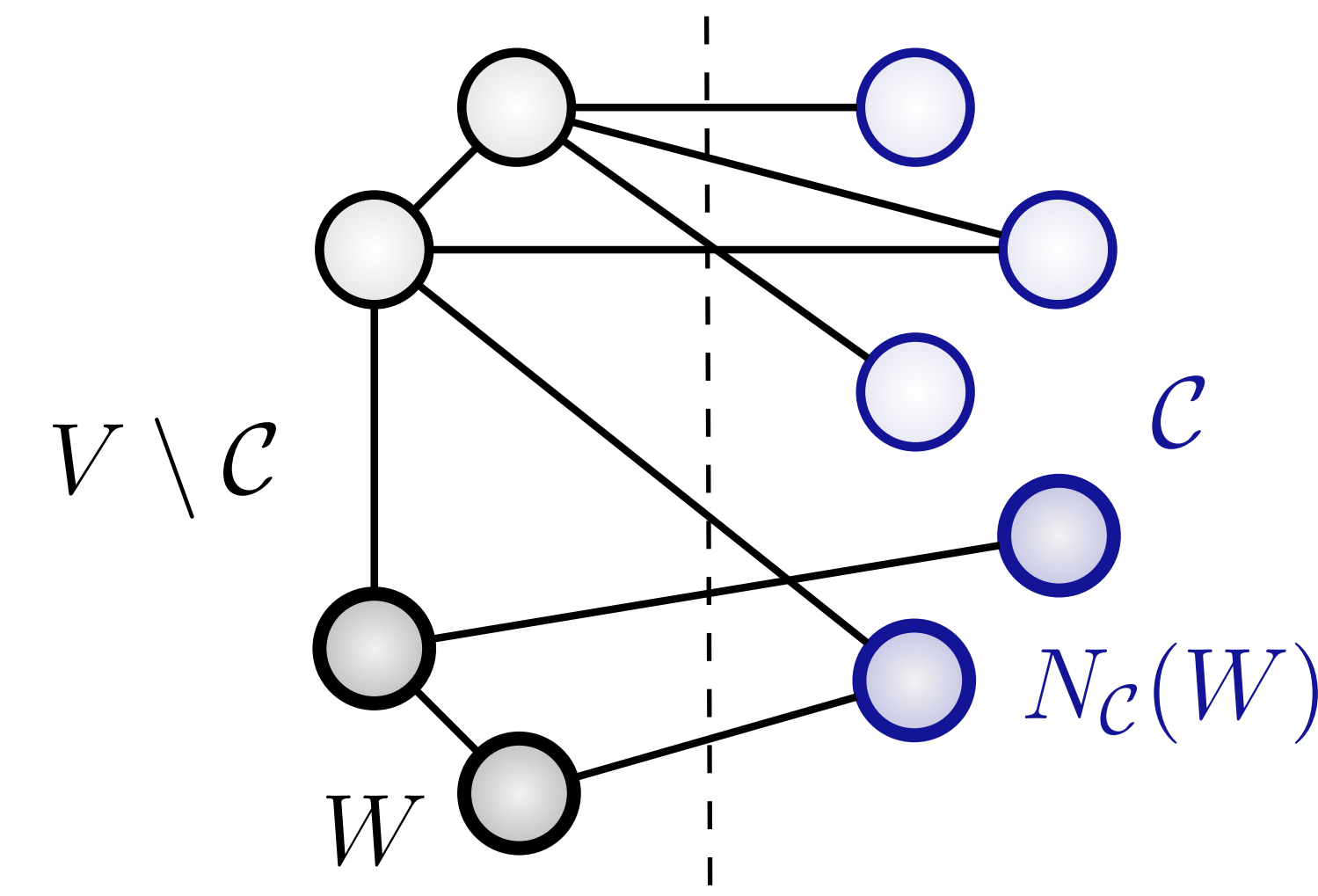
$$p_x \geq p_y \quad \text{for all } y \sim x.$$

- The **polytope of  $\mathcal{C}$ -modes** in  $G$  is the set of probability distributions

$$\mathbf{M}(G, \mathcal{C}) := \{p \in \Delta(V) : \text{each } x \in \mathcal{C} \text{ is a mode}\}.$$

### Proposition 1. (Vertices)

1.  $\mathbf{M}(G, \mathcal{C})$  is the convex hull of  $\{e_C^W : \emptyset \neq W \subseteq V \setminus \mathcal{C}\} \cup \{\delta_x : x \in \mathcal{C}\}$ .
2. For any  $x \in \mathcal{C}$ , the distribution  $\delta_x$  is a vertex of  $\mathbf{M}(G, \mathcal{C})$ .
3.  $e_C^W$  is a vertex of  $\mathbf{M}(G, \mathcal{C})$  iff for any  $x, y \in W$ ,  $x \neq y$ , there is a path  $x = x_0 \sim x_1 \sim \dots \sim x_r = y$  in  $G$  with  $x_0, x_2, \dots \in W$  and  $x_1, x_3, \dots \in N_{\mathcal{C}}(W)$ .



$$N_{\mathcal{C}}(W) := \{y \in \mathcal{C} : y \sim W\}$$

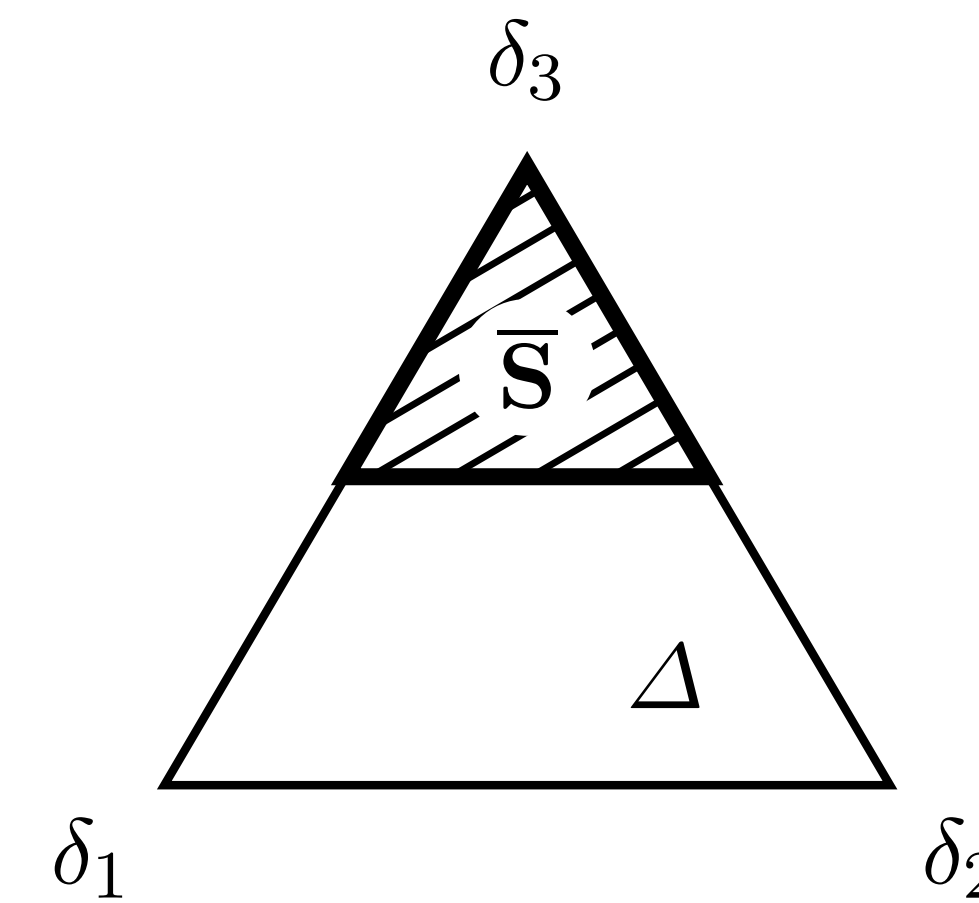
$$e_C^W := \text{uniform on } N_{\mathcal{C}}(W) \cup W$$

### Proposition 2. (Facets)

1. For any  $x \in V \setminus \mathcal{C}$ , the positivity inequality  $p_x \geq 0$  defines a facet.
2. If  $x \in \mathcal{C}$ , then  $p_x \geq 0$  defines a facet iff  $x$  is isolated in  $G$ .
3. For any  $x \in \mathcal{C}$  and  $y \sim x$ , the mode inequality  $p_x \geq p_y$  defines a facet.

**Proposition 3. (Volume)** Let  $\Sigma$  be the set of linear extensions of the partial order associated to  $(G, \mathcal{C})$ . Then  $\text{vol}(\mathbf{M}(G, \mathcal{C})) = \frac{|\Sigma|}{|V|!} \text{vol}(\Delta(V))$ .

## THE POLYTOPE OF STRONG MODES



- A point  $x \in V$  is a **strong mode** of a probability distribution  $p \in \Delta(V)$  if

$$p_x \geq \sum_{y \sim x} p_y \quad \text{for all } x.$$

- The **polytope of strong  $\mathcal{C}$ -modes** in  $G$  is the set of probability distributions

$$\mathbf{S}(G, \mathcal{C}) := \{p \in \Delta(V) : \text{each } x \in \mathcal{C} \text{ is a strong mode}\}.$$

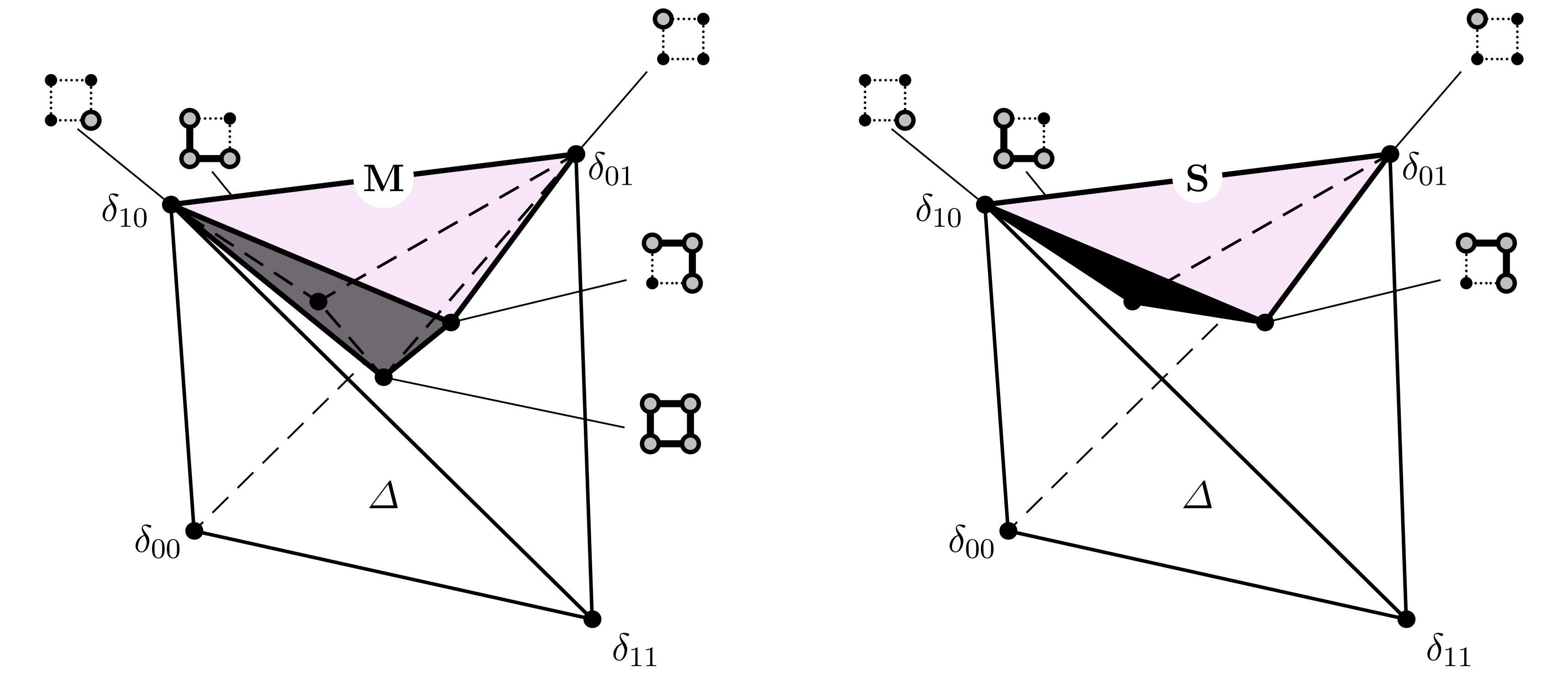
**Proposition 4. (Vertices)**  $\mathbf{S}(G, \mathcal{C})$  is a  $(|V| - 1)$ -simplex with vertices  $e_C^x$ ,  $x \in V$ .

### Proposition 5. (Facets)

1. If  $x \in \mathcal{C}$ , the mode inequality  $p_x \geq \sum_{y \sim x} p_y$  defines a facet.
2. If  $x \in V \setminus \mathcal{C}$ , the positivity inequality  $p_x \geq 0$  defines a facet.

**Proposition 6. (Volume)**  $\text{vol}(\mathbf{S}(G, \mathcal{C})) = \left( \prod_{x \in V} \frac{1}{|N_{\mathcal{C}}(x)| + 1} \right) \text{vol}(\Delta(V))$ .

## EXAMPLES



**Example 7.** The figure shows  $\mathbf{M}(G, \mathcal{C})$  and  $\mathbf{S}(G, \mathcal{C})$  for

Vertices	$V = \{0, 1\}^2$
Edges	$E = \{(00, 01), (00, 10), (01, 11), (10, 11)\}$
Modes	$\mathcal{C} = \{01, 10\}$

**Example 8.** Let  $G$  be the edge graph of an  $n$ -cube with vertices  $V = \{0, 1\}^n$  and edges between pairs of Hamming distance one.

- If  $\mathcal{C} \subseteq V$  has cardinality  $|\mathcal{C}| = k$  and minimum distance 3, then
  - $\mathbf{S}$  has  $2^n$  vertices and  $\text{vol}(\mathbf{S}) = 2^{-kn} \text{vol}(\Delta)$ .
  - $\mathbf{M}$  has  $k(2^n - 1) + 2^n - kn$  vertices and  $\text{vol}(\mathbf{M}) = \frac{|\Sigma|}{2^n!} \text{vol}(\Delta) \geq k! 2^{-kn} \text{vol}(\Delta)$ .
- If  $\mathcal{C}$  is the set of all even-parity strings, then
  - $\mathbf{S}$  has  $2^n$  vertices and  $\text{vol}(\mathbf{S}) = (n + 1)^{-2^{n-1}} \text{vol}(\Delta)$ .
  - $\mathbf{M}$  has  $2^{2^{n-1}} - 1 + 2^{n-1}$  vertices and  $\text{vol}(\mathbf{M}) = \frac{|\Sigma|}{2^n!} \text{vol}(\Delta) \geq \binom{2^n}{2^{n-1}}^{-1} \text{vol}(\Delta)$ .
  - For  $n = 2$  and  $n = 3$  we have  $|\Sigma| = 4$  and  $|\Sigma| = 720$ . Next open case is  $n = 4$ .

## RELATION TO ORDER POLYTOPES

- The patterns of modes that are possible in a given vicinity structure define special types of partial orders in the coordinates of the probability vectors.
- The *order polytope* of a partial order arises by looking at subsets of the unit hypercube instead of subsets of the probability simplex:

$$\mathcal{O}(\succeq) := \{p \in [0, 1]^V : p_x \geq p_y \text{ whenever } x \succeq y\}.$$

One can show that  $\mathbf{M}(\succeq)$  is the vertex figure of  $\mathcal{O}(\succeq)$  at the vertex 0.

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