Mode Poset Probability Polytopes

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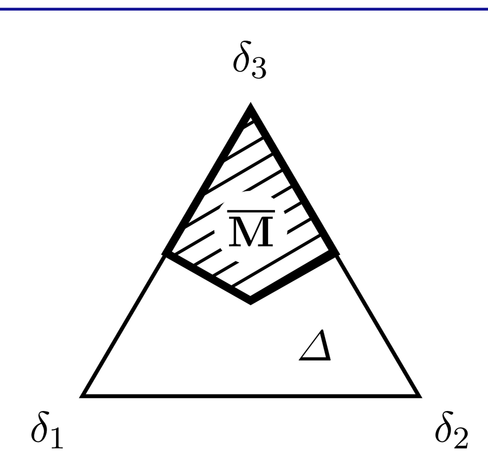
Introduction

- A mode of a probability vector is a local maximum with respect to some vicinity structure on the set of elementary events. The mode inequalities cut out a polytope from the simplex of probability vectors.
- Statistical models are usually constrained in the patterns of modes that they can realize. The set of patterns of modes, or strong modes, realizable by a statistical model gives a combinatorial characteristic of the model.
- The mode characteristic of a statistical model can be used to describe a portion of its complement and represents a coarse form of implicit semialgebraic description.
- We study the vertices, facets, and volume of mode poset probability polytopes, depending on the sets of (strong) modes and the vicinity structures.

Settings

Finite set of elementary events:	V
Probability simplex:	$\Delta(V)$
Vicinity structure:	$G = (V, E)$, a simple graph
Set of (strong) modes:	$\mathcal{C} \subseteq V$, an independent set in G

The Polytope of Modes



- A point $x \in V$ is a **mode** of a probability distribution $p \in \Delta(V)$ if

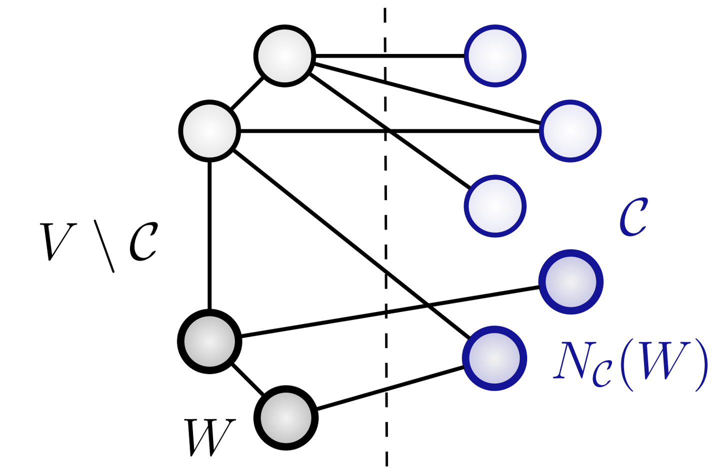
$$p_x \geq p_y \quad \text{for all } y \sim x.$$

- The **polytope of $\mathcal{C}$-modes** in G is the set of probability distributions

$$\mathbf{M}(G, \mathcal{C}) := \{p \in \Delta(V) \colon \text{each } x \in \mathcal{C} \text{ is a mode}\}.$$

Proposition 1. (Vertices)

1. $\mathbf{M}(G, \mathcal{C})$ *is the convex hull of* $\{e_{\mathcal{C}}^W : \emptyset \neq W \subseteq V \setminus \mathcal{C}\} \cup \{\delta_x : x \in \mathcal{C}\}$.
2. *For any* $x \in \mathcal{C}$, *the distribution* δ_x *is a vertex of* $\mathbf{M}(G, \mathcal{C})$.
3. $e_{\mathcal{C}}^W$ *is a vertex of* $\mathbf{M}(G, \mathcal{C})$ *iff for any* $x, y \in W$, $x \neq y$, *there is a path* $x = x_0 \sim x_1 \sim \cdots \sim x_r = y$ *in* G *with* $x_0, x_2, \dots \in W$ *and* $x_1, x_3, \dots \in N_{\mathcal{C}}(W)$.



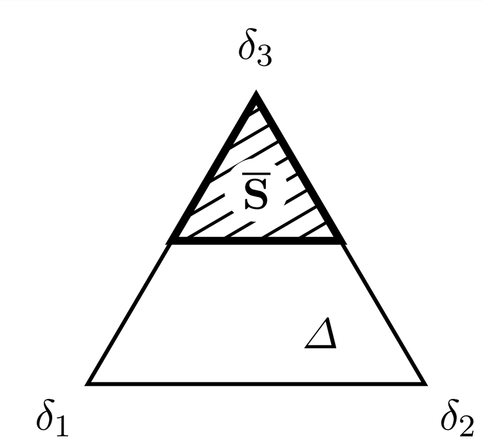
$$N_{\mathcal{C}}(W) := \{y \in \mathcal{C} \colon y \sim W\}$$
$$e_{\mathcal{C}}^W := \text{uniform on } N_{\mathcal{C}}(W) \cup W$$

Proposition 2. (Facets)

1. *For any* $x \in V \setminus \mathcal{C}$, *the positivity inequality* $p_x \geq 0$ *defines a facet.*
2. *If* $x \in \mathcal{C}$, *then* $p_x \geq 0$ *defines a facet iff* x *is isolated in* G.
3. *For any* $x \in \mathcal{C}$ *and* $y \sim x$, *the mode inequality* $p_x \geq p_y$ *defines a facet.*

Proposition 3. (Volume) *Let* Σ *be the set of linear extensions of the partial order associated to* $(G, \mathcal{C})$. *Then* $\operatorname{vol}(\mathbf{M}(G, \mathcal{C})) = \frac{|\Sigma|}{|V|!} \operatorname{vol}(\Delta(V))$.

The Polytope of Strong Modes



- A point $x \in V$ is a **strong mode** of a probability distribution $p \in \Delta(V)$ if

$$p_x \geq \sum_{y \sim x} p_y \quad \text{for all } x.$$

- The **polytope of strong $\mathcal{C}$-modes** in G is the set of probability distributions

$$\mathbf{S}(G, \mathcal{C}) := \{p \in \Delta(V) \colon \text{each } x \in \mathcal{C} \text{ is a strong mode}\}.$$

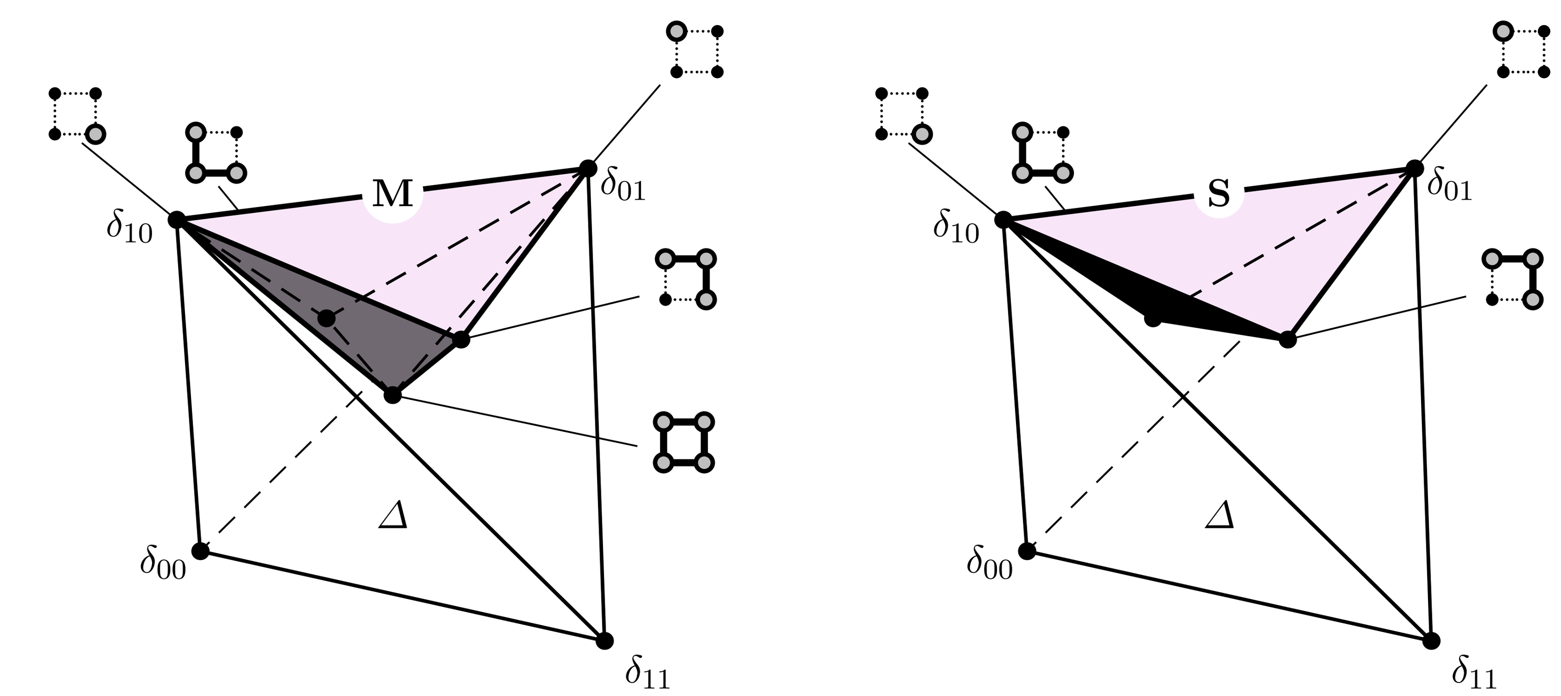
Proposition 4. (Vertices) $\mathbf{S}(G, \mathcal{C})$ *is a* $(|V| - 1)$*-simplex with vertices* $e_{\mathcal{C}}^x$, $x \in V$.

Proposition 5. (Facets)

1. *If* $x \in \mathcal{C}$, *the mode inequality* $p_x \geq \sum_{y \sim x} p_y$ *defines a facet.*
2. *If* $x \in V \setminus \mathcal{C}$, *the positivity inequality* $p_x \geq 0$ *defines a facet.*

Proposition 6. (Volume) $\operatorname{vol}(\mathbf{S}(G, \mathcal{C})) = \left(\prod_{x \in V} \frac{1}{|N_{\mathcal{C}}(x)| + 1} \right) \operatorname{vol}(\Delta(V))$.

Examples



Example 7. The figure shows $\mathbf{M}(G, \mathcal{C})$ and $\mathbf{S}(G, \mathcal{C})$ for

Vertices	$V = \{0, 1\}^2$
Edges	$E = \{(00, 01), (00, 10), (01, 11), (10, 11)\}$
Modes	$\mathcal{C} = \{01, 10\}$

Example 8. Let G be the edge graph of an n-cube with vertices $V = \{0, 1\}^n$ and edges between pairs of Hamming distance one.

- If $\mathcal{C} \subseteq V$ has cardinality $|\mathcal{C}| = k$ and minimum distance 3, then
  - $\mathbf{S}$ has 2^n vertices and $\operatorname{vol}(\mathbf{S}) = 2^{-kn} \operatorname{vol}(\Delta)$.
  - $\mathbf{M}$ has $k(2^n - 1) + 2^n - kn$ vertices and $\operatorname{vol}(\mathbf{M}) = \frac{|\Sigma|}{2^n!} \operatorname{vol}(\Delta) \geq k! 2^{-kn} \operatorname{vol}(\Delta)$.
- If $\mathcal{C}$ is the set of all even-parity strings, then
  - $\mathbf{S}$ has 2^n vertices and $\operatorname{vol}(\mathbf{S}) = (n + 1)^{-2^{n-1}} \operatorname{vol}(\Delta)$.
  - $\mathbf{M}$ has $2^{2^{n-1}} - 1 + 2^{n-1}$ vertices and $\operatorname{vol}(\mathbf{M}) = \frac{|\Sigma|}{2^n!} \operatorname{vol}(\Delta) \geq \binom{2^n}{2^{n-1}}^{-1} \operatorname{vol}(\Delta)$.
  - For $n = 2$ and $n = 3$ we have $|\Sigma| = 4$ and $|\Sigma| = 720$. Next open case is $n = 4$.

Relation to Order Polytopes

- The patterns of modes that are possible in a given vicinity structure define special types of partial orders in the coordinates of the probability vectors.
- The *order polytope* of a partial order arises by looking at subsets of the unit hypercube instead of subsets of the probability simplex:

$$\mathcal{O}(\succeq) := \{p \in [0, 1]^V : p_x \geq p_y \text{ whenever } x \succeq y\}.$$

  One can show that $\mathbf{M}(\succeq)$ is the vertex figure of $\mathcal{O}(\succeq)$ at the vertex 0.

Acknowledgments

G.M. is grateful to the Santa Fe Institute, Santa Fe, NM, USA, for hosting him during the initial work on this article.

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Algebraic Statistics 2015, Genoa, Italy, June 8–11, 2015.