RESEARCH PROJECT

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My research works and projects lie at the intersection between geometric structures and dynamical systems.

In the one hand, I am interested with rigidity phenomena for hyperbolic or partially hyperbolic dynamical systems whose invariant distributions are highly regular. In this setting, I obtained in [MM22b] a classification result for three-dimensional partially hyperbolic diffeomorphisms of contact type with smooth invariant distributions, presented in Theorem A below. This result is obtained by studying an invariant rigid geometric structure called *path geometry*, with the tools of *Cartan geometries*. Both of these notions are introduced in paragraph 1.3, together with a related rigidity result obtained in collaboration with Elisha Falbel and Jose Miguel Veloso in [FMMV21], see Theorem B. Following these two results, my first broad research project is to pursue a systematic study of the rigidity of partially hyperbolic diffeomorphisms having smooth invariant distributions, by the means of rigid geometric structures and Cartan geometries. I will present in paragraphs 2.1 and 2.2 two ongoing projects in this direction.

I am also interested with the dual problem, aiming at describing those compact rigid geometric structures that have a non-compact automorphism group. The research of new such examples in the case of *flat path geometries* led me to construct in [MM22a] a geometric compactification of the geodesic flow of complete and non-compact hyperbolic surfaces, see Theorem C. These are examples of closed three-manifolds locally modelled on the *flag space* $\mathrm{PGL}_3(\mathbb{R})/\mathbf{P}_{min}$ (with $\mathbf{P}_{min}$ the Borel subgroup of upper triangular matrices), a geometry which is still quite poorly understood. In a tentative to broader available examples, I will present in paragraph 2.3 an ongoing project with Elisha Falbel aiming at constructing *non-Kleinian*, *i.e.* exotic examples of such flag structures.

The third aspect of my current research described in paragraphs 2.4 and 2.5, is interested with the interaction of a *singular* kind of locally de-Sitter Lorentzian metrics on the torus, with the topological dynamics of the lightlike foliations that they define, and with three-dimensional Anosov flows. The interaction between geometry and dynamics goes in the other direction in this setting, and I prove in a paper in preparation that in some cases, the dynamics of the foliations essentially describe the geometry.

1. Contributions

1.1. **Contact-Anosov flows.**

Let us recall that a non-singular flow (φ^t) of class $\mathcal{C}^\infty$ of a closed manifold M is called *Anosov*, if its differential preserves a splitting $\mathrm{T}M = E^s \oplus E^c \oplus E^u$ of the tangent bundle, where E^c is the direction of the flow and E^s and E^u are non-trivial distributions verifying the following estimates (with respect to any Riemannian metric on M).

(1) The *stable distribution* E^s is *uniformly contracted* by (φ^t), *i.e.* there are two constants $C > 0$ and $0 < \lambda < 1$ such that for any $t \in \mathbb{R}$ and $x \in M$:

$$\left\| \mathrm{D}_x \varphi^t |_{E^s} \right\| \leq C\lambda^t. \tag{1.1}$$

(2) The *unstable distribution* E^u is *uniformly expanded* by (φ^t), *i.e.* uniformly contracted by (φ^{-t}).

Important examples of three-dimensional Anosov flows are given by the geodesic flows of closed hyperbolic surfaces Σ, acting on their unitary tangent bundle $\mathrm{T}^1\Sigma$. These flows have the following specific properties among Anosov flows: their stable and unstable distributions are $\mathcal{C}^\infty$ (while they are in general only Hölder continuous), and the sum $E^s \oplus E^u$ is furthermore a *contact distribution*. We recall that a plane field ξ of a three-dimensional manifold is called *contact* if it is nowhere integrable, or more precisely if it is locally the kernel of a *contact form* θ, *i.e.* a one-forme such

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that $\theta \wedge d\theta$ does not vanish. A beautiful result of Étienne Ghys in [Ghy87] says that, up to finite coverings and orbit equivalence[1], the geodesic flows of closed hyperbolic surfaces are in fact the only examples of three-dimensional Anosov flows whose stable and unstable distributions are $\mathcal{C}^\infty$ and such that $E^s \oplus E^u$ is a contact distribution. Ghys actually proves that all these flows are smoothly conjugated to algebraic examples (the right diagonal flow on compact quotients of $\mathrm{PSL}_2(\mathbb{R})$), and we will thus call them the *algebraic contact-Anosov flows*.

1.2. **Partially hyperbolic diffeomorphisms of contact type.** The result of [Ghy87] is a striking expression of the dynamical rigidity that can be deduced from geometrical assumptions for the case of flows, *i.e.* *continuous-time* dynamical systems. A thrilling question is then to know if these results generalize to *discrete-time* dynamical systems. Natural discrete-time analogs for the Anosov flows are the diffeomorphisms f of closed manifolds M, whose differential preserves a splitting $\mathrm{T}M = E^s \oplus E^c \oplus E^u$ (within non-zero distributions) such that E^s (respectively E^u) is uniformly contracted (resp. expanded) by $\mathrm{D}f$. These diffeomorphisms are called *partially hyperbolic*[2] (see [CP15] for a comprehensive introduction, and [HP18] for a general survey about the classification problem) and received a lot of attention in the last decades. In this setting, I obtained the following result.

Theorem A ([MM22b, Theorem A])**.** *Let f be a partially hyperbolic diffeomorphism of a three-dimensional connected compact manifold M, whose invariant distributions E^s, E^u and E^c are smooth, such that $E^s \oplus E^u$ is a contact distribution, and whose non-wandering set $NW(f)$ equals M. Then, up to finite coverings and iterates, f is $\mathcal{C}^\infty$-conjugated to one of the following examples:*
1. *the time-one map of a three-dimensional algebraic contact-Anosov flow;*
2. *or a partially hyperbolic affine automorphism of a nil-*$\mathrm{Heis}(3)$*-manifold.*

Note that any diffeomorphism preserving a volume form satisfies the assumption $NW(f) = M$. The second family of examples are defined on compact quotients $\Gamma\backslash\mathrm{Heis}(3)$ of the three-dimensional Heisenberg group by cocompact lattices, and induced by affine automorphisms of $\mathrm{Heis}(3)$ preserving Γ (see for instance [MM22b, §1.1] or [Sma67, Ham13] for a description of such algebraic examples).

Actually, the Theorem A does not rely on any uniformity concerning the contraction (respectively expansion) of E^s (resp. E^u) by $\mathrm{D}f$. More precisely, let f be a diffeomorphism of a three-dimensional closed manifold M having a dense orbit in M (this replaces the hypothesis $NW(f) = M$), preserving a smooth splitting $\mathrm{T}M = E^\alpha \oplus E^c \oplus E^\beta$ with $E^\alpha \oplus E^\beta$ a contact distribution, and assume that for any $x \in M$ we have, for $\varepsilon = \alpha$ and $\varepsilon = \beta$:

$$\lim_{n\to+\infty} \|\mathrm{D}_x f^n|_{E^\varepsilon}\| = 0 \text{ or } \lim_{n\to-\infty} \|\mathrm{D}_x f^n|_{E^\varepsilon}\| = 0 \tag{1.2}$$

with respect to some Riemannian metric on M. Then the conclusions of Theorem A hold on f (see [MM22b, Theorem B]). Let us emphasize that the assumption (1.2) is related to, though different from, the notion of *quasi-Anosov diffeomorphism* of Mañé in [Mañ77].

1.3. **Path geometries and Cartan geometries.** The triplet $\mathcal{S} = (E^s, E^c, E^u)$ preserved by a partially hyperbolic diffeomorphism f of contact type as in Theorem A happens to be a *rigid geometric structure*, and the rough idea is that the dynamical properties of the *automorphism* f of $\mathcal{S}$ will imply a geometrical classification of $\mathcal{S}$, giving in return a dynamical classification of f.

On a three-dimensional manifold, a pair $\mathcal{L} = (E^\alpha, E^\beta)$ of transverse line fields whose sum is a contact distribution is called a *path geometry*. These structures are intimately linked with the homogeneous space $\mathbf{X}$ of *full flags of* $\mathbb{R}^3$, endowed with a natural path geometry invariant under the natural action of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$. The flag space $\mathbf{X}$ plays for path geometries the role played by the euclidean space for Riemannian metrics: it is the *flat model*. The notion of *Cartan geometry* (originally due to Élie Cartan, see [Car10, Sha97, ČS09]) allows indeed to give a precise meaning to the following idea: every three-dimensional path geometry $\mathcal{L}$ is a "curved version" of the homogeneous space $\mathbf{X}$, and enjoys a *curvature* whose vanishing is equivalent to $\mathcal{L}$ being *flat*,

[1] Two flows are *orbit equivalent* if there exists a diffeomorphism conjugating their orbits.

[2] The denomination partially hyperbolic actually refers in the litterature to the case where the invariant splitting $E^s \oplus E^c \oplus E^u$ is furthermore *dominated*. This assumption being however unnecessary in Theorem A and elsewhere in this text, we allow ourselves to elude it to simplify the terminology, and refer the interested reader to [CP15].

i.e. locally isomorphic to $\mathbf{X}$. The tools of Cartan geometries play a crucial role in the classification of Theorem A.

In Theorem A, even if the diffeomorphisms are only assumed to preserve the triplet (E^s, E^u, E^c), the classification shows *a posteriori* that they preserve in fact a *global* contact form θ of kernel $E^s \oplus E^u$. In other words, they preserve the triplet $\mathcal{T} = (E^s, E^u, \theta)$, that we call a *strict path structure*. In [GD91], a general program was introduced for studying, and possibly classifying those compact rigid geometric structures having a non-compact automorphism group. In this direction, we obtained with Elisha Falbel and Jose Miguel Veloso the following result concerning strict path structures.

Theorem B ([FMMV21, Theorem 1.1])**.** *Let $(M, \mathcal{T})$ be a three-dimensional closed and connected strict path structure, whose automorphism group is non-compact and has a dense orbit. Then $(M, \mathcal{T})$ is isomorphic to one of the family of examples appearing in Theorem A.*

1.4. **Compactifications of path geometries.** All the diffeomorphisms of Theorem A are *conservative* (*i.e.* preserve a volume form), and moreover preserve a line field E^c transverse to the contact distribution $E^s \oplus E^u$. A first reasonable problem to understand the diversity of path geometries with large automorphism groups is thus to exhibit path geometries enjoying non-conservative automorphisms that are *non-equicontinuous* (*i.e.* generate a non-compact subgroup of the automorphism group) and moreover *essential*: they preserve no line field transverse to the contact distribution. For any (complete) hyperbolic surface Σ, the unitary tangent bundle $\mathrm{T}^1\Sigma$ is endowed with a natural path geometry $\mathcal{L}_\Sigma$ invariant by the geodesic flow, for which I obtained the following.

Theorem C ([MM22a, Theorem A])**.** *Let $g_1, \dots, g_d$ be hyperbolic elements of $\mathrm{PSL}_2(\mathbb{R})$ with pairwise distinct fixed points on the boundary $\partial_\infty \mathbf{H}^2$. Then there exists integers $r_i > 0$ such that the hyperbolic surface $\Sigma = \langle g_1^{r_1}, \dots, g_d^{r_d} \rangle \backslash \mathbf{H}^2$ verifies the following.*

(1) *The path geometry $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ admits a compactification $(M, \mathcal{L})$.*
(2) *Furthermore, the geodesic flow of $\mathrm{T}^1\Sigma$ extends to a non-equicontinuous, non-conservative and essential automorphism flow of $(M, \mathcal{L})$.*

The first statement of this theorem relies on the study of the action of "Schottky" discrete subgroups of $\mathrm{PGL}_3(\mathbb{R})$ on the flag space $\mathbf{X}$, which provides an independent and elementary proof of the existence of open subsets of the flag space with proper and cocompact action of these Schottky subgroups. These domains of discontinuity, also provided by general results about Anosov representations in [GW12, KLP18, BPS19], are here obtained by constructing explicit fundamental domains for the action. This is done by a precise analysis of the dynamics of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$, allowing to obtain the dynamical properties of the compactified geodesic flow in the second statement.

2. Ongoing and future projects

2.1. **Rigidity of three-dimensional partially hyperbolic diffeomorphisms.** Ghys actually classifies in [Ghy87] all three-dimensional Anosov flows with smooth stable and unstable distributions. A natural project is then to extend Theorem A by classifying all the three-dimensional partially hyperbolic diffeomorphisms having smooth invariant distributions E^s, E^c and E^u – wether $E^s \oplus E^u$ is contact or not. A first result was obtained in this direction in [CPRH20] under the following strong additional restrictions on the partially hyperbolic diffeomorphism f: $\mathrm{D}f$ reads as a constant (diagonal) matrix in some global frame of vector fields generating (E^s, E^c, E^u), and f has a dense orbit. This result was recently precised in [AM21][3] with a new geometrical proof, which is a motivation to look at the general question with geometrical eyes, *i.e.* to consider (E^s, E^c, E^u) as a geometric structure whose behaviour differs between the open subset $O \subset M$ where $E^s \oplus E^u$ is contact and its complement. The case $O = M$ is the one of contact-type partially hyperbolic diffeomorphisms classified in Theorem A. The only known examples for which O is a strict open subset of M are $\mathcal{C}^\infty$-conjugated to the suspension of an Anosov automorphism of the two-torus, in which case the distribution $E^s \oplus E^u$ is integrable and O is thus empty. This suggests that $O \neq \varnothing$ should imply that $O = M$, in other words that "if $E^s \oplus E^u$ is contact somewhere, then it is contact

[3]In [AM21], the framing only needs to be $\mathcal{C}^1$ instead of $\mathcal{C}^2$ and the topological transitivity assumption is dropped.

everywhere". While I am able to obtain a very precise geometric description of O, even in the absence of compacity, the main difficulty of this problem is to find a *global* information to extract from the partially hyperbolic behaviour in order to study points of ∂O.

2.2. **Higher-dimensional partially hyperbolic diffeomorphisms of contact type.** Theorem A is an analog for partially hyperbolic diffeomorphisms of Ghys classification in [Ghy87] of three-dimensional contact-Anosov flows with smooth invariant distributions. In 1992, Ghys theorem was generalized in higher dimensions by Benoist, Foulon and Labourie in [BFL92]: any contact-Anosov flow with smooth stable and unstable distributions is, up to finite coverings and orbit equivalence, the geodesic flow of a closed locally symmetric Riemannian manifold of strictly negative curvature. It is thus natural to look for a discrete-time analog of this classification, that is for partially hyperbolic diffeomorphisms f in any (odd) dimension, having a dense orbit, smooth invariant distributions, and for which $E^s \oplus E^u$ is a contact distribution. The known examples of such contact-type partially hyperbolic diffeomorphisms are of two kind: the time-one maps of the geodesic flows cited above, and higher-dimensional versions of nil-manifold automorphisms appearing in Theorem A. The pair (E^s, E^u) of integrable distributions defines again in this case a rigid geometric structure, called *Lagrangian contact structure* and equivalent to *Cartan geometries* modelled on some higher-dimensional flag space $\mathbf{X}_{2n+1}$, homogeneous under the action of $\mathrm{PGL}_{n+2}(\mathbb{R})$. Going from the case of Anosov flows to the one of partially hyperbolic diffeomorphisms makes however the situation deeply different in various regards. The algebraic dichotomy between the two families of examples, respectively coming from simple and nilpotent Lie groups, reflects for instance in dimensions ≥ 5 into a geometric dichotomy. Indeed, while the path geometries of both kind of examples are flat in dimension 3, the Lagrangian-contact structure preserved by the geodesic flows is not flat in dimension ≥ 5, whereas the one of nil-manifolds is. Furthermore, topologically transitive Anosov flows enjoy numerous strong global dynamical properties (among which the density of periodic points and of stable and unstable leaves) which disappear in the partially hyperbolic case. These remarks illustrate the need to preceed any global arguments (of homogeneous dynamics especially) by a detailed study of the *curvature* of the Lagrangian-contact structure (E^s, E^u), which is currently my main focus.

2.3. **Surgeries of flag structures on closed 3-manifolds.** The examples constructed in Theorem C are a motivation to construct new flat path geometries, *i.e.* $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-structures which we will call *flag structures*, on closed 3-manifolds. A basic and important question about any (G, X)-structure on a closed manifolds, is the one of the existence of *non-Kleinian* structures, *i.e.* one which cannot be written as a cocompact quotient $\Gamma\backslash\Omega$ of an open set of the model by a properly discontinuous action. The flag structures constructed by Barbot in [Bar10] are Kleinian structures with Anosov holonomies, and it is not known wether the examples constructed by Falbel and Thebaldi in [FT15] are Kleinian or not. To the best of our knowledge, only one family of non-Kleinian flag structures on closed 3-manifolds is in fact known. In an ongoing work with Elisha Falbel, we try to construct non-Kleinian examples through surgeries of flag manifolds.

2.4. **Rigidity of singular de-Sitter tori with respect to their lightlike foliations.** Because of a Lorentzian version of Gauss-Bonnet formula, the only constant curvature Lorentzian metrics that can arise on the torus are *flat*. Inspired from flat Riemannian metrics with conical singularities, it is thus natural to look at *singular de-Sitter* Lorentzian metrics on $\mathbf{T}^2$, namely constant curvature 1 Lorentzian metrics, defined on $\mathbf{T}^2$ but at a finite number of *singularities* around which the metric has a non-trivial holonomy called the *angle*. In other words, a singular de-Sitter metric is a constant curvature 1 Lorentzian metric on a finitely-punctured torus, with standard models for the neighbourhoods of the punctures. Our interest for these singular de-Sitter tori is motivated by the existence of topological foliations, which extend at the singularities the lightlike foliations of the metric. This allows for an interaction between the geometry of the singular metric and the topological dynamics of its lightlike foliations. These foliations form a natural isometry-invariant beside the angles of the singularities and in a paper in preparation, we consider the case of a single singularity and prove then that in some cases (including the one of minimal lightlike foliations), these are the only invariants of de-Sitter tori with one singularity. Namely, that two de-Sitter tori with one singularity, having the same angles and *topologically* equivalent lightlike foliations, are isometric. In an ongoing work with Selim Ghazouani, we pursue this work in the case of multiple singularities.

2.5. **Singular de-Sitter surfaces and Anosov flows.** According to a work of Ghys, transitive three-dimensional Anosov flows with a transverse de-Sitter structure are essentially geodesic flows of closed hyperbolic surfaces, and it is an important open question of *Fried* to know if any three-dimensional transitive Anosov flow is connected to such a geodesic flow by finitely many *surgeries*. Using the singular de-Sitter Lorentzian metrics introduced in the previous paragraph, our first goal with Pierre Dehornoy and Selim Ghazouani, is to use the *Birkhoff sections* of a transitive Anosov flow and the one-dimensional foliations that its strong stable and unstable foliations print on these surfaces, to define a singular de-Sitter structure *transverse to the flow*. Our aim would then be to show that a surgery of the flow links to a desingularization of this structure, opening the road to a strategy to investigate Fried's conjecture.

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