RESEARCH PROGRAMME

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My research work lies at the intersection between geometric structures and dynamical systems.

In the one hand, I am interested with rigidity phenomena for hyperbolic or partially hyperbolic dynamical systems whose invariant distributions are highly regular. In this setting, I obtained in [MM22b] a classification result for three-dimensional partially hyperbolic diffeomorphisms of contact type with smooth invariant distributions, presented in Paragraph 1.2 below (see Theorem A). This result is obtained by studying an invariant rigid geometric structure called *path structure*, with the tools of *Cartan geometries*. The two latter notions are introduced in Paragraph 1.3, together with two rigidity results obtained in collaboration with Elisha Falbel and Jose Miguel Veloso in [FMMV21, FMMV24] (see Theorems B and C). In the continuity of these results, my first broad research project is to pursue a systematic study of the rigidity of partially hyperbolic diffeomorphisms and Anosov flows having smooth invariant distributions, by the means of rigid geometric structures and Cartan geometries. I will present in Paragraph 2.1 two ongoing projects in this direction, partly joint with Karin Melnick and Elisha Falbel.

The *flat* path structures are three-dimensional manifolds locally modelled on the *flag space* $\mathrm{PGL}_3(\mathbb{R})/\mathbf{P}_{min}$ (with $\mathbf{P}_{min} \subset \mathrm{PGL}_3(\mathbb{R})$ the subgroup of upper triangular matrices), which led me to study the geometry of $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-structures, that are called *flag structures*. I will present in Paragraph 1.4 two results obtained on this subject in [MM22a, FMM24], respectively concerning a geometric compactification of the geodesic flow of non-compact hyperbolic surfaces (see Theorem D), and a surgery method developed with Elisha Falbel yielding new non-uniformizable examples (see Theorem E).

In Paragraph 1.5, I describe the third aspect of my current research and present a rigidity result obtained in [MM24], where I show that de-Sitter tori with a single *singularity* are determined by the topological type of their minimal lightlike foliations (see Theorem F). I present in Paragraph 2.2 three projects concerning this geometric rigidity for multiple singularities, and its application to circle diffeomorphisms with breaks and to three-dimensional Anosov flows. These projects are respectively in collaboration with Selim Ghazouani, Konstantin Khanin and Pierre Dehornoy.

1. Contributions

1.1. Literature on contact-Anosov flows.

Let us recall that a non-singular flow (φ^t) of class $\mathcal{C}^\infty$ of a closed manifold M is called *Anosov*, if its differential preserves a splitting $\mathrm{T}M = E^s \oplus E^c \oplus E^u$ of the tangent bundle, where E^c is the direction of the flow and E^s and E^u are non-trivial distributions verifying the following estimates (with respect to any Riemannian metric on M).

(1) The *stable distribution* E^s is *uniformly contracted* by (φ^t), *i.e.* there are two constants $C > 0$ and $0 < \lambda < 1$ such that for any $t \in \mathbb{R}$ and $x \in M$:

$$\left\| \mathrm{D}_x \varphi^t |_{E^s} \right\| \leq C\lambda^t. \tag{1.1}$$

(2) The *unstable distribution* E^u is *uniformly expanded* by (φ^t), *i.e.* uniformly contracted by (φ^{-t}).

Important examples of three-dimensional Anosov flows are given by the geodesic flows of closed hyperbolic surfaces Σ, acting on their unitary tangent bundle $\mathrm{T}^1\Sigma$. These flows have the following specific properties among Anosov flows: their stable and unstable distributions are $\mathcal{C}^\infty$ (while they are in general only Hölder continuous), and the sum $E^s \oplus E^u$ is furthermore a *contact distribution*. We recall that a plane field ξ of a three-dimensional manifold is called *contact* if it is nowhere integrable, or more precisely if it is locally the kernel of a *contact form* θ, *i.e.* a one-forme such that $\theta \wedge d\theta$ does not vanish. A beautiful result of Étienne Ghys in [Ghy87] says that, up to

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finite coverings and orbit equivalence[1], the geodesic flows of closed hyperbolic surfaces are in fact the only examples of three-dimensional Anosov flows whose stable and unstable distributions are $\mathcal{C}^\infty$ and such that $E^s \oplus E^u$ is a contact distribution. Ghys actually proves that all these flows are smoothly conjugated to algebraic examples (the right diagonal flow on compact quotients of $\mathrm{PSL}_2(\mathbb{R})$), and we will thus call them the *algebraic contact-Anosov flows*. In 1992, Ghys theorem was generalized in higher dimensions by Benoist, Foulon and Labourie in [BFL92]: any contact-Anosov flow with $\mathcal{C}^\infty$ stable and unstable distributions is, up to finite coverings and orbit equivalence, the geodesic flow of a closed locally symmetric Riemannian manifold of strictly negative curvature.

1.2. **Rigidity of partially hyperbolic diffeomorphisms of contact type.** The result of [Ghy87] is a striking expression of the dynamical rigidity that can be deduced from geometrical assumptions in the case of flows, *i.e.* of *continuous-time* dynamical systems. A thrilling question is then to know if these results generalize to *discrete-time* dynamical systems. Natural discrete-time analogs of Anosov flows are the diffeomorphisms f of closed manifolds M, whose differential preserves a splitting $\mathrm{T}M = E^s \oplus E^c \oplus E^u$ (within non-zero distributions) and such that E^s (respectively E^u) is uniformly contracted (resp. expanded) by $\mathrm{D}f$ in the sense of (1.1). These diffeomorphisms are called *partially hyperbolic*[2] (see [CP15] for a comprehensive introduction, and [HP18] for a general survey about the classification problem) and received a lot of attention in the last decades. In this setting, I obtained the following result, partially answering the question in dimension three.

Theorem A ([MM22b, Theorem A])**.** *Let f be a partially hyperbolic diffeomorphism of a three-dimensional connected closed manifold M, whose invariant distributions E^s, E^u and E^c are smooth, such that $E^s \oplus E^u$ is a contact distribution, and whose non-wandering set $NW(f)$ equals M. Then, up to finite coverings and iterates, f is $\mathcal{C}^\infty$-conjugated to one of the following examples:*

(1) the time-one map of a three-dimensional algebraic contact-Anosov flow;
(2) or a partially hyperbolic affine automorphism of a nil-$\mathrm{Heis}(3)$*-manifold.*

Note that any diffeomorphism preserving a volume form satisfies the assumption $NW(f) = M$. The second family of examples are defined on compact quotients $\Gamma\backslash\mathrm{Heis}(3)$ of the three-dimensional Heisenberg group by cocompact lattices, and are induced by affine automorphisms of $\mathrm{Heis}(3)$ preserving Γ (see for instance [MM22b, §1.1] or [Ham13] for a description of these algebraic examples).

1.3. **Path structures, Cartan geometries and rigidity.** The triplet $\mathcal{S} = (E^s, E^c, E^u)$ of distributions preserved by a partially hyperbolic diffeomorphism f of contact type as in Theorem A happens to be a *rigid geometric structure*, and the rough idea is that the dynamical properties of the *automorphism* f of $\mathcal{S}$ will imply a geometrical classification of $\mathcal{S}$, yielding in return a dynamical classification of f.

On a three-dimensional manifold, a pair $\mathcal{L} = (E^\alpha, E^\beta)$ of transverse $\mathcal{C}^\infty$ line fields whose sum is a contact distribution is called a *path structure*. These structures are intimately linked with the homogeneous space

$$\mathbf{X} \coloneqq \left\{(l, P) \,\middle|\, \dim l = 1, \dim P = 2, l \subset P \subset \mathbb{R}^3\right\}$$

of *full flags of* $\mathbb{R}^3$. The latter is endowed with a natural path structure having $\mathrm{PGL}_3(\mathbb{R})$ as group of automorphisms (which naturally act on a flag $(l, P) \in \mathbf{X}$ by $g \cdot (l, P) = (g(l), g(P))$). The flag space $\mathbf{X}$ plays for path structures the role played by the euclidean space for Riemannian metrics: it is the *flat model* of path structures. The notion of *Cartan geometry* (originally due to Élie Cartan, see [Car10, Sha97]) indeed allows to associate to every three-dimensional path structure $\mathcal{L}$ a *curvature*, whose vanishing is equivalent to $\mathcal{L}$ being locally isomorphic to $\mathbf{X}$. We say in this

[1]Two flows are *orbit equivalent* if there exists a diffeomorphism conjugating their orbits.

[2]The denomination partially hyperbolic actually refers in the litterature to the case where the invariant splitting $E^s \oplus E^c \oplus E^u$ is furthermore *dominated*. This assumption being however unnecessary in Theorem A and elsewhere in this text, we allow ourselves to elude it to simplify the terminology, and refer the interested reader to [CP15].

case that $\mathcal{L}$ is *flat*. The tools offered by Cartan geometries play a crucial role in the proof of Theorem A.

Even though the diffeomorphisms are only assumed to preserve the triplet (E^s, E^u, E^c), Theorem A shows *a posteriori* that they preserve in fact a *global* contact form θ of kernel $E^s \oplus E^u$. In other words, they preserve the triplet $\mathcal{T} = (E^s, E^u, \theta)$, that we call a *strict path structure*. In [GD91], a general programme was suggested for studying, and possibly classifying the compact rigid geometric structures having a non-compact automorphism group. In this direction, we obtained with Elisha Falbel and Jose Miguel Veloso the following result concerning strict path structures.

Theorem B ([FMMV21, Theorem 1.1]). *Let $(M, \mathcal{T})$ be a three-dimensional closed and connected strict path structure, whose automorphism group is non-compact.*[3] *Then $(M, \mathcal{T})$ is isomorphic to one of the family of examples appearing in Theorem A.*

In an other direction, it is natural to investigate those path structures which have a "large" group of automorphisms not anymore in terms of the dynamics (namely non-compactness), but of the dimension. This question makes sense locally by looking at the Lie algebra $\mathfrak{kill}_x^{loc}(\mathcal{L})$ of local *Killing fields*, which are the vector fields defined in the neighbourhood of a point x and whose flow preserves the path structure. According to a classical result of Tresse [Tre96], $\dim \mathfrak{kill}_x^{loc}(\mathcal{L}) \leq 3$ for non-flat path structures. While $\dim \mathfrak{kill}_x^{loc}(\mathcal{L}) = 3$ does *a priori* not force the path structure to be locally homogeneous in the neighbourhood of x (since some Killing fields could vanish at x), we obtained with E. Falbel and J. M. Veloso the following result.

Theorem C ([FMMV24, Theorem 1.1]). *At a point where $\dim \mathfrak{kill}_x^{loc}(\mathcal{L}) > 2$, a three-dimensional path structure whose curvature does not vanish is locally isomorphic to a left-invariant path structure on a three-dimensional Lie group.*[4]

1.4. **Three-dimensional flag structures.** The diffeomorphisms appearing in Theorem A are *conservative* (*i.e.* preserve a volume form), and moreover preserve a line field E^c transverse to the contact distribution $E^s \oplus E^u$. A first reasonable problem to understand the diversity of path structures with large automorphism groups is thus to exhibit path structures enjoying non-conservative automorphisms that are not only *non-equicontinuous* (*i.e.* generate a non-compact subgroup of the automorphism group) but also *essential*: they preserve no line field transverse to the contact distribution. For any (complete) hyperbolic surface Σ, the unitary tangent bundle $\mathrm{T}^1\Sigma$ is endowed with a natural flat path structure $\mathcal{L}_\Sigma$ invariant by the geodesic flow, for which I obtained the following.

Theorem D ([MM22a, Theorem A]). *Let $g_1, \dots, g_d$ be hyperbolic elements of $\mathrm{PSL}_2(\mathbb{R})$ with pairwise distinct fixed points on the boundary $\partial_\infty \mathbf{H}^2$. Then there exists integers $r_i > 0$ such that the hyperbolic surface $\Sigma = \langle g_1^{r_1}, \dots, g_d^{r_d} \rangle \backslash \mathbf{H}^2$ verifies the following.*

(1) The path structure $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ admits a flat compactification $(M, \mathcal{L})$.
(2) Furthermore, the geodesic flow of $\mathrm{T}^1\Sigma$ extends to a non-equicontinuous, non-conservative and essential automorphism flow of $(M, \mathcal{L})$.

This theorem relies on the study of the action of "Schottky" discrete subgroups of $\mathrm{PGL}_3(\mathbb{R})$ on the flag space $\mathbf{X}$, and provides an independent and elementary proof of the existence of open subsets of the flag space with proper and cocompact action of these Schottky subgroups. These domains of discontinuity, also provided by general results about *Anosov representations* in [GW12, KLP18], are here obtained by constructing explicit fundamental domains for the action.

Since a flat path structure on a three-manifold M is locally isomorphic to $\mathbf{X}$, it is described by an atlas of charts from M to $\mathbf{X}$ whose transition functions are restrictions of elements of $\mathrm{PGL}_3(\mathbb{R})$. Such a maximal atlas is called a $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-*structure*, which we will henceforth call a *flag structure*. To the best of our knowledge, all of the closed flag structures which appeared so

[3]The published version of this result assumes the existence of a dense orbit of this group, but a revision made to the pre-published version on arXiv shows that this assumption is in fact superfluous.

[4]This statement is a corollary of [FMMV24, Theorem 1.1] which will be explicitly stated in the future version of the preprint, to be updated on arXiv.

far in the literature were *Kleinian*, *i.e.* isomorphic to the quotient $\Gamma\backslash\Omega$ of an open set $\Omega \subset \mathbf{X}$ by a properly discontinuous action of a discrete subgroup $\Gamma \subset \mathrm{PGL}_3(\mathbb{R})$ (see for instance [Bar10]). The only general method to produce examples of closed (G, X)-structures is the *Ehresman-Thurston* principle, asserting in our setting that the set of morphisms from $\pi_1(M)$ to $\mathrm{PGL}_3(\mathbb{R})$ that are holonomy morphisms of a flag structure on a closed manifold M, is open. However, Ehresman-Thurston principle does not automatically provide non-Kleinian examples. In [FMM24], we introduce with Elisha Falbel a notion of geometric surgery for flag structures, allowing to combine two previously known flag structures to obtain a new one. Using such surgeries we provide new examples of flag structures of both Kleinian and non-Kleinian types, and obtain in particular the following result.

Theorem E ([FMM24, Theorem D])**.** *Let M be a closed three-manifold endowed with a flag structure:*

- *containing an open set U isomorphic to the neighborhood of an $\alpha - \beta$ bouquet of two circles,*
- *and whose holonomy group contains a loxodromic element.*

There there exists a closed three-manifold with a non-Kleinian flag structure, into which $M \setminus U$ embedds.

1.5. **Rigidity of singular de-Sitter tori.** A Lorentzian metric on a surface induces a pair of *lightlike foliations*, and the Poincaré-Hopf theorem therefore implies that the torus is the only closed and orientable Lorentzian surface. An analog of the Gauß-Bonnet formula shows moreover that the only constant curvature Lorentzian metrics on the torus are actually *flat*. It is then natural to try to widen this class of geometries, to obtain structures which are locally modelled on the *de-Sitter space*, namely the two-dimensional non-flat Lorentzian homogeneous space $\mathbf{dS}^2 \equiv \mathrm{PSL}_2(\mathbb{R})/A$ (with $A = \{a^t\}_{t\in\mathbb{R}} \subset \mathrm{PSL}_2(\mathbb{R})$ the diagonal one-dimensional subgroup). This is not possible without removing some points, and a natural way to do this is to proceed as in the Riemannian case, by considering Lorentzian metrics locally isometric to $\mathbf{dS}^2$ and defined on the complement of finitely many points in a surface, where the metric has *standard singularities*. These local singularities, already appearing in [BBS11], are constructed as in the Riemannian case by considering a nontrivial isometry $a^\theta \in A$ fixing a point $x \in \mathbf{dS}^2$ and two geodesic rays $\gamma_+, \gamma_- = a^\theta(\gamma_+)$ emanating from x, and by gluing the two boundary components of the sector delimited by γ_- and γ_+ by a^θ. While the metric is not defined at the point x in the quotient (for the holonomy of a small loop around x is $a^\theta \neq \mathrm{id}$), its lightlike foliations extend however at the singularity to two transverse one-dimensional topological foliations $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$, that we call the *lightlike bi-foliation* of the singular de-Sitter surface.

The seminal work of Troyanov [Tro86, Tro91] describes the main global rigidity properties of Riemannian surfaces with conical singularities. Troyanov proves therein that for any fixed set of singularities and angles on a closed orientable surface, any conformal class contains a unique metric of a given curvature having the prescribed singularities (with necessary conditions relating the angles, the constant curvature and the Euler characteristic of the surface, given by the Gauß-Bonnet formula). On the other hand, it is easily checked that two Lorentzian metrics μ_1 and μ_2 on a surface are conformal if, and only if, they have identical lightlike bi-foliations. It is then natural to investigate the relation of singular constant curvature Lorentzian surfaces to their lightlike bi-foliations and in [MM24], I obtain the following result in the case of a single singularity.

Theorem F ([MM24, Theorem A])**.** *Let S_1, S_2 be two closed singular $\mathbf{dS}^2$-surfaces having a unique singularity of the same angle. Assume that the lightlike bi-foliations of S_1 and S_2 are minimal*[5] *and topologically equivalent*[6]*. Then S_1 and S_2 are isometric.*

I show moreover in [MM24, Theorem B] that closed singular $\mathbf{dS}^2$-surfaces with minimal lightlike bifoliations indeed exist, and that any topological types of such bi-foliations can be realized.

[5] *I.e.* have all of their leaves dense.

[6] *I.e.* there exists a homeomorphism $f\colon S_1 \to S_2$ such that $f(\mathcal{F}^{S_1}_{\alpha/\beta}(x)) = \mathcal{F}^{S_2}_{\alpha/\beta}(f(x))$ for any $x \in S_1$.

2. Research projects

2.1. **Rigidity of (partially) hyperbolic dynamics with smooth invariant distributions.**

For a three-dimensional Anosov flow whose stable and unstable ditributions are $\mathcal{C}^\infty$, the plane field $E^s \oplus E^u$ is either integrable or contact. In the former case, the flow is the suspension of an Anosov diffeomorphism according to a work of Plante [Pla72], and in the latter one, it is orbit-equivalent to a geodesic flow according to Ghys [Ghy87]. In other words, the above "*integrable vs contact*" dichotomy for $E^s \oplus E^u$ essentially concludes the classification of three-dimensional Anosov flows with $\mathcal{C}^\infty$ invariant distributions. Fortunately or not, this automatic dichotomy stops both for higher-dimensional Anosov flows and three-dimensional partially hyperbolic diffeomorphisms. The extreme situation where $E^s \oplus E^u$ is contact is preferred because the pair (E^s, E^u) is then (in most cases) a rigid geometric structure invariant by the flow. The literature [Ghy87, BFL92, MM22b] focused so far uniquely on this contact case, or on similar cases where the existence of a rigid geometric structure invariant by the dynamics is assumed *a priori* (as in [Fan05] for instance).

2.1.1. *Beyond the contact case.* The first part of my research programme on the rigidity of hyperbolic dynamics is to go beyond this setting, by looking for rigidity results when the invariant distributions are $\mathcal{C}^\infty$, but without any *ad hoc* assumption of existence of an invariant rigid geometric structure.

In a first project in collaboration with Karin Melnick, we investigate 5-dimensional Anosov flows with $\mathcal{C}^\infty$ invariant distributions, for which $E^s \oplus E^u$ is neither contact nor integrable. In this case, there is no rigid geometric structure invariant by the Anosov flow "for free", and our main goal is to show that we can nevertheless extract enough informations from the pair (E^s, E^u) to obtain a rigidity. We are currently studying a geometric structure transversal to a natural foliation, the latter measuring in a sense the "non-contactness" of $E^s \oplus E^u$.

Another difficulty arising in the non-contact case is that the local geometry of $E^s \oplus E^u$ is in this case *not* homogeneous on the manifold. This problem is already encountered when studying three-dimensional partially hyperbolic diffeomorphisms having $\mathcal{C}^\infty$ invariant distributions and such that $E^s \oplus E^u$ is neither contact nor integrable, a question that I currently study. There is in this case a non-empty open subset $O \subset M$ where $E^s \oplus E^u$ is contact, and the main difficulty is to understand the behaviour of the geometry on ∂O. A first classification result was obtained for partially hyperbolic diffeomorphisms with $\mathcal{C}^\infty$ invariant distributions in [CPRH20, AM24], but under the following strong additional restriction on the partially hyperbolic diffeomorphism f: $\mathrm{D}f$ reads as a *constant* (diagonal) matrix in some global frame of vector fields generating (E^s, E^c, E^u).

2.1.2. *Higher-dimensional partially hyperbolic diffeomorphisms of contact type.* The second part of my research programme concerning the rigidity of hyperbolic dynamics, is conversely to investigate partially hyperbolic diffeomorphisms in dimension > 3, having smooth invariant distributions and for which $E^s \oplus E^u$ is a contact distribution. The pair (E^s, E^u) defines again in this case a rigid geometric structure called *Lagrangian contact structure*, equivalent to Cartan geometries modelled on some higher-dimensional flag space $\mathbf{X}_{2n+1}$ homogeneous under the action of $\mathrm{PGL}_{n+2}(\mathbb{R})$. Going from Anosov flows to partially hyperbolic diffeomorphisms, one loses however many rigid dynamical properties, which forces to preceed any global argument by a detailed study of the local geometry of the Lagrangian-contact structure (E^s, E^u).

2.2. **Singular Lorentzian surfaces and applications.**

Theorem F presented above can be seen as a *geometric rigidity* result for de-Sitter tori with a single singularity, in the sense that a topological equivalence between their pairs of lightlike foliations forces the existence of an isometry between them (hence of a smooth foliation equivalence). I present now three projects concerning this rigidity phenomenon, and its application to two different dynamical problems.

2.2.1. *Rigidity of de-Sitter tori with multiple singularities.* It is first of all natural to wonder what happens of Theorem F for multiple singularities. We strongly expect the same rigidity to be true in this setting, in which most of the methods developed in [MM24] moreover persist. However as explained with more details in [MM24, §1.4], a crucial argument of one-dimensional dynamics fails for multiple singularities.

We study this general case in an ongoing work with Selim Ghazouani, and develop a new approach to treat its structural difficulty.

2.2.2. *Smoothness of conjugacies for circle diffeomorphisms with breaks.* The first-return maps of lightlike foliations of singular $\mathbf{dS}^2$-surfaces are not only continuous but are actually *circle diffeomorphisms with breaks*, and while it may appear as a technical detail, this regularity actually gives a crucial dynamical information on the first-return map T. Indeed, the seminal work of Denjoy [Den32] implies then that T does not have an exceptional minimal set, and is thus topologically conjugated to a rigid rotation of the circle if it has an irrational *rotation number*. Since T is piecewise smooth, it is natural to wonder at this point if T is actually *smoothly* conjugated to a rotation. But as naive as it may seem, this question is an old and deep one which remains still open in its full generality. If T is $\mathcal{C}^\infty$ and its rotation number *Diophantine*, Herman showed in [Her79] that it is $\mathcal{C}^\infty$-conjugated to a rigid rotation, following the initial work of Arnol'd [Arn64] on this question. The problem remains unsolved for general circle diffeomorphisms with breaks, about which the optimal result up to date appears in [KKM17] and answers the question in the case of a single singularity.

In a current work in progress with Selim Ghazouani and Konstantin Khanin, we are investigating this connection further, with the aim of showing that the geometric rigidity of singular $\mathbf{dS}^2$-tori implies the smoothness of conjugacies for circle diffeomorphisms with breaks.

2.2.3. *Singular de-Sitter surfaces transversal to Anosov flows.* According to a work of Ghys [Ghy92], the transitive three-dimensional Anosov flows bearing a transversal constant curvature Lorentzian metric are either orbit equivalent to geodesic flows of closed hyperbolic surfaces, or to suspensions of hyperbolic automorphisms of the torus. While Anosov flows generally admit no transversal Lorentzian metrics, their Birkhoff sections bear in some cases natural *singular* Lorentzian metrics, which are likely to be helpful for their study. Among the questions that one may try to approach through such transversal singular Lorentzian metrics, is an important open question of Ghys asking wether any two three-dimensional transitive Anosov flows are connected by finitely many *Fried surgeries.*

In collaboration with Pierre Dehornoy, we plan to use the singular Lorentzian metrics existing on Birkhoff sections for the study of Anosov flows. More precisely, two important steps in this research programme would be to understand Fried surgeries through the geometrical framework of singular Lorentzian metrics, and to try and apply the previously described geometrical rigidity of the latter (see Theorem F) at the level of Anosov flows.

Mario Shannon very recently informed us that he is also currently studying singular Lorentzian metrics on Birkhoff sections of Anosov flows. The simultaneous interest of different members of the community for these methods is promising, and as it appears from our first discussion, our methods and points of view on this subject seem furthermore to be different and could be complementary. This suggests a possible fruitful collaboration with Mario Shannon on this subject in the future.

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