DISTRIBUTIONS INVARIANT BY PARTIALLY HYPERBOLIC DIFFEOMORPHISMS AND ANOSOV FLOWS

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1. For partially hyperbolic diffeomorphisms

Let f be a $\mathcal{C}^1$ diffeomorphism of a smooth closed manifold M, endowed with an auxiliary Riemannian metric (with respect to which all of the above definitions are independent). A splitting $E^1 \oplus E^2$ within two non-zero $\mathrm{D}f$-invariant distributions[1] is said to be a *dominated splitting*, if there exists $\lambda \in]0\,;1[$ and $C > 0$ such that for any $x \in M$, any unit vectors $v_1 \in E^1_x$, $v_2 \in E^2_x$, and any $n \in \mathbb{N}$, we have:

$$\|\mathrm{D}_x f^n(v_1)\| \leq C\lambda^n \|\mathrm{D}_x f^n(v_2)\|\,. \tag{1.1}$$

If $E^1 \oplus E^2$ is a dominated splitting, then E^1 and E^2 are automatically continuous (see for instance [CP15, Proposition 2.5]). More generally, a $\mathrm{D}f$-invariant splitting $\oplus_{i=1}^d E^i$ is dominated if $E^i \oplus E^{i+1}$ is dominated for any $1 \leq i \leq d-1$. A non-zero $\mathrm{D}f$-invariant distribution E of M is *uniformly contracted* by $\mathrm{D}f$ if there exists $\lambda \in]0\,;1[$ and $C > 0$ such that for any $x \in M$, any unit vectors $v \in E_x$ and any $n \in \mathbb{N}$, we have:

$$\|\mathrm{D}_x f^n(v)\| \leq C\lambda^n; \tag{1.2}$$

and E is *uniformly expanded* if it is uniformly contracted by $\mathrm{D}f^{-1}$. Finally, f is *partially hyperbolic*, if there exists a $\mathrm{D}f$-invariant splitting $\mathrm{T}M = E^s \oplus E^c \oplus E^u$ within three non-zero distributions, respectively called the *stable*, *central* and *unstable* distributions of f, such that:

(1) E^s is uniformly contracted by $\mathrm{D}f$,
(2) E^u is uniformly expanded by $\mathrm{D}f$,
(3) and $E^s \oplus E^c \oplus E^u$ is a dominated splitting.

Note that partially hyperbolic diffeomorphisms are sometime allowed to have zero stable or unstable distributions, while we will assume here each of the three distributions E^s, E^c and E^u to be non-zero (which is sometime called *strong* partial hyperbolicity). Note also that if f is a partially hyperbolic diffeomorphism, then f^{-1} is also partially hyperbolic, and that its stable and unstable distributions are exchanged with the ones of f, while their central distribution is the same. We denote $E^{sc} \coloneqq E^s \oplus E^c$, $E^{cu} \coloneqq E^c \oplus E^u$ and $E^{su} = E^s \oplus E^u$. For $P \subset M$ we denote $\omega(P) \coloneqq \{\lim f^{n_k}(x) \mid x \in P, n_k \to +\infty\}$.

Lemma 1.1. *Let f be a diffeomorphism of a manifold M, $E = E^1 \oplus E^2$ be a $\mathrm{D}f$-invariant dominated splitting on a compact f-invariant subset $P \subset M$, and H be a continuous $\mathrm{D}f$-invariant distribution on P contained in E. Then for any $y \in \omega(P)$: $H_y = (H_y \cap E^1_y) \oplus (H_y \cap E^2_y)$.*

Proof. Let $y = \lim f^{n_k}(x) \in P$ with $x \in P$ and $\lim n_k = +\infty$, so that $\lim \mathrm{D}_x f^{n_k}(H_x) = H_y$ by continuity. With $d = \dim H$ and $e = \dim H_x \cap E^1_x$, let $u_i \in E^1_x$, $i = 1, \dots, d$, and $v_i \in E^2_x$, $i = e+1, \dots, d$ such that $\|v_i\| = 1$, satisfying $H_x = V^1 \oplus V^2$ with $V^1 = \mathrm{Vect}\{u_i \mid i = 1, \dots, e\} = H_x \cap E^1_x$ and $V^2 = \mathrm{Vect}\{u_i + v_i \mid i = e+1, \dots, d\}$. The sequence $V^1_k \coloneqq \mathrm{D}_x f^{n_k}(V^1)$ converges to a subspace V^1_∞ of $H_y \cap E^1_y$ of dimension e, the sequence $V^2_k \coloneqq \mathrm{D}_x f^{n_k}(V^2)$ converges to a subspace V^2_∞ of $H_y \cap (E^1_y \oplus E^2_y)$ of dimension $d-e$, and $V^1_k \oplus V^2_k$ converges to H_y, hence $V^1_\infty + V^2_\infty \subset H_y$. By domination, we have

$$\frac{\|\mathrm{D}_x f^{n_k}(u_i)\|}{\|\mathrm{D}_x f^{n_k}(v_i)\|} \leq C(\max_{1 \leq i \leq d} \|u_i\|)\lambda^{n_k} \underset{k\infty}{\to} 0$$

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[1]We call *distribution* a field of subspaces of the tangent spaces of M of a given constant dimension.

for any $i = e+1, \dots, d$, hence with $u_i^k = \frac{\mathrm{D}_x f^{n_k}(u_i)}{\|\mathrm{D}_x f^{n_k}(v_i)\|}$ and $v_i^k = \frac{\mathrm{D}_{x_k} f^{n_k}(v_i)}{\|\mathrm{D}_{x_k} f^{n_k}(v_i)\|}$, $\|u_i^k\| \to 0$ and the limit V_∞^2 of $V_k^2 = \mathrm{Vect}(\{u_i^k + v_i^k \mid e+1 \leq i \leq d\}) \subset \mathrm{D}_x f^{n_k}(H_x)$ equals thus the one of $\mathrm{Vect}(\{v_k^i \mid e+1 \leq i \leq d\}) \subset E^2_{f^{n_k}(x)}$, which is contained in E_y^2 by continuity of E^2. In the end $V_\infty^2 \subset E_y^2$ and the sum $V_\infty^1 \oplus V_\infty^2 \subset H_y$ is direct, implying that $H_y = V_\infty^1 \oplus V_\infty^2$ by equality of dimensions, therefore $H_y \subset (H_y \cap E_y^1) \oplus (H_y \cap E_y^2)$ which concludes the proof. □

Corollary 1.2. *Let f be a topologically transitive diffeomorphism of a closed manifold M, and $E = E^1 \oplus E^2 \oplus E^3$ a $\mathrm{D}f$-invariant dominated splitting on M. Then for any ordering (i, j, k) of $(1, 2, 3)$, $E^i \oplus E^j$ is the only continuous $\mathrm{D}f$-invariant distribution H on M contained in E and such that: $\dim H = \dim E - \dim E^k$ and $E^k \cap H = \{0\}$.*

Proof. Let H be such a distribution, and let us denote $H^{lm} := H \cap (E^l \oplus E^m)$. Since every point of M is a ω-limit point of f and $H \cap E^k = \{0\}$, Lemma 1.1 shows that $H^{ik} = H^{ik} \cap E^i$ $(= H \cap E^i)$, $H^{jk} = H^{jk} \cap E^j (= H \cap E^j)$. A computation of dimensions shows then that $\dim(H \cap E^i) + \dim(H \cap E^j) = \dim(E^i \oplus E^j)$, and thus that $H = E^i \oplus E^j$. □

Corollary 1.3. *Let f be a topologically transitive partially hyperbolic diffeomorphism of a three-dimensional closed manifold M, of invariant distributions E^s, E^c, E^u. Then the only continuous $\mathrm{D}f$-invariant plane fields of M are E^{sc}, E^{cu} and E^{su}.*

Proof. Let H be a continuous $\mathrm{D}f$-invariant plane field, and assume by contradiction that it is none of these three ones. Let us first assume that E^c is not included in H by contradiction. By continuity of H, there exists thus a non-empty open set U such that for any $x \in U$, $H_x \cap E_x^c = \{0\}$. If H was by contradiction equal to E^{su} on U, then by topological transitivity, a dense orbit of f would meet U and H would thus be equal to E^{su} on M, contradicting our hypothesis. Possibly restricting U, we can thus assume that for any $x \in U$, we have furthermore $H_x \neq E_x^{su}$. Hence H_x cannot contain both E_x^s and E_x^u, and thus, possibly restricting U again and replacing f by f^{-1}, it holds now that for any $x \in U$, the line $D_x := H_x \cap E_x^{sc}$ is distinct from E_x^s and from E_x^c. Now f being transitive, the set of recurrent points of f is dense in M, and there exists thus a recurrent point $x \in U$. But Lemma 1.1 implies then $D_x = \{0\}$, which is a contradiction.

Hence $E^c \subset H$. Since H is distinct from both E^{sc} and E^{cu}, and arguing as before, there exists by continuity of H a non-empty open set U such that for every $x \in U$, the line $D_x = H_x \cap E_x^{su}$ is neither E_x^s nor E_x^u. Then as before, U contains a recurrent point x of f and Lemma 1.1 implies $D_x = \{0\}$. This last contradiction concludes the proof. □

2. For Anosov flows

A non-singular flow (φ^t) of a smooth closed manifold M whose derivative vector field X is of class $\mathcal{C}^1$ will be called *Anosov* if there exists a $\mathrm{D}\varphi^t$-invariant splitting $\mathrm{T}M = E^s \oplus \mathbb{R}X \oplus E^u$, where and E^s and E^u are two non-zero distributions for which there exists $\lambda \in]0\,;1[$ and $C > 0$ such that for any $x \in M$ and $t \in \mathbb{R}^+$:

$$\left\|\mathrm{D}_x\varphi^t|_{E_x^s}\right\| \leq C\lambda^t, \left\|\mathrm{D}_x\varphi^{-t}|_{E_x^u}\right\| \leq C\lambda^t. \tag{2.1}$$

Note that any non-zero time of an Anosov flow is a partially hyperbolic diffeomorphism, with the same stable and unstable distributions than φ^t, and the central distribution being equal to $\mathbb{R}X$. The stable and unstable distributions E^s and E^u of an Anosov flow are automatically continuous. Moreover if $\dim E^s = 1$ then: $E^{cu} := E^c \oplus E^u$ is $\mathcal{C}^1$, and if (φ^t) moreover preserves a continuous volume form then both E^{uc} and $E^{sc} := E^s \oplus E^c$ are $\mathcal{C}^1$ with Hölder derivatives (see [Has94, Corollary 1.8 and 1.9]). In particular if $\dim M = 3$, then E^{sc} and E^{uc} are always $\mathcal{C}^1$.

The following consequence of Livšic results in [LS72, Liv74] (see also [LMM86]) on the cohomological equation for Anosov flows is well known (see for instance [HK90, Theorem 2.3]), but we explain it again here for sake of completeness.

Lemma 2.1. *Let ω be a continuous field of densities on M and $\mu(U) = \int_U \omega$ its associated Borel measure. If μ is preserved by a $\mathcal{C}^2$ Anosov flow (φ^t), then either $\omega = 0$, or:*

- *ω does not vanish, is (φ^t)-invariant, and is of class $\mathcal{C}^1$ (and even $\mathcal{C}^\infty$ if (φ^t) is $\mathcal{C}^\infty$);*

– *and moreover* (φ^t) *is topologically transitive.*

Proof. If ω non-zero, then $|\omega|$ defines a (φ^t)-invariant finite non-zero Borel measure μ, having a continuous density with respect to the measure m defined by a Riemannian metric on M. According to Poincaré recurrence Theorem, the set of recurrent points of (φ^t) contains the support of μ, which has non-empty interior since μ has a continuous density. But an Anosov flow whose non-wandering set has non-empty interior is topologically transitive, as shown in [Pla72, Lemma 4.2]. Now according to Livšic and de la Llave-Marco-Moriyon results, a $\mathcal{C}^2$ topologically transitive Anosov flow has at most one invariant Borel probability measure which is absolutely continuous with respect to the Riemannian volume m. And moreover if such a measure exists, then it actually has a non-vanishing density with respect to m, which is $\mathcal{C}^1$ and even $\mathcal{C}^\infty$ if (φ^t) is $\mathcal{C}^\infty$ (see for instance [LMM86, Corollary 2.1], and also [Bow08, Corollary 4.13 and Theorem 4.14] for the case of Anosov diffeomorphisms). There exists thus on M a $\mathcal{C}^1$ field d of densities (which is $\mathcal{C}^\infty$ if (φ^t) is), whose integration on any open set equals the one of $|\omega|$. With $f \geq 0$ the continuous function such that $|\omega| = fd$, if we had $f(x) \neq 1$ at some point by contradiction, say for instance $f(x) > 1$, then $f > 1$ on a non-empty open set U (by continuity) which would contradict $\int_U fd = \int_U |\omega| = \int_U d$. Hence $\omega = d$ does not vanish, and has the regularity properties stated above. □

For H a codimension-one distribution of M invariant by the Anosov flow (φ^t) and transverse to it, we define the *canonical one-form* θ *of* H by $\theta(X) \equiv 1$ and $\theta|_H \equiv 0$ (and the *canonical one-form of* (φ^t) as the canonical one-form of $E^{su} = E^s \oplus E^u$). Note that the canonical one-form of a (φ^t)-invariant distribution is by construction itself (φ^t)-invariant, and that $d\theta(X, \cdot) \equiv 0$ thanks to Cartan's formula (since X preserves H). The following is due to Plante in [Pla72, Theorem 3.1], and to Franks-Newhouse in [Fra70, New70].

Theorem 2.2. *Let* $k \in \mathbb{N}^* \cup \{\infty\}$, (φ^t) *be a* $\mathcal{C}^k$ *Anosov flow of a closed manifold* M, *and* H *be a* φ^t*-invariant codimension-one continuous distribution on* M *transverse to* (φ^t).

(1) *If* (φ^t) *is topologically transitive, then* $H = E^s \oplus E^u$.
(2) *Let assume that* H *is* $\mathcal{C}^k$. *Then with* θ *the canonical form of* H: H *is integrable if, and only if* $d\theta|_H \equiv 0$ *if, and only if* $d\theta \equiv 0$ *if, and only if* $\theta \wedge d\theta \equiv 0$. *Moreover if* H *is integrable, then:*
   (a) (φ^t) *is* $\mathcal{C}^k$*-orbit equivalent to the suspension of an Anosov diffeomorphism.*
   (b) *If* (φ^t) *is codimension-one (for instance if* $\dim M = 3$*), then* $H = E^s \oplus E^u$ *and* (φ^t) *is* $\mathcal{C}^0$*-conjugated to the suspension of a hyperbolic toral automorphism, hence is topologically transitive.*

Proof. 1. This a consequence of Corollary 1.2.
2. This is a direct consequence of Cartan's formula, which yields $d\theta(X,Y) = -\theta([X,Y])$ for any $\mathcal{C}^1$ sections X, Y of H, showing that $[X,Y]$ has values in $E^s \oplus E^u$ for any such pair (and thus that this distribution is integrable) if, and only if $d\theta|_{E^s \oplus E^u} = 0$, if, and only if $d\theta = 0$ since $d\theta(X, \cdot) \equiv 0$.
2.a). Plante proves in [Pla72, Proposition 2.3 and Corollary 2.11] that if a $\mathcal{C}^k$ flow (φ^t) of a compact $\mathcal{C}^\infty$ manifold M preserves a $\mathcal{C}^k$ codimension-one foliation transverse to (φ^t), then (φ^t) admits a global $\mathcal{C}^\infty$ transverse section S, and is thus $\mathcal{C}^k$-orbitally equivalent to the suspension of the first return map R of (φ^t) on S, which is an Anosov diffeomorphism if (φ^t) is Anosov.
2.b). If (φ^t) is codimension-one, then so is the (Anosov) first-return map R, which is thus $\mathcal{C}^0$-conjugated to a hyperbolic toral automorphism according to Franks-Newhouse theorem [Fra70, New70]. In particular, (φ^t) is topologically transitive, hence $H = E^s \oplus E^u$ according to 1. In this case, the suspension manifold M of R satisfies $\dim \mathrm{H}_1(M, \mathbb{R}) = 1$, which implies that the integral leaves of $E^s \oplus E^u = H$ are compact according to [Pla72, Corollary 2.5]. Consequently, (φ^t) is in fact conjugated to the first return map on these leaves (the time of the first return on a given leave being this time constant since it is part of an invariant foliation), showing that (φ^t) is $\mathcal{C}^0$-conjugated to the suspension of a hyperbolic toral automorphism. □

The following consequence of Livšic results is proved in [HK90, Theorem 2.3] (see also related results [Pla72, Ghy87]).

Corollary 2.3. *Let (φ^t) be a $\mathcal{C}^2$ Anosov flow of a closed manifold of dimension $2n+1$, and H be a φ^t-invariant $\mathcal{C}^1$ codimension-one distribution transverse to (φ^t). Then with θ the canonical one-form of H, we have the following dichotomy:*

(1) *either $\theta \wedge (d\theta)^n = 0$;*
(2) *or θ is a contact form,* i.e. *$\theta \wedge (d\theta)^n = 0$ does not vanish.*

If moreover $\dim M = 3$, then the following holds.

(1) *In both cases of the above dichotomy: $H = E^s \oplus E^u$, and (φ^t) is topologically transitive.*
(2) *In the first case, $H = E^s \oplus E^u$ is integrable and (φ^t) is $\mathcal{C}^0$-conjugated to the suspension of an hyperbolic automorphism of the torus.*
(3) *If moreover $\dim M = 3$ and (φ^t) is $\mathcal{C}^\infty$, then $H = E^s \oplus E^u$ is also $\mathcal{C}^\infty$.*

Proof. The dichotomy is simply a reformulation of Lemma 2.1 for the $2n+1$-form $\theta \wedge (d\theta)^n$. We now assume that $\dim M = 3$. If $\theta \wedge d\theta$ does not vanish, then (φ^t) is volume-preserving and thus topologically transitive, implying that $H = E^s \oplus E^u$ according to the first claim of Theorem 2.2. If $\theta \wedge d\theta \equiv 0$, the second claim is just a reformulation of the last one of Theorem 2.2, and in particular: (φ^t) is topologically transitive and $H = E^s \oplus E^u$. For the last claim, see [HK90, Theorem 2.3]. □

Bibliography

[Bow08] Rufus Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes Math.* Springer, Berlin, 2nd revised ed. edition, 2008. ISSN: 0075-8434.

[CP15] Sylvain Crovisier and Rafael Potrie. Introduction to partially hyperbolic dynamics, July 2015. Available on the web-pages of the authors.

[Fra70] John Franks. Anosov diffeomorphisms. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 61–93. Amer. Math. Soc., Providence, R.I., 1970.

[Ghy87] Étienne Ghys. Flots d'Anosov dont les feuilletages stables sont différentiables. *Annales scientifiques de l'École normale supérieure*, 20(2):251–270, 1987.

[Has94] Boris Hasselblatt. Regularity of the Anosov splitting and of horospheric foliations. *Ergodic Theory and Dynamical Systems*, 14(4):645–666, 1994.

[HK90] S. Hurder and Anatoly Katok. Differentiability, rigidity and Godbillon-Vey classes for Anosov flows. *Publications Mathématiques de l'IHÉS*, 72:5–61, 1990.

[Liv74] A. N. Livshits. Cohomology of dynamical systems. *Mathematics of the USSR. Izvestiya*, 6:1278–1301, 1974.

[LMM86] R. De la Llave, J. M. Marco, and R. Moriyon. Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation. *Annals of Mathematics. Second Series*, 123:537–611, 1986.

[LS72] A. N. Livshits and Ya. G. Sinai. On invariant measures compatible with the smooth structure for transitive U-systems. *Soviet Mathematics. Doklady*, 13:1656–1659, 1972.

[New70] S. E. Newhouse. On Codimension One Anosov Diffeomorphisms. *American Journal of Mathematics*, 92(3):761–770, 1970.

[Pla72] Joseph F. Plante. Anosov flows. *American Journal of Mathematics*, 94:729–754, 1972.

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