ON SIMULTANEOUS CONJUGACIES OF PAIRS OF TRANSVERSE FOLIATIONS OF THE TORUS

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Abstract. We prove in this note that two pairs of transverse minimal topological foliations of the torus are individually conjugated if, and only if they are simultaneously conjugated.

1. Introduction

It is known since Poincaré [Poi85] that the *rotation number* (introduced in Proposition-Definition 2.2 below) is the only invariant of minimal, orientation-preserving homeomorphisms of the circle $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$ under topological conjugacies. For a one-dimensional oriented topological foliation $\mathcal{F}$ of the torus $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the global conjugacy invariant which naturally replaces the rotation number is the *projective asymptotic cycle* due to Schwartzman [Sch57] (see (2.1)). The latter is a half-line $A(\mathcal{F})$ in the first homology group $\mathrm{H}_1(\mathbf{T}^2, \mathbb{R})$ of the torus, which gives a rigorous idea to the "asymptotic direction of $\mathcal{F}$ in homology", and reflects the dynamics of the foliation in multiple ways. For instance, the asymptotic cycle of a minimal foliation of $\mathbf{T}^2$ is *irrational* in the sense that the half-line $A(\mathcal{F})$ does not intersect $\pi_1(\mathbf{T}^2) \equiv \mathbb{Z}^2 \subset \mathbb{R}^2 \equiv \mathrm{H}_1(\mathbf{T}^2, \mathbb{R})$. Moreover in the same way than the rotation number for minimal circle homeomorphisms, the projective asymptotic cycle is known to be a complete conjugacy invariant for minimal foliations. Any minimal oriented topological foliation of $\mathbf{T}^2$ is indeed conjugated to the linear foliation defined by its asymptotic cycle $A(\mathcal{F})$, by a homeomorphism isotopic to the identity (see Proposition 2.5).

We study in this note the *minimal topological bi-foliations* of the torus $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, defined as the pairs $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ of minimal oriented topological foliations of $\mathbf{T}^2$ which are *transverse* (in a sense made precise in Definition 3.1). Such objects appear naturally in different dynamical contexts. For instance, any closed surface S which is a transverse section of a three-dimensional Anosov flow is endowed with a bi-foliation, whose leaves are the intersections of S with the weak stable and unstable leaves of the Anosov flow. While suspensions are the only three-dimensional Anosov flows admitting such a transverse section, Birkhoff sections [Bir17] constitute a natural generalization of them which is an important tool for the classification of Anosov flows, and any genus one Birkhoff section of a transitive three-dimensional Anosov flow inherits in the same way a minimal topological bi-foliation. In a recent work of the author [MM24], bi-foliations of the torus are studied from a geometrical point of view in the setting of *singular de-Sitter Lorentzian metrics*. Isotopy classes of singular de-Sitter metrics of $\mathbf{T}^2$ are indeed shown to be equivalent to the *simultaneous* conjugacy classes of their minimal lightlike bi-foliations (see [MM24, Theorem A]). Two bi-foliations $(\mathcal{F}^1_\alpha, \mathcal{F}^1_\beta)$ and $(\mathcal{F}^2_\alpha, \mathcal{F}^2_\beta)$ are said to be *simultaneously conjugated* by a homeomorphism ϕ of $\mathbf{T}^2$, if $\mathcal{F}^2_\alpha = \phi_* \mathcal{F}^1_\alpha$ and $\mathcal{F}^2_\beta = \phi_* \mathcal{F}^1_\beta$.

If two bi-foliations $(\mathcal{F}^1_\alpha, \mathcal{F}^1_\beta)$ and $(\mathcal{F}^2_\alpha, \mathcal{F}^2_\beta)$ of the torus have the same pairs of projective asymptotic cycles, then we already know that $\mathcal{F}^1_\alpha$ and $\mathcal{F}^2_\alpha$ in the one hand, and that $\mathcal{F}^1_\beta$ and $\mathcal{F}^2_\beta$ in the other hand, are individually conjugated. It is then natural to ask if $(\mathcal{F}^1_\alpha, \mathcal{F}^1_\beta)$ and $(\mathcal{F}^2_\alpha, \mathcal{F}^2_\beta)$ are actually *simultaneously* conjugated. The main goal of the present note is to answer this question positively with the following result.

Theorem A. *Let $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ be a minimal topological bi-foliation of the torus and $p \in \mathbf{T}^2$. Then $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ is simultaneously conjugated to the linear bi-foliation $(\mathcal{F}_{A(\mathcal{F}_\alpha)}, \mathcal{F}_{A(\mathcal{F}_\beta)})$ defined by its asymptotic cycles, by a homeomorphism isotopic to the identity relatively to p.*

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I was informed, after posting this preprint on arXiv, that Theorem A was already proved in [AGK03].

Theorem A describes thus a "*dynamical* Teichmüller space": the space of bi-foliations of the torus modulo simultaneous conjugacies isotopic to the identity. More precisely, it identifies the subset of minimal bi-foliations in this Teichmüller space (which are the most dynamically relevant ones) with the subset $\{l_\alpha \neq l_\beta\}$ of $(\mathbb{R}\mathbf{P}^1)^2$, which is particularly useful to study the dynamics of the *mapping class group* $\mathrm{Mod}^+(\mathbf{T}^2)$ of the torus on its space of bi-foliations. We recall that $\mathrm{Mod}^+(\mathbf{T}^2)$ is the quotient of the group of homeomorphisms of $\mathbf{T}^2$ by the subgroup of homeomorphisms isotopic to the identity, which acts naturally on the space of bi-foliations of $\mathbf{T}^2$ modulo isotopies. Theorem A intertwines thus (in restriction to minimal bi-foliations and irrational lines) the *a priori* complicated dynamics of $\mathrm{Mod}^+(\mathbf{T}^2)$ on the space of bi-foliations with the *explicit* diagonal action of $\mathrm{PSL}_2(\mathbb{Z})$ on $(\mathbb{R}\mathbf{P}^1)^2$. The latter dynamics being entirely known, one can hope to use this description to deduce new informations.

This idea applies for instance to the *geometrical* Teichmüller space $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ of singular de-Sitter metrics of $\mathbf{T}^2$ studied in [MM24]. It allows to reformulate its main result to identify a part of $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ with the subset $\{l_\alpha \neq l_\beta \text{ irrational}\}$ of $(\mathbb{R}\mathbf{P}^1)^2$, which intertwines (on relevant subsets) the action of $\mathrm{Mod}^+(\mathbf{T}^2)$ on $\mathsf{Def}_\theta(\mathbf{T}^2, 0)$ with the diagonal action of $\mathrm{PSL}_2(\mathbb{Z})$ on $(\mathbb{R}\mathbf{P}^1)^2$. In a joint work in progress, Florestan Martin-Baillon and the author use this description to study the dynamics of $\mathrm{Mod}^+(\mathbf{T}^2)$ on relative character varieties of the one-holed torus with values in $\mathrm{PSL}_2(\mathbb{R})$.

As a direct consequence of Theorem A, we obtain the following result by applying an argument due to Ghys-Sergiescu [GS80].

Corollary B. *Let f be a homeomorphism of $\mathbf{T}^2$ which is isotopic to the identity relatively to a point, and preserves a minimal topological bi-foliation. Then f is the identity.*

Corollary B strengtens the main result of [MM24], by showing that any *topological* conjugacy between the minimal lightlike bi-foliations of two singular de-Sitter metrics (having a unique singularity of the same angle) is actually an *isometry.* This implies in particular a geometric rigidity result for this class of dynamical systems: any topological conjugacy between such minimal lightlike bi-foliations is actually piecewise smooth.

2. Preliminaries

2.1. **Foliations of the torus and suspensions.** We recall the basic definitions of foliations, refering to [CLN85, Chapter II] and [HH86, Chapter I] for more details.

Definition 2.1. Let S be a topological surface. A *foliated (topological) atlas* of S is a continuous atlas $\mathcal{A}$ of S satisfying the following conditions.

(1) For any chart $(U, \varphi) \in \mathcal{A}$, $\varphi(U) = I \times J \subset \mathbb{R}^2$ with I and J open intervals in $\mathbb{R}$.
(2) For any $(U_i, \varphi_i), (U_j, \varphi_j) \in \mathcal{A}$ such that $U_i \cap U_j \neq \varnothing$, the transition map $\varphi_{i,j} \coloneqq \varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is of the form
$$\varphi_{i,j}(x, y) = (\alpha_{i,j}(x, y), \gamma_{i,j}(y)).$$

A *(topological) foliation* $\mathcal{F}$ of S is a maximal foliated topological atlas of S, the charts of which are called the *foliation charts* of $\mathcal{F}$. In a domain U of foliated chart of $\mathcal{F}$ and for any $p = \varphi^{-1}(x, y) \in U$, $P^U_{\mathcal{F}}(p) = \varphi^{-1}(I \times \{y\})$ is called the *plaque* of x for $\mathcal{F}$. The set of all plaques is a basis of a topology on S whose connected components are called the *leaves* of $\mathcal{F}$, and the leaf containing x is denoted by $\mathcal{F}(x)$. $\mathcal{F}$ is said *oriented* by the choice of a sub-atlas of $\mathcal{A}$ such that $\alpha_{i,j}(\cdot, y)$ is an orientation-preserving map for any (i, j) and y. In this case, each leaf of $\mathcal{F}$ inherits the orientation given in any foliated chart by the identification of the plaque $\varphi^{-1}(I \times \{y\})$ with $I \subset \mathbb{R}$. Henceforth, all the foliations will implicitly be topological and oriented.

A *topological conjugacy* between two topological foliations $(S_1, \mathcal{F}_1)$ and $(S_2, \mathcal{F}_2)$ is a homeomorphism $\phi \colon S_1 \to S_2$ such that $\mathcal{F}_2 = \phi_* \mathcal{F}_1$, namely $\phi(\mathcal{F}_1(x)) = \mathcal{F}_2(x)$ for any $x \in S_1$.

Let $\mathcal{F}$ be a foliation of $\mathbf{T}^2$ which admits a closed *section*, *i.e.* a simple closed curve $\gamma \subset \mathbf{T}^2$ transverse to $\mathcal{F}$ (in the sense of Definition 3.1 below) and intersecting all of its leaves. Then the *first-return map*

$$P^\gamma_{\mathcal{F}}\colon \gamma \to \gamma$$

of $\mathcal{F}$ on γ is well-defined, $P^\gamma_{\mathcal{F}}(x)$ being the first intersection point of the oriented leaf $\mathcal{F}(x)$ with γ after x. We can now describe $\mathcal{F}$ in terms of $P^\gamma_{\mathcal{F}}$ in the following way. Given an orientation-preserving homeomorphism T of the circle, the *suspension* $\mathcal{F}_T$ of T is the oriented foliation of the topological torus

$$M_T := [0\,;1] \times \mathbf{S}^1/\{(1,x) \sim (0,T(x))\}$$

defined by the projection of the horizontals $[0\,;1] \times \{x\}$. Then $(S,\mathcal{F})$ is clearly topologically conjugated to the suspension of the first-return map $P^\gamma_{\mathcal{F}}$, and the dynamical properties of these two dynamical systems are the same. For instance $\mathcal{F}$ is *minimal* (namely has all of its leaves dense) if and only if $P^\gamma_{\mathcal{F}}$ is *minimal* (namely has all of its orbits dense). Moreover if two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on S are conjugated by a homeomorphism isotopic to the identity, then they admit freely homotopic sections on which their first-return maps will be topologically conjugated. We recall that two circle homeomorphisms f and g are *topologically conjugated* if there exists a homeomorphism φ of $\mathbf{S}^1$ such that $g = \varphi \circ f \circ \varphi^{-1}$.

In conclusion, any topological conjugacy invariant of circle homeomorphisms will yield an isotopy invariant for foliations. We now define the only such invariant.

2.2. **Circle homeomorphisms and rotation numbers.** We denote by $x \in \mathbb{R} \mapsto [x] \in \mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$ the canonical projection onto the circle, and by $R_\theta\colon x \in \mathbf{S}^1 \mapsto x+\theta \in \mathbf{S}^1$ the *rotation by* $\theta \in \mathbf{S}^1$. We refer for instance to [dFG22, §1.1 & 2.1] and [dMvS93, I.1] for a proof of the following classical results due to Henri Poincaré [Poi85].

Proposition-Definition 2.2 (Poincaré)**.** Let $f \in \mathrm{Homeo}^+(\mathbf{S}^1)$ be an orientation-preserving homeomorphism of the circle.

(1) For any lift F of f, the limit $\tau(F) := \lim\limits_{n\to\pm\infty} \frac{F^n(x)-x}{n}$ exists for any $x \in \mathbb{R}$ and is independent of x. If $G = F + d$ is another lift of f $(d \in \mathbb{Z})$, then $\tau(G) = \tau(F) + d$, and

$$\rho(f) := [\tau(F)] \in \mathbf{S}^1$$

is thus a well-defined point called the *rotation number* of f.

(2) $\rho(f)$ is invariant under topological conjugacies. The rotation number of any orientation-preserving homeomorphism g of an oriented topological circle is thus well-defined by the relation $\rho(g) := \rho(g_0) \in \mathbf{S}^1$, with $g_0 \in \mathrm{Homeo}^+(\mathbf{S}^1)$ conjugated to g by an orientation-preserving map.

(3) If $f \in \mathrm{Homeo}^+(\mathbf{S}^1)$ is minimal, then it is topologically conjugated to the rotation $R_{\rho(f)}$.

Recall that any half-line $l \in \mathbf{P}^+(\mathbb{R}^2)$ induces an oriented *linear foliation* $\mathcal{F}_l$ on $\mathbf{T}^2$ defined by $\mathcal{F}_l[x] = [x+l]$ for any $[x] \in \mathbf{T}^2$ (with $x \in \mathbb{R}^2 \mapsto [x] \in \mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ the canonical projection). According to Proposition 2.2.(3), if a foliation $\mathcal{F}$ of $\mathbf{T}^2$ admits a section γ on which the first return map has rotation number $[\theta] \in \mathbf{S}^1$, then $\mathcal{F}$ is topologically conjugated to the linear foliation $\mathcal{F}_{\mathbb{R}(1,\theta)}$. However the description of $\mathcal{F}$ up to *isotopy* demands more than the rotation number of its first-return maps.

Let indeed γ' be a simple closed curve freely homotopic to γ and disjoint from it, and D be a positive Dehn twist around γ' whose support is disjoint from γ. Then the first-return map of $D_*\mathcal{F}$ on γ is equal to the one of $\mathcal{F}$, although $D_*\mathcal{F}$ is *not* isotopic to $\mathcal{F}$. This motivates the introduction of a finer invariant, a "global version of the rotation number" which will detect the action of the mapping class group of $\mathbf{T}^2$.

2.3. **Asymptotic cycles.** Originally introduced by Schwartzman in [Sch57] for topological flows of closed manifolds M, the notion of *asymptotic cycle* fulfills this role. It associates to any suitable orbit O of the flow the "best approximation of O by a closed loop in homology". This notion has a natural projective counterpart for an oriented topological foliation $\mathcal{F}$ of $\mathbf{T}^2$ that we now quickly describe, referring to [Sch57, Yan85] for more details. We consider an auxiliary smooth Riemannian metric g on $\mathbf{T}^2$ and its induced distance $d_{\mathcal{F}}$ on the leaves of $\mathcal{F}$. For $x \in \mathbf{T}^2$ and $T \in \mathbb{R}$ we denote by $\gamma_{T,x}$ the closed curve of $\mathbf{T}^2$ obtained by first following $\mathcal{F}(x)$ from x to the unique point $y \in \mathcal{F}(x)$ such that $d_{\mathcal{F}}(x,y) = T$, and then closing the curve by following the minimal geodesic of g from y to x. Following [Sch57, Yan85], the *projective asymptotic cycle* of $\mathcal{F}$ at x is then defined as the half-line

$$A_{\mathcal{F}}(x) \coloneqq \mathbb{R}^+ \left(\lim_{T \to +\infty} \frac{1}{T} [\gamma_{T,p}] \right) \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R})) \tag{2.1}$$

in the first homology group of $\mathbf{T}^2$, if this limit exists and does not vanish. This cycle is by definition constant on leaves, does not depend on the auxiliary Riemannian metric, and is moreover natural with respect to any homeomorphism f:

$$A_{f_* \mathcal{F}}(f(x)) = f_*(A_{\mathcal{F}}(x)).$$

In particular, any homeomorphism isotopic to the identity acts trivially on projective asymptotic cycles (see [Sch57, Theorem p.275]). While the asymptotic cycles have *a priori* no reason to exist at any point, they are easily described for foliations of the torus by the following result which is a reformulation of [Yan85, Theorem 6.1 and Theorem 6.2]. We identify henceforth $\mathrm{H}_1(\mathbf{T}^2, \mathbb{R})$ with $\mathbb{R}^2$ through the isomorphism induced by the canonical projection $\mathbb{R}^2 \to \mathbf{T}^2$.

Proposition 2.3 ([Yan85])**.** *Let $\mathcal{F}$ be an oriented topological foliation of $\mathbf{T}^2$ which is the suspension of a circle homeomorphism.*

(1) $A_{\mathcal{F}}(x)$ exists at any $x \in \mathbf{T}^2$. It is moreover constant on $\mathbf{T}^2$ and will be denoted by $A(\mathcal{F})$.
(2) For any $l \in \mathbf{P}^+(\mathbb{R}^2)$, $A(\mathcal{F}_l) = l$.

Asymptotic cycles play their expected role of "global version of the rotation number", precisely formulated by the following result which is folklore in the literature.

Lemma 2.4. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two oriented topological foliations of $\mathbf{T}^2$. Then $A(\mathcal{F}_1) = A(\mathcal{F}_2)$ if, and only if for any respective sections γ_1 and γ_2 of $\mathcal{F}_1$ and $\mathcal{F}_2$ which are freely homotopic, we have $\rho(P^{\gamma_1}_{\mathcal{F}_1}) = \rho(P^{\gamma_2}_{\mathcal{F}_2})$.*

Using the previous Lemma, one easily obtains the following classification result usually attributed to Poincaré.

Proposition 2.5. *Let $\mathcal{F}$ be a minimal oriented topological foliation of $\mathbf{T}^2$. Then $\mathcal{F}$ is conjugated to the linear foliation defined by its asymptotic cycle $A(\mathcal{F}) \in \mathbf{P}^+(\mathbb{R}^2)$, by a homeomorphism isotopic to the identity.*

3. Proof of Theorem A

Definition 3.1. A pair $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ of topological foliations of $\mathbf{T}^2$ is said *transverse* if for any $p \in \mathbf{T}^2$ there exists a connected open neighborhood U of p, two open intervals $I, J \subset \mathbb{R}$ containing 0 and a homeomorphism $\varphi \colon U \to I \times J$, which sends every plaque of $\mathcal{F}_\alpha$ (respectively $\mathcal{F}_\beta$) in U to a horizontal interval $I \times \{y\}$ (resp. vertical interval $\{x\} \times J$). Such a homeomorphism will be called a *simultaneous foliated chart* of $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$.

We now prove Theorem A. According to Proposition 2.5, we can assume without lost of generality that $\mathcal{F}_\beta$ is a linear foliation. We denote by $\mathcal{F}^0_\alpha$ the linear foliation defined by $A(\mathcal{F}_\alpha) \in \mathbf{P}^+(\mathrm{H}_1(\mathbf{T}^2, \mathbb{R}))$, fix $p \in \mathbf{T}^2$ and denote $F_\alpha \coloneqq \mathcal{F}_\alpha(p)$ and $F^0_\alpha \coloneqq \mathcal{F}^0_\alpha(p)$.

(a) Flowing along $\mathcal{F}_\beta$ on $\mathcal{F}_\alpha(p)$. Let U be the domain of a simultaneous foliated chart of $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ around p. Then for any point x in the plaque $P^U_{\mathcal{F}_\alpha}(p)$ of p, there exists a unique point $\phi(x) \coloneqq P^U_{\mathcal{F}^0_\alpha}(p) \cap P^U_{\mathcal{F}_\beta}(x)$ belonging both to the plaque of p for $\mathcal{F}^0_\alpha$ (defined as the connected component of $\mathcal{F}^0_\alpha(p) \cap U$ containing p) and to the plaque of x for $\mathcal{F}_\beta$ in U. Let us now assume ϕ to be defined on an interval $I = [a\,; b]$ of F_α with values in F^0_α. Let $\gamma \colon [0\,; 1] \to \mathcal{F}_\beta(b)$ denote a continuous parametrization of the (unique) interval of $\mathcal{F}_\beta(b) \setminus \{b, \phi(b)\}$ whose closure contains both $b = \gamma(0)$ and $\phi(b) = \gamma(1)$. Then with U and V the domains of respective foliated charts of $\mathcal{F}_\alpha$ at b and of $\mathcal{F}^0_\alpha$ at $\phi(b)$, we extend ϕ on $P^U_{\mathcal{F}_\alpha}(b)$ to be equal to the holonomy map of $\mathcal{F}_\beta$ along γ from $P^U_{\mathcal{F}_\alpha}(b)$ to $P^V_{\mathcal{F}^0_\alpha}(\phi(b))$ (which is well-defined, possibly diminishing U). The subset of F_α on which ϕ is uniquely defined in this way is thus non-empty, open and closed, hence equal to F_α.

(b) Extending ϕ to $\mathbf{T}^2$. The only possible candidate for a continuous extension (denoted in the same way) of ϕ is of course

$$\phi(\lim x_n) := \lim \phi(x_n) \tag{3.1}$$

for any converging sequence $x_n \in F_\alpha$. Our main task is thus to show first that for any converging sequence $x_n \in F_\alpha$, the sequence $\phi(x_n)$ converges, and to show moreover that for two converging sequences x_n^1 and x_n^2 in F_α having the same limit x, we have $\lim \phi(x_n^1) = \lim \phi(x_n^2)$. Let y_n^i ($i = 1$ or 2) denote the first intersection point of $\mathcal{F}_\alpha(x_n^i)$ with $\mathcal{F}_\beta(x)$, and let us temporarily admit that $\phi(y_n^{1/2})$ are convergent and have the same limit. For any large enough n, let $J_{+,n}^i$ denote the segment of leave of $\mathcal{F}_\beta(x)$ from y_n^i to $\phi(y_n^i)$, $I_{-,n}^i$ denote the segment of leave of $\mathcal{F}_\alpha(x_n^i)$ from x_n^i to y_n^i, and $I_{+,n}^i$ denote the segment of leave of $\mathcal{F}_\alpha(\phi(x_n^i))$ from $\phi(x_n^i)$ to $\phi(y_n^i)$. Then by definition of ϕ, there exists a unique segment $J_{-,n}^i$ of $\mathcal{F}_\beta(x_n^i)$ such that there exists an embedded rectangle $\mathcal{R}$ of boundary $I_{-,n}^i \cup J_{+,n}^i \cup I_{+,n}^i \cup J_{-,n}^i$, and foliated by segments of $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$. This shows that $\phi(x_n^i) = J_{-,n}^i \cap I_{+,n}^i$ is entirely described by x_n^i and $\phi(y_n^i)$, that the convergence of $x_n^{1/2}$ and $\phi(y_n^{1/2})$ together imply the one of $\phi(x_n^{1/2})$, and that the equalities $\lim x_n^1 = \lim x_n^2$ and $\lim \phi(y_n^1) = \lim \phi(y_n^2)$ imply $\lim \phi(x_n^1) = \lim \phi(x_n^2)$.

To prove that $\lim \phi(y_n^1)$ and $\lim \phi(y_n^2)$ exist and are equal, we can assume without lost of generality that $y_n^{1/2}$ converges to x from above on $\mathcal{F}_\beta(x)$, and that the sequence $y_n^{1/2}$ is also increasing on the oriented topological line $\mathcal{F}_\alpha(p)$. Let a denote a (long) segment of $\mathcal{F}_\beta(x)$ containing x, y_n^1, y_n^2, $\phi(y_n^1)$ and $\phi(y_n^2)$ for any large enough n. We can then close a with a (short) path b to obtain a simple closed curve $\gamma = a \cdot b$ transverse to $\mathcal{F}_\alpha$. Furthermore since $\mathcal{F}_\alpha$ and $\mathcal{F}_\alpha^0$ are isotopic and both transverse to $\mathcal{F}_\beta$, there exists a (short) path b^0 and a segment a^0 of $\mathcal{F}_\beta(x)$ with $a \cap a^0$ connected and containing x, y_n^1, y_n^2, $\phi(y_n^1)$ and $\phi(y_n^2)$ for any large enough n, such that $\gamma^0 = a^0 \cdot b^0$ is a simple closed curve transverse to $\mathcal{F}_\alpha^0$ and homotopic to γ. Then with P and P_0 the respective first-return maps of $\mathcal{F}_\alpha$ and $\mathcal{F}_\alpha^0$ on γ and γ^0, we have

$$[\theta] := \rho(P) = \rho(P_0) \in \mathbf{S}^1 = \mathbb{R}/\mathbb{Z}, \tag{3.2}$$

since $A(\mathcal{F}_\alpha) = A(\mathcal{F}_\alpha^0)$ and γ is homotopic to γ^0. Let y be the first intersection point of the oriented leaf $\mathcal{F}_\alpha(p)$ with γ. Since $y_n^{1/2}$ is increasing on $\mathcal{F}_\alpha(p)$, there exists two increasing sequences $k_n^1, k_n^2 \in \mathbb{N}$ such that $y_n^{1/2} = P^{k_n^{1/2}}(y)$. With y^0 the first intersection point of $\mathcal{F}_\alpha^0(p)$ with γ, we have then $\phi(y_n^{1/2}) = P_0^{k_n^{1/2}}(y^0)$ by the very definition of ϕ.

We now make a naive but useful general remark. Let f be a minimal orientation-preserving circle homeomorphism of rotation number $[\theta] \in \mathbf{S}^1$. Then since f is topologically conjugated to the rotation R_θ, for any $x \in \mathbf{S}^1$ and $k_n \in \mathbb{N}$, the sequence $(f^{k_n}(x))$ is converging if, and only if $[k_n\theta]$ is converging in $\mathbf{S}^1$. Moreover for $k_n^1, k_n^2 \in \mathbb{N}$ such that $(f^{k_n^1}(x))_n$ and $(f^{k_n^2}(x))_n$ are converging, $\lim f^{k_n^1}(x) = \lim f^{k_n^2}(x)$ if, and only if $\lim[(k_n^1 - k_n^2)\theta] = [0] \in \mathbf{S}^1$. Since P and P^0 have the same rotation numbers according to (3.2), the convergence of $y_n^1 = P^{k_n^1}(y)$ and $y_n^2 = \lim P^{k_n^2}(y)$ and the equality of their limits is thus equivalent to the convergence of $\phi(y_n^1) = P^{k_n^1}(y)$ and $\phi(y_n^2) = P^{k_n^2}(y)$ and to the equality of their limits. Therefore $\lim \phi(y_n^1) = \lim \phi(y_n^2)$, and we can thus extend ϕ as desired, to an application well-defined on $\mathbf{T}^2$ by the relation (3.1).

Note that we incidentally proved that ϕ is "continuous along F_α", in the sense that the equality (3.1) holds for any sequence $x_n \in F_\alpha$ converging to a point of F_α. Moreover, we also proved that $\lim \phi(x_n^1) = \lim \phi(x_n^2)$ implies that $x^1 = x^2$ for any two sequences x_n^1 and x_n^2 in F_α respectively converging to x^1 and x^2, hence that ϕ is injective by our definition of ϕ.

(c) ϕ is a homeomorphism. Let us first show that ϕ is continuous. For any sequence $x_n \in \mathbf{T}^2$ converging to x, let $x_n^k \in F_\alpha$ denote for any n a sequence converging to $x_n = \lim\limits_{k \to +\infty} x_n^k$. Then by the definition (3.1) of ϕ, we have $\phi(x_n) = \lim\limits_k \phi(x_n^k)$. Possibly extracting a subsequence of $(x_n^k)_k$, we can furthermore assume that

$$d(\phi(x_n^k), \phi(x_n)) \leq \frac{1}{k+1} \tag{3.3}$$

for any n and k, with d a distance defining the topology of $\mathbf{T}^2$. There exists now an increasing sequence $k_n \in \mathbb{N}$ such that $x_n^{k_n} \in F_\alpha$ converges to x, and we have thus $\phi(x) = \lim \phi(x_n^{k_n})$ by the definition (3.1) of ϕ. But since $d(\phi(x_n^{k_n}), \phi(x_n)) \leq \frac{1}{k_n+1}$ according to (3.3), we obtain $\phi(x) = \lim\ \phi(x_n^{k_n}) = \lim\ \phi(x_n)$ which proves the continuity of ϕ.

Since ϕ is an injective and continuous map defined from the topological surface $\mathbf{T}^2$ to itself, the Invariance of domain theorem of Brouwer shows that $\phi(\mathbf{T}^2)$ is an open (and non-empy) subset of $\mathbf{T}^2$. Since it is also closed by compactness of $\mathbf{T}^2$, we eventually have $\phi(\mathbf{T}^2) = \mathbf{T}^2$ by connectedness of $\mathbf{T}^2$, hence ϕ is a homeomorphism of $\mathbf{T}^2$ as desired.

(d) ϕ preserves $\mathcal{F}_\beta$. By construction, the restriction of ϕ to F_α consists by flowing along leaves of $\mathcal{F}_\beta$. We have therefore $\phi(y) \in \mathcal{F}_\beta(\phi(x))$ for any $x \in F_\alpha$ and $y \in F_\alpha \cap \mathcal{F}_\beta(x)$. By continuity of the foliations and of ϕ, this relation extends to $\mathbf{T}^2$, hence ϕ preserves $\mathcal{F}_\beta$.

(e) ϕ is isotopic to id rel p. Moreover ϕ fixes p by construction, and acts trivially on $\pi_1(\mathbf{T}^2, p)$. Indeed, let (a, b) be a pair of simple closed curves based at p which are the concatenations $a = a_\alpha a_\beta$ and $b = b_\alpha b_\beta$ of segments $a_{\alpha/\beta}$ (respectively $b_{\alpha/\beta}$) of leaves of $\mathcal{F}_{\alpha/\beta}$, and whose homotopy classes define a basis $([a], [b])$ of $\pi_1(\mathbf{T}^2, p)$. Since ϕ is defined on $\mathcal{F}_\alpha(p)$ by flowing along leaves of $\mathcal{F}_\beta$, the paths a_α (respectively b_α) are isotopic to $\phi \circ a_\alpha$ (resp. $\phi \circ b_\alpha$), their arrival point flowing along $\mathcal{F}_\beta(p)$. But the paths a_β (resp. b_β) are for the same reason isotopic to $\phi \circ a_\beta$ (resp. $\phi \circ b_\beta$), their departure point flowing along $\mathcal{F}_\beta(p)$, and therefore a (resp. b) is isotopic to $\phi \circ a$ (resp. $\phi \circ b$). Since ϕ acts trivially on $\pi_1(\mathbf{T}^2, p)$, it is isotopic to the identity by a classical result of Epstein in [Eps66] (see also [BCLR20, Proposition 1.6 and Theorem 2]).

In conclusion, ϕ is a homeomorphism isotopic to id relatively to p, preserving $\mathcal{F}_\beta$ and redressing $\mathcal{F}_\alpha$ on $\mathcal{F}_\alpha^0$, *i.e.* is a conjugation of $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ with $(\mathcal{F}_\alpha^0, \mathcal{F}_\beta)$. This concludes the proof of Theorem A.

4. Proof of Corollary B

We conclude this note with a proof of Corollary B. According to Theorem A, the minimal bi-foliation preserved by f is conjugated to a linear bi-foliation by a homeomorphism isotopic to the identity. We can thus assume without lost of generality that f preserves the bi-foliation defined by two transverse irrational lines l and l' of the plane, and is isotopic to the identity relatively to $[0] \in \mathbf{T}^2$. Let

$$F(x, y) = (F_1(x, y), F_2(x, y))$$

denote the lift of f to $\mathbb{R}^2$ which fixes the origin. According to [GS80, Lemme 4], we have then

$$\begin{cases} F_2(x, y) = \delta F_1(x, y) + a(y - \delta x) + b \\ F_2(x, y) = \delta' F_1(x, y) + a'(y - \delta' x) + b' \end{cases}$$

with δ and δ' the respective slopes of l and l'. A direct computation yields then

$$\begin{cases} F_1(x, y) = \frac{1}{\delta - \delta'}\left((a\delta - a'\delta')x + (a' - a)y + (b' - b)\right) \\ F_2(x, y) = \frac{1}{\delta - \delta'}\left(\delta\delta'(a - a')x + (\delta a' - \delta' a)y + (\delta b' - \delta' b)\right). \end{cases}$$

In other words: $F(x, y) = M(x, y) + \frac{1}{\delta - \delta'}(b' - b, \delta b' - \delta' b)$ with

$$M = \frac{1}{\delta - \delta'}\begin{pmatrix} a\delta - a'\delta' & a' - a \\ \delta\delta'(a - a') & \delta a' - \delta' a \end{pmatrix}.$$

Since $F(0, 0) = (0, 0)$ and $\delta \neq \delta'$, we have $b = b' = 0$. Moreover $M \in \mathrm{GL}_2(\mathbb{Z})$ since F induces the map f on $\mathbf{T}^2$, and the action of f on $\pi_1(\mathbb{Z}^2)$ is thus given by the matrix M. Since f is isotopic to the identity, this implies $M = \mathrm{id}$ and therefore $F = \mathrm{id}$, which shows our claim and concludes the proof of Corollary B.

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