Invitation to Nonlinear Algebra

Version of February 20, 2020

Mateusz Michałek
Bernd Sturmfels

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany
E-mail address: michalek@mis.mpg.de

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany
E-mail address: bernd@mis.mpg.de
2010 Mathematics Subject Classification. Primary
Exercises 68

Chapter 5. Linear Spaces and Grassmannians 71
§ 5.1. Coordinates for Linear Spaces 71
§ 5.2. Plücker Relations 74
§ 5.3. Schubert Calculus 77
Exercises 82

Chapter 6. Nullstellensätze 85
§ 6.1. Certificates for Infeasibility 85
§ 6.2. Hilbert’s Nullstellensatz 87
§ 6.3. Let’s Get Real 90
Exercises 94

Chapter 7. Tropical Algebra 97
§ 7.1. Arithmetic and Valuations 97
§ 7.2. Linear Algebra 102
§ 7.3. Tropical Varieties 107
Exercises 111

Chapter 8. Toric Varieties 113
§ 8.1. The Affine Story 113
§ 8.2. Varieties from Polytopes 118
§ 8.3. The World is Toric 124
Exercises 127

Chapter 9. Tensors 129
§ 9.1. Eigenvectors 129
§ 9.2. Tensor Rank 136
§ 9.3. Matrix Multiplication 141
Exercises 144

Chapter 10. Representation Theory 147
§ 10.1. Groups, Representations and Characters 147
§ 10.2. Invertible Matrices and Permutations 153
§ 10.3. Exploiting Symmetry 158
Exercises 164

Chapter 11. Invariant Theory 167
§ 11.1. Finite Groups 167
Contents

§11.2. Classical Invariant Theory 171
§11.3. Geometric Invariant Theory 174
Exercises 178

Chapter 12. Semidefinite Programming 181
§12.1. Spectrahedra 181
§12.2. Optimization and Duality 185
§12.3. Sums of Squares 190
Exercises 194

Chapter 13. Combinatorics 195
§13.1. Matroids 195
§13.2. Lattice Polytopes 199
§13.3. Generating Functions 204
Exercises 209

Bibliography 211
Index 215
Preface

This book grew out of the lecture notes for a graduate course we taught during the Summer Semester 2018 at the Max-Planck Institute (MPI) for Mathematics in the Sciences in Leipzig, Germany. This was part of the general lecture series (called Ringvorlesung in German) which is offered biannually by the International Max-Planck Research School (IMPRS). The aim of our course was to introduce the theme of Nonlinear Algebra, which is also the name of the research group that started at MPI Leipzig in early 2017.

Linear algebra is the foundation of much of mathematics, particularly in applied mathematics. Numerical linear algebra is the basis of scientific computing, and its importance for the sciences and engineering can hardly be overestimated. The ubiquity of linear algebra masks the fairly recent growth of nonlinear models across the mathematical sciences. There has been a proliferation of methods based on systems of multivariate polynomial equations and inequalities. This is fueled by recent theoretical advances, efficient software, and an increased awareness of these tools. At the heart of this lies algebraic geometry, but there are links to many other branches, such as combinatorics, algebraic topology, commutative algebra, convex and discrete geometry, tensors and multilinear algebra, number theory, representation theory, and symbolic and numerical computation. Application areas include optimization, statistics, complexity theory, among many others.

Nonlinear algebra is not simply a rebranding of algebraic geometry. It is a recognition that a focus on computation and applications, and the theoretical needs that this requires, results in a body of inquiry that is complementary to the existing curriculum. The term nonlinear algebra is intended to capture these trends, and to be more friendly to applied scientists. A special research semester with that title, held in the fall of 2018 at the Institute
for Computational and Experimental Research in Mathematics (ICERM) in Providence, Rhode Island, explored the theoretical and computational challenges that have arisen, and it charted the course for the future. This book supports this effort by offering a warm welcome to nonlinear algebra.

Our presentation is structured into 13 chapters, one for each week in a semester. Many of our chapters are rather ambitious in that they promise a first introduction to an area of mathematics that would normally be covered in a full-year course. But what we offer is really just an invitation. Our readers are hence encouraged to go further in their studies with other sources. We think that students will enjoy our presentation. We hope that nonlinear algebra will encourage them to think critically and deeply, and to question the historic boundaries between “pure” and “applied” mathematics.

Mateusz Michalek and Bernd Sturmfels
Acknowledgements

A two-week intense course on the book manuscript took place at MPI Leipzig in June 2019. We are grateful to the following participants for their lectures and comments: Zachary Adams, Yulia Alexandr, Tobias Boege, Marie-Charlotte Brandenburg, Madeline Brandt, Türkü Çelik, Rodica Dinu, Eliana Duarte, Yassine El Maazouz, Yuhan Jiang, Paul Görlach, Alex Heaton, Nidhi Kaihnsa, Max Kölbl, Andras Lorincz, Orlando Marigliano, Orlich Milo, Yue Ren, Jose Samper, Mahsa Sayyary, Emre Sertöz, Tim Seynnaeve, Stefana Sorea, Isabelle Shankar, Martin Vodicka, Maddie Weinstein.

In addition we are grateful for valuable comments we received from the following readers: Sara Billey, Stark Ledbetter, András Lörincz, Diane Maclagan, Anna-Laura Sattelberger, Mima Stanojkovski.
After a course in linear algebra one often encounters abstract algebra. In that course one studies algebraic structures such as fields, rings and ideals. In our first chapter we introduce the corresponding basics, with focus on polynomials and Gröbner basis. We show how to use these for computing invariants of a polynomial ideal, like dimension or degree. The formalism we develop now will be applied to geometric situations in later chapters.

1.1. Ideals

Our most basic algebraic structure is that of a field. The elements of the field serve as numbers, also called scalars. We can add, subtract, multiply and divide them. It is customary to denote fields by the letter $K$, for the German word Körper. Our favorite field is the set $K = \mathbb{Q}$ of rational numbers. Another important field is the set $K = \mathbb{R}$ of real numbers. In practise, these two fields are very different. Numbers in $\mathbb{Q}$ can be manipulated by exact symbolic computation, whereas numbers in $\mathbb{R}$ are approximated by floating point representations and manipulated by numerical computation.

Other widely used fields are the complex numbers $\mathbb{C}$ and the finite field $\mathbb{F}_q$ with $q$ elements. If $K$ is not algebraically closed then we write $\overline{K}$ for its algebraic closure. This is the smallest field in which every non-constant polynomial with coefficients in $K$ has a root. For instance, $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}_q}$ are the algebraic closures of the two fields above. Another important example is the field of rational functions $\mathbb{Q}(t)$. Its algebraic closure $\overline{\mathbb{Q}(t)}$ is contained in the field of Puiseux series, denoted $\mathbb{C}\{\{t\}\}$, which is also algebraically closed.
The last two fields are less familiar to most readers. Their importance is seen in Chapter 7 when we pass from classical algebra to tropical algebra.

In this section we study the ring of polynomials in \( n \) variables \( x_1, \ldots, x_n \) with coefficients in our field \( K \). It is denoted \( K[x] = K[x_1, \ldots, x_n] \). If the number \( n \) is small then we typically use letters without indices to denote the variables. For instance, we often write \( K[x], K[x, y], \) or \( K[x, y, z] \) for the polynomial ring when \( n = 1, 2, 3 \).

Many of the constructions we present work not just for the polynomial ring \( K[x] \) but for an arbitrary commutative ring \( R \) with unit 1. We allow \( 1 = 0 \), i.e. \( R \) as a set may contain just one element 0. For the most part, the reader may assume \( R = K[x] \). But, it would not hurt to peruse a standard text book on abstract algebra and look up the axioms of a ring and the formal definitions of commutative and unit. Important examples of commutative rings are the integers \( \mathbb{Z} \), the polynomial ring of the integers \( \mathbb{Z}[x] \), or the quotient of a polynomial ring by an ideal. The latter will be discussed soon.

The polynomial ring \( K[x] \) is an infinite-dimensional \( K \)-vector space. A distinguished basis of this vector space consists of the monomials \( x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \). There is one monomial for each nonnegative integer vector \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n \). Every polynomial \( f \in K[x] \) is written uniquely as a finite \( K \)-linear combination of monomials:

\[
  f = \sum_a c_a x^a.
\]

The degree of \( f \) is the maximum of the quantities \( |a| = a_1 + \cdots + a_n \), where \( c_a \neq 0 \). For polynomials of degree 1, 2, 3, 4, 5, 6 we use the words linear, quadratic, cubic, quartic, quintic, sextic. These can be adjectives or nouns. It is also common to use the term quadric for a quadratic polynomial.

For example, the following is a cubic polynomial in \( n = 3 \) variables:

\[
  (1.1) \quad f = \det \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} = 2xyz - x^2 - y^2 - z^2 + 1.
\]

The zero set of \( f \) is the surface in \( \mathbb{R}^3 \). It consists of all points at which the rank of the \( 3 \times 3 \)-matrix in (1.1) drops. It has four singular points, namely the points \( (1, 1, 1), (1, -1, -1), (-1, 1, -1), \) and \( (-1, -1, 1) \). These points are the common zeros in \( \mathbb{R}^3 \) of the cubic \( f \) and its three partial derivatives

\[
  \frac{\partial f}{\partial x} = 2yz - 2x, \quad \frac{\partial f}{\partial y} = 2xz - 2y, \quad \frac{\partial f}{\partial z} = 2xy - 2z.
\]

These are the points at which the rank of the \( 3 \times 3 \)-matrix in (1.1) equals 1. More information, including a formal definition of singular points is presented at the end of Chapter 2.1.
Figure 1. A cubic surface with four singular points.

Figure 1 is meant to illustrate the cubic surface \( \{ f = 0 \} \). However, the picture is drawn in a different coordinate system. Namely, we divide each of the three coordinates by \( \frac{1}{3} (1 - x - y - z) \), and we clear denominators to get \( g = 8x^3 + 6x^2y + 6x^2z + \cdots - 3z + 1 \). Figure 1 shows the part of the surface \( \{ g = 0 \} \) that lies in the box \(-1.5 < x, y, z < 1.5\). This change of coordinates amounts to applying a projective transformation to our surface. From vantage point of projective geometry, to be adopted in Section 2.2, it is natural to regard two varieties as the same if they differ by a projective transformation. We will therefore, from now on, assert that Figure 1 shows the surface \( \{ f = 0 \} \). This serves as a running example throughout this book.

Definition 1.1. All rings in this book are commutative and have a unit 1. An ideal in such a ring \( R \) is a nonempty subset \( I \) of \( R \) such that

(a) if \( f \in R \) and \( g \in I \) then \( fg \in I \);

(b) if \( f, g \in I \) then \( f + g \in I \).

If \( R = K[x] \) then an ideal \( I \) is a nonempty subset of \( K[x] \) that is closed under taking linear combinations with polynomial coefficients. An alternative definition is as follows: A subset \( I \) of a ring \( R \) is an ideal if and only if there exists a ring homomorphism \( \phi : R \to S \) whose kernel \( \ker \phi = \phi^{-1}(0) \) is equal to \( I \). For instance, if \( R = \mathbb{Z} \) then the set \( I \) of even integers is an ideal. It is the kernel of the ring homomorphism \( \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \) that takes an integer to either 0 or 1, depending on its parity.

Ideals in a ring play the same role as normal subgroups in a group. They are the subobjects used to define quotients. Consider the quotient of abelian groups \( R/I \). Its elements are the congruence classes \( f + I \) modulo \( I \). The axioms (a) and (b) in Definition 1.1 ensure that the following identities hold:

\[
(f + I) + (g + I) = (f + g) + I \quad \text{and} \quad (f + I)(g + I) = fg + I.
\]

1.1. Ideals
Proposition 1.2. If \( I \subset R \) is an ideal then the quotient \( R/I \) is a ring.

Given any subset \( \mathcal{F} \) of a ring \( R \), we write \( \langle \mathcal{F} \rangle \) for the smallest ideal containing \( \mathcal{F} \). This is the ideal generated by \( \mathcal{F} \). If \( R = K[x] \) then the ideal \( \langle \mathcal{F} \rangle \) is the set of all polynomial linear combinations of finite subsets of \( \mathcal{F} \).

Proposition 1.3. If \( I \) and \( J \) are ideals in a ring \( R \) then the following subsets of \( R \) are ideals as well: the sum \( I + J \), the intersection \( I \cap J \), the product \( IJ \), and the quotient \( (I : J) \). The latter two subsets of \( R \) are defined as follows:

\[
IJ = \{ fg : f \in I, g \in J \} \quad \text{and} \quad (I : J) = \{ f \in R : fJ \subseteq I \}.
\]

Proof. The product \( IJ \) is an ideal by definition. For the others one checks that conditions (a) and (b) hold. We shall carry this out for the ideal quotient \( (I : J) \). To show (a), suppose that \( f \in R \) and \( g \in (I : J) \). We have:

\[
(fg)J = f(gJ) \subseteq fI \subseteq I.
\]

For (b), suppose \( f \) and \( g \) are in \( (I : J) \). We have:

\[
(f + g)J \subseteq fJ + gJ \subseteq I + I = I.
\]

This implies \( f + g \in (I : J) \). We have shown that \( (I : J) \) is an ideal. \( \Box \)

The Euclidean algorithm works in the polynomial ring \( K[x] \) in one variable \( x \) over a field \( K \). This implies that \( K[x] \) is a principal ideal domain (PID), i.e. every ideal \( I \) in \( K[x] \) is generated by one element. That generator can be uniquely factored into irreducible polynomials.

Unique factorization of polynomials also holds when the number of variables satisfies \( n \geq 2 \). We say that the polynomial ring \( K[x] \) is a unique factorization domain (UFD). However, \( K[x] \) is not a PID when \( n \geq 2 \). For instance, for \( n = 2 \), the ideal \( \langle x_1, x_2 \rangle \) is not principal. But, let’s first go back to the univariate case in order to illustrate the operations in Proposition 1.3

Example 1.4 \((n = 1)\). Consider the following two ideals in \( \mathbb{Q}[x] \):

\[
I = \langle x^3 + 6x^2 + 12x + 8 \rangle \quad \text{and} \quad J = \langle x^2 + x - 2 \rangle.
\]

We compute the four ideals in Proposition 1.3. For this, it helps to factor:

\[
I = \langle (x + 2)^3 \rangle \quad \text{and} \quad J = \langle (x - 1)(x + 2) \rangle.
\]

The four new ideals are

\[
I \cap J = \langle (x - 1)(x + 2)^2 \rangle \quad IJ = \langle (x - 1)(x + 2)^4 \rangle \\
I + J = \langle x + 2 \rangle \quad I : J = \langle (x + 2)^2 \rangle.
\]

We see that arithmetic in \( \mathbb{Q}[x] \) is just like arithmetic in the ring of integers \( \mathbb{Z} \).

A non-zero element \( f \) in a ring \( R \) is called

- a nilpotent if \( f^m = 0 \) for some positive integer \( m \),
• a zero divisor if there exists $0 \neq g \in R$ such that $gf = 0$.

A ring $R$ is called an integral domain if it has no zero divisors and $1 \neq 0$ in $R$. For instance, the set $\{0\}$ is a ring but it is not an integral domain.

We examine these properties for the quotient ring $R/I$ where $I$ is an ideal in $R$. Properties of the ideal $I$ correspond to properties of the ring $R/I$. This summarized in the following table:

<table>
<thead>
<tr>
<th>property</th>
<th>definition</th>
<th>the quotient ring $R/I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$ is maximal</td>
<td>no other proper ideal contains $I$</td>
<td>is a field</td>
</tr>
<tr>
<td>$I$ is prime</td>
<td>$fg \in I \Rightarrow f \in I$ or $g \in I$</td>
<td>is an integral domain</td>
</tr>
<tr>
<td>$I$ is radical</td>
<td>$(\exists s : f^s \in I) \Rightarrow f \in I$</td>
<td>has no nilpotent elements</td>
</tr>
<tr>
<td>$I$ is primary</td>
<td>$fg \in I$ and $g \notin I \Rightarrow (\exists s : f^s \in I)$ all zero divisors are nilpotent</td>
<td></td>
</tr>
</tbody>
</table>

Maximal, prime and primary ideals are proper. In other words, the ring $R$ itself is an ideal in $R$, but it is neither maximal, nor prime, nor primary.

**Example 1.5.** The ideal $I = \langle x^2 + 10x + 34, 3y - 2x - 13 \rangle$ is maximal in the polynomial ring $\mathbb{R}[x, y]$. The field $\mathbb{R}[x, y]/I$ is isomorphic to the field of complex numbers $\mathbb{C} = \mathbb{R}[i]/(i^2 + 1)$. One isomorphism is gotten by sending $i = \sqrt{-1}$ to $\frac{1}{4}(x + 5y)$. The square of that expression is $-1$ mod $I$. The principal ideal $J = \langle x^2 + 10x + 34 \rangle$ is prime but not maximal in $\mathbb{R}[x, y]$. The quotient $\mathbb{R}[x, y]/J$ is an integral domain. It is isomorphic to $\mathbb{C}[y]$.

Examples for the other two classes of ideals are given in the next proof.

**Proposition 1.6.** We have the following implications for an ideal $I$ in $R$:

$I$ maximal $\Rightarrow$ $I$ prime $\Rightarrow$ $I$ radical, $\Rightarrow$ $I$ primary.

None of these implications is reversible. However, every ideal that is both radical and primary is prime. Every intersection of prime ideals is radical.

**Proof.** The first implication holds because there are no zero divisors in a field. To see that prime implies radical, we take $g = f^{s-1}$ and we use induction on $s$. That prime implies primary is clear. To prove that every radical primary ideal is prime assume $fg \in I$ and $f \notin I$. Then, as $I$ is primary, we have $g^s \in I$ for some $s \in \mathbb{N}$. As $I$ is radical, we conclude $g \in I$.

To see that no implication is reversible, we consider the following three ideals in the polynomial ring $\mathbb{R}[x, y]$ with $n = 2$ variables:

- $I = \langle x^2 \rangle$ is primary but not radical,
- $I = \langle x(x - 1) \rangle$ is radical but not primary,
- $I = \langle x \rangle$ is prime but not maximal.

The last statement holds since intersections of radical ideals are radical.  \(\square\)
1. Polynomial Rings

We now revisit the surface in Figure 1 from the perspective of ideals.

**Example 1.7 (n = 3).** We consider the ideal generated by the partial derivatives of the cubic \( f = 2xyz - x^2 - y^2 - z^2 + 1 \). This is the ideal

\[
I = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle yz - x, xz - y, xy - z \rangle \subset \mathbb{R}[x, y, z].
\]

The cubic \( f \) is not in this ideal because every polynomial in \( I \) has zero constant term. The ideal \( I \) is radical because we can write it as the intersection of five maximal ideals. Namely, using a computer algebra system, we find

\[ (1.3) \quad I = \langle x, y, z \rangle \cap \langle x - 1 \rangle \cap \langle y - 1 \rangle \cap \langle z - 1 \rangle. \]

This is a primary decomposition of \( I \), as discussed in detail in Chapter 3.

The cubic \( f \) lies in the last four maximal ideals. Their intersection equals \( I + (f) \). The zero set of the radical ideal \( I + (f) \) consists of the four singular points on the surface seen in Figure 1. The **Chinese Remainder Theorem** for rings implies that the quotient ring is a product of fields. Namely, we have an isomorphism \( \mathbb{R}[x, y, z]/I \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \). It takes each polynomial modulo \( I \) to its residue classes modulo in the intersectands in (1.3).

1.2. Gröbner Bases

Every ideal has many different generating sets. There is no canonical notion of basis for an ideal. For instance, the set \( F = \{x^6 - 1, x^{10} - 1, x^{15} - 1\} \) minimally generates the ideal \( \langle x - 1 \rangle \) in the polynomial ring \( \mathbb{Q}[x] \) in one variable. Of course, the singleton \( \{x - 1\} \) is a preferable generating set for that ideal. Recall that every ideal in \( \mathbb{Q}[x] \) is principal for \( n = 1 \). The **Euclidean algorithm** transforms the set \( F \) into the set \( \{x - 1\} \).

Here is a certificate for the fact that \( x - 1 \) is in the ideal generated by \( F \):

\[
x^5 \cdot (x^6 - 1) - (x^5 + x) \cdot (x^{10} - 1) + 1 \cdot (x^{15} - 1) = x - 1.
\]

Such identities can be found with the **Extended Euclidean Algorithm**. Please google this. Finding certificates for ideal membership when \( n \geq 2 \) is a harder problem. This topic comes up when we discuss Hilbert’s Nullstellensatz in Chapter 3. In this section we introduce the basics for computing with ideals.

**Gaussian elimination** is familiar from linear algebra. It gives a process for manipulating ideals that are generated by linear polynomials. For example, the following two ideals are identical in the polynomial ring \( \mathbb{Q}[x, y, z] \):

\[
\langle 2x + 3y + 5z + 7, 11x + 13y + 17z + 19, 23x + 29y + 31z + 37 \rangle
= \langle 7x - 16, 7y + 12, 7z + 9 \rangle.
\]

Undergraduate linear algebra taught us how to transform the three generators on the left into the simpler ones on the right. This is the process
of solving a system of linear equations. In our example there is a unique solution, namely the point \((\frac{16}{7}, -\frac{12}{7}, -\frac{9}{7})\) in \(\mathbb{R}^3\).

We next introduce Gröbner bases. The framework of Gröbner bases offers practical methods for computing with ideals in a polynomial ring \(K[x]\) in \(n\) variables. Here \(K\) is a field whose arithmetic we can compute. Implementations of Gröbner bases are available in many computer algebra systems. We strongly encourage our readers to experiment with these tools.

Informally, we can think of computing Gröbner bases as a version of the Euclidean algorithm for polynomials in \(n \geq 2\) variables, or as a version of Gaussian elimination for polynomials of degree \(\geq 2\). Computing Gröbner bases for ideals in \(K[x]\) is fundamental in nonlinear algebra, just like Gaussian elimination for matrices is fundamental in linear algebra. The premise of this book is that nonlinear algebra is a next step after linear algebra.

We identify the set \(\mathbb{N}^n\) of non-negative integer vectors with the monomial basis of the polynomial ring \(K[x]\). The coordinatewise partial order on \(\mathbb{N}^n\) corresponds to divisibility of monomials. To be precise, we have \(a \leq b\) in the poset \(\mathbb{N}^n\) if and only if the monomial \(x^a\) divides the monomial \(x^b\).

**Theorem 1.8** (Dickson’s Lemma). Any infinite subset of \(\mathbb{N}^n\) contains a pair \(\{a, b\}\) that satisfies \(a \leq b\).

**Corollary 1.9.** For any nonempty set \(\mathcal{M} \subset \mathbb{N}^n\), its subset of coordinatewise minimal elements is finite and nonempty.

**Proof.** The fact that it is nonempty follows by induction on \(n\). The set is finite by Dickson’s Lemma. 

**Proof of Theorem 1.8** We proceed by induction on \(n\). The statement is trivial for \(n = 1\). Any subset of cardinality at least two in \(\mathbb{N}\) contains a comparable pair. Suppose now that Dickson’s Lemma has been proved for \(n - 1\), and consider an infinite subset \(\mathcal{M}\) of \(\mathbb{N}^n\). For each \(i \in \mathbb{N}\) let \(\mathcal{M}_i\) denote the set of all vectors \(a \in \mathbb{N}^{n-1}\) such that \((a, i) \in \mathcal{M}\). If some \(\mathcal{M}_i\) is infinite then we are done by the induction hypothesis. Hence suppose that each \(\mathcal{M}_i\) is a finite subset of \(\mathbb{N}^{n-1}\), and we have \(\mathcal{M}_i \neq \emptyset\) for infinitely many \(i\).

The infinite subset \(\bigcup_{i=0}^{\infty} \mathcal{M}_i\) of \(\mathbb{N}^{n-1}\) satisfies the assertion. This means that its subset of minimal elements with respect to the coordinatewise order is finite. Here we are using Corollary 1.9 which holds for \(n - 1\) by induction hypothesis. Hence there exists an index \(j\) such that all minimal elements are contained in the finite set \(\bigcup_{i=0}^{j} \mathcal{M}_i\). Pick any element \(b \in \mathcal{M}_k\) for \(k > j\). Since \(b\) is not minimal in \(\bigcup_{i=0}^{\infty} \mathcal{M}_i\), there exists an index \(i\) with \(i \leq j < k\) and an element \(a \in \mathcal{M}_i\) with \(a \leq b\). We have \((a, i) \leq (b, k)\) in \(\mathcal{M}\). 

**Definition 1.10.** Consider a total ordering \(<\) of the set \(\mathbb{N}^n\). We write \(a \preceq b\) if \(a < b\) or \(a = b\). The ordering \(<\) is a monomial order if, for all \(a, b, c \in \mathbb{N}^n\),
• \((0,0,\ldots,0) \preceq a;\)
• \(a \preceq b\) implies \(a + c \preceq b + c.\)

This gives a total order on monomials in \(K[x].\) Three standard examples are:

• the lexicographic ordering: we set \(a \prec_{\text{lex}} b\) if the leftmost non-zero entry of \(b - a\) is positive. This ordering is important for elimination.

• the degree lexicographic order: we set \(a \prec_{\text{deglex}} b\) if either \(|a| < |b|,\) or \(|a| = |b|\) and the leftmost non-zero entry of \(b - a\) is positive.

• the degree reverse lexicographic order: \(a \prec_{\text{revlex}} b\) if either \(|a| < |b|,\) or \(|a| = |b|\) and the rightmost non-zero entry of \(b - a\) is negative.

All three orders satisfy \(x_1 \succ x_2 \succ \cdots \succ x_n,\) but they differ on monomials of higher degree. We recommend that the reader list the 10 quadratic monomials for \(n = 4\) in each of the three orderings above.

Throughout this book we specify a monomial order by giving the name of the order and how the variables are sorted. For instance, we might say: “let \(\prec\) denote the degree lexicographic order on \(K[y,z]\) given by \(y \prec z \prec x\)”.

Other choices of monomial orders are obtained by assigning positive weights to the variables. See [11, Exercise 11 in §2.4]. We also note that any monomial order is a refinement of the coordinatewise partial order on \(\mathbb{N}^n:\)

\[
\text{if } x^a \text{ divides } x^b \text{ then } a \preceq b.
\]

**Remark 1.11.** Fix a monomial order \(\prec\) and let \(M\) be any nonempty subset of \(\mathbb{N}^n.\) Then \(M\) has a unique minimal element with respect to \(\prec.\) To show this, we apply Dickson’s Lemma as in Corollary [1.9]. Our set \(M\) has a finite, nonempty subset of minimal elements in the componentwise order on \(\mathbb{N}^n.\) This finite subset is linearly ordered by \(\prec.\) We select its minimal element.

We now fix a monomial order \(\prec.\) Given any nonzero polynomial \(f \in K[x],\) its initial monomial \(\text{in}_\prec(f)\) is the \(\prec\)-largest monomial \(x^a\) among those that appear in \(f\) with non-zero coefficient. To illustrate this for the orders above, let \(n = 3\) with variable order \(x \succ y \succ z:\) Fix the polynomial \(f = x^2 + xz^2 + y^3.\) Then \(\text{in}_{\text{lex}}(f) = x^2, \text{ in}_{\text{deglex}}(f) = xz^2\) and \(\text{in}_{\text{revlex}}(f) = y^3.\)

For any ideal \(I \subset K[x],\) we define the initial ideal of \(I\) with respect to a given monomial order \(\prec\) as follows:

\[
\text{in}_\prec(I) = \langle \text{in}_\prec(f) : f \in I \rangle.
\]

This is a monomial ideal, i.e. it is generated by a set of monomials. A priori, this generating set is infinite. However, it turns out that we can always choose a finite subset that suffices to generate this monomial ideal.
Proposition 1.12. Fix a monomial order $\prec$. Every ideal $I$ in the polynomial ring $K[x]$ has a finite subset $G$ such that
\[ \text{in}_\prec(I) = \langle \text{in}_\prec(f) : f \in G \rangle. \]
Such a finite subset $G$ of $I$ is called a Gröbner basis for $I$ with respect to $\prec$.

Proof. Suppose no such finite set $G$ exists. Then we can create a list of infinitely many polynomials $f_1, f_2, f_3, \ldots$ in $I$ such that none of the initial monomials $\text{in}_\prec(f_i)$ divides any other initial monomial $\text{in}_\prec(f_j)$. This would be a contradiction to Dickson’s Lemma.

We next show that every Gröbner basis actually generates its ideal.

Theorem 1.13. If $G$ is a Gröbner basis for an ideal $I$ in $K[x]$ then $I = \langle G \rangle$.

Proof. Suppose that $G$ does not generate $I$. Among all elements in the set-theoretic difference $I \setminus \langle G \rangle$, there exists a polynomial $f$ whose initial monomial $x^b = \text{in}_\prec(f)$ is minimal with respect to $\prec$. This follows from Remark 1.11. Since $x^b \in \text{in}_\prec(I)$, there exists an element $g \in G$ whose initial monomial divides $x^b$, say $x^b = x^c \cdot \text{in}_\prec(g)$. Now, $f - x^c g$ is a polynomial with strictly smaller initial monomial. It lies in $I$ but it does not lie in the ideal $\langle G \rangle$. This is a contradiction to the choice of $f$.

Corollary 1.14 (Hilbert’s Basis Theorem). Every ideal $I$ in the polynomial ring $K[x]$ is finitely generated.

Proof. Fix any monomial order $\prec$. By Proposition 1.12, the ideal $I$ has a finite Gröbner basis $G$. By Theorem 1.13, the finite set $G$ generates $I$.

Gröbner bases are not unique. If $G$ is a Gröbner basis of an ideal $I$ with respect to a monomial order $\prec$ then so is every other finite subset of $I$ that contains $G$. In that sense, Gröbner bases differ from the bases we know from linear algebra. The issue of minimality and uniqueness is addressed next.

Definition 1.15. Fix $I$ and $\prec$. A Gröbner basis $G$ is reduced if the following two conditions hold:

(a) The leading coefficient of each polynomial $g \in G$ is 1.
(b) For distinct $g, h \in G$, no monomial in $g$ is a multiple of $\text{in}_\prec(h)$.

In what follows we fix an ideal $I \subset K[x]$ and a monomial order $\prec$.

Theorem 1.16. The ideal $I$ has a unique reduced Gröbner basis for $\prec$.

Proof idea. We refer to §2.7, Theorem 5]. The idea is as follows. We start with any Gröbner basis $G$ and we turn it into a reduced Gröbner basis by applying the following steps. First we divide each $g \in G$ by its leading
Let $S_<(I)$ be the set of all monomials $x^b$ that are not in the initial ideal $\text{in}_<(I)$. We call these $x^b$ the standard monomials of $I$ with respect to $\prec$.

**Theorem 1.17.** The set $S_<(I)$ of standard monomials is a basis for the $K$-vector space $K[x]/I$.

**Proof.** The image of $S_<(I)$ in $K[x]/I$ is linearly independent because every non-zero polynomial $f$ whose image is zero in $K[x]/I$ lies in the ideal $I$. Any such $f$ has at least one monomial, namely in$_<(f)$, that is not in $S_<(I)$.

We next prove that $S_<(I)$ spans $K[x]/I$. Suppose not. Then there exists a monomial $x^c$ which is not in the $K$-span of $S_<(I)$ modulo $I$. We may assume that $x^c$ is minimal with respect to the monomial order $\prec$. Since $x^c$ is not in $S_<(I)$, it lies in the initial ideal $\text{in}_<(I)$. Hence there exists $h \in I$ with $\text{in}_<(h) = x^c$. Each monomial in $h$ other than $x^c$ is smaller with respect to $\prec$, so it lies in the $K$-span of $S_<(I)$ modulo $I$. Hence $x^c$ lies in the $K$-span of $S_<(I)$ modulo $I$. This is a contradiction. $\square$

Software for Gröbner bases rests on Buchberger’s Algorithm [11 §2.7]. This is implemented in all major computer algebra systems. It takes as its input a monomial order $\prec$ and a finite set $\mathcal{F}$ of polynomials in $K[x]$. The output of Buchberger’s Algorithm is the unique reduced Gröbner basis $\mathcal{G}$ for the ideal $I = \langle \mathcal{F} \rangle$ with respect to $\prec$. Experimenting with such an implementation is an essential step for any student in nonlinear algebra.

In what follows we present some examples of input-output pairs $(\mathcal{F}, \mathcal{G})$ for $n = 3$. Here we take the lexicographic monomial order with $x \succ y \succ z$.

**Example 1.18.** A computer algebra system, like Maple, Mathematica, Magma, Macaulay2, or Singular, transforms the input $\mathcal{F} \subset \mathbb{Q}[x,y,z]$ into its reduced Gröbner basis $\mathcal{G}$. The initial monomials are always underlined:

- For $n = 1$, computing the reduced Gröbner basis means computing the greatest common divisor of the input polynomials:

  $\mathcal{F} = \{x^3-6x^2-5x-14, 3x^3+8x^2+11x+10, 4x^4+4x^3+7x^2-x-2\}$,
  $\mathcal{G} = \{x^2+x+2\}$.

- For linear polynomials, running Buchberger’s algorithm amounts to Gaussian elimination: For $\mathcal{F} = \{2x + 3y + 5z + 7, 11x + 13y + 14z + 9\}$, the reduced Gröbner basis is $\mathcal{G} = \{x+1, y+2, z+3\}$. 


\[ \frac{1}{4} \]
17z + 19, 23x + 29y + 31z + 37}, the reduced Gröbner basis is found to be \( G = \{ x - \frac{16}{7}, y + \frac{12}{7}, z + \frac{9}{7} \} \).

- The input \( F = \{ xy - z, xz - y, yz - x \} \) yields the output \( G = \{ x - yz, y^3 - 2, yz^2 - y, z^3 - z \} \). There are precisely five standard monomials: \( S_\prec(I) = \{ 1, y, z, yz, z^2 \} \). Note that the number five also occurred in Example 1.7, we saw that \( F \) has five zeros in \( \mathbb{C}^3 \).

- This input is a curve in the \( (y, z) \)-plane parametrized by two cubics in one variable \( x \). We write this as \( F = \{ y - x^3 - 4x, z - x^3 - x + 1 \} \). The Gröbner basis has the implicit equation of this curve as its second element: \( G = \{ x + \frac{1}{5}y + \frac{1}{5}z - \frac{1}{5}, y^3 - 3y^2z - 3y^2 + 3yz^2 + 6yz + 28y - z^3 - 3z^2 + 97z + 99 \} \).

- Let \( z \) be the sum of \( x = \sqrt[7]{7} \) and \( y = \sqrt[5]{5} \). We encode this in the set \( F = \{ x^3 - 7, y^4 - 5, z - x - y \} \). The real number \( z = \sqrt[7]{7} + \sqrt[5]{5} \) is algebraic of degree 12 over \( \mathbb{Q} \). Its minimal polynomial is the first element in the Gröbner basis \( G = \{ z^{12} - 28z^9 - 15z^8 + 294z^6 - 1680z^5 + 75z^4 - 1372z^3 - 7350z^2 - 2100z + 2276, \ldots \} \).

- The elementary symmetric polynomials \( F = \{ x + y + z, xy + xz + yz, xyz \} \) have the reduced Gröbner basis \( G = \{ x + y + z, y^2 + yz + z^2, z^3 \} \). There are six standard monomials. The quotient \( \mathbb{Q}[x, y, z]/I \) is the regular representation of the symmetric group \( S_3 \).

For each of the six ideals above, what is the reduced Gröbner basis for the degree lexicographic order? What are the possible initial monomial ideals?

Many such examples boil down to the fact that lexicographic Gröbner bases are useful for eliminating variables. We shall see this in Theorem 4.5.

In general, the choice of monomial order can make a huge difference in the complexity of the reduced Gröbner basis, even for two input polynomials.

**Example 1.19** (Intersecting two quartic surfaces in projective 3-space \( \mathbb{P}^3 \)). A random homogeneous polynomial of degree four in \( n = 4 \) variables has 35 monomials. Consider the ideal \( I \) generated by two such random polynomials. If \( \prec \) is the degree reverse lexicographic order then the reduced Gröbner basis \( G \) contains 5 elements of degree up to 7. If \( \prec \) is the lexicographic order then \( G \) contains 150 elements of degree up to 73.

Naturally, one uses a computer to find the 150 polynomials above. Many computer algebra systems offer an implementation of Buchberger’s algorithm for Gröbner bases. We reiterate: our readers are strongly encouraged to experiment with a computer algebra system while studying this book.

For an introduction to Buchberger’s algorithm and many further details regarding Gröbner bases, the reader is referred to the textbooks by Cox-Little-O’Shea [11], Greuel-Pfister [26] and Kreuzer-Robbiano [34]. In later
chapters we shall freely use concepts from this area, like S-polynomials and Buchberger’s Criterion. After all, our book is nothing but an “Invitation”.

1.3. Dimension and Degree

The two most important invariants of an ideal $I$ in a polynomial ring $K[x]$ are its dimension and degree. We shall define these invariants, starting with the case of monomial ideals. In this section we focus on combinatorial aspects. The geometric interpretation will be presented in Chapter 2.

**Definition 1.20** (Hilbert function). Let $I \subset K[x]$ be a monomial ideal. The *Hilbert function* $h_I$ takes nonnegative integers to nonnegative integers. The value $h_I(q)$ is the number of monomials of degree $q$ not belonging to $I$.

A convenient way to represent a function $\mathbb{N} \to \mathbb{N}$ is by its generating function. This is a formal power series with nonnegative integer coefficients. The generating function for the Hilbert function is known as *Hilbert series*.

**Definition 1.21** (Hilbert series). Let $I \subset K[x]$ be a monomial ideal. We fix a formal variable $z$. The Hilbert series of $I$ is the generating function

$$HS_I(z) = \sum_{q=0}^{\infty} h_I(q) z^q.$$  

We begin with the zero ideal $I = \{0\}$. We count all monomials in $K[x]$.

**Example 1.22.** The Hilbert series of the zero ideal is the rational function

$$HS_{\{0\}}(z) = \frac{1}{(1-z)^n} = \sum_{q=0}^{\infty} \binom{n+q-1}{n-1} z^q.$$  

The number of monomials of degree $q$ in $n$ variables equals $h_I(q) = \binom{n+q-1}{n-1}$. Note that the Hilbert function $h_{\{0\}}(q)$ is a polynomial in $q$ of degree $n - 1$ in $q$.

We next consider the case of a principal ideal.

**Example 1.23.** Let $I = \langle x_1^{a_1} \cdots x_n^{a_n} \rangle$, where $\sum_{i=1}^{n} a_i = e$. We must count monomials of degree $q$ that are not divisible by the generator of $I$. To do this, we count all monomials and then subtract those that are in $I$. This yields

$$HS_I(z) = \frac{1 - z^e}{(1-z)^n} = \sum_{q=0}^{\infty} \left[ \binom{n+q-1}{n-1} - \binom{n+q-e-1}{n-1} \right] z^q.$$  

The second binomial coefficient is zero when $q < e$. For all $q \geq e$, the Hilbert function $h_I(q) = \binom{n+q-1}{n-1} - \binom{n+q-e-1}{n-1}$ is a polynomial in $q$ of degree $n - 2$.

Our third example concerns ideals generated by two monomials:
1.3. Dimension and Degree

Example 1.24. Fix an ideal $I = \langle m_1, m_2 \rangle$ in $K[x]$, where $m_i$ is a monomial of degree $e_i$ for $i = 1, 2$. We count the monomials in $I$ of degree $q$ by

1. computing the number of monomials divisible by $m_1$,
2. adding the number of monomials divisible by $m_2$,
3. subtracting the number of monomials divisible both by $m_1$ and $m_2$.

Case (3) concerns monomials that are divisible by the least common multiple $m_{12} = \text{lcm}(m_1, m_2)$. Let $e_{12}$ denote the degree of $m_{12}$. The Hilbert series is

$$\text{HS}_I(z) = \frac{1 - z^{e_1} - z^{e_2} + z^{e_{12}}}{(1 - z)^n}.$$  

Therefore, the Hilbert function is an alternating sum of binomial coefficients:

$$h_I(q) = \sum_{\tau \subseteq \{1, 2, \ldots, r\}} (-1)^{\vert \tau \vert} \binom{e_{12}}{n-1} - \binom{e_1}{n-1} - \binom{e_2}{n-1} + \binom{e_{12}}{n-1}.$$  

This expression agrees with a polynomial in $q$, provided $q \geq e_{12}$.

Theorem 1.25. The Hilbert series of a monomial ideal $I \subset K[x]$ equals

$$\text{HS}_I(z) = \frac{\kappa_I(z)}{(1 - z)^n},$$

where $\kappa_I(z)$ is a polynomial with integer coefficients and $\kappa_I(0) = 1$. There exists a polynomial $\text{HP}_I$ in one unknown $q$ of degree $\leq n - 1$, known as the Hilbert polynomial of the ideal $I$, such that $\text{HP}_I(q) = h_I(q)$ for all values of the integer $q$ that are sufficiently large.

Proof. We prove this result by counting monomials using inclusion-exclusion, as hinted at in the three examples above. Let $m_1, m_2, \ldots, m_r$ be the monomials that minimally generate $I$. For any subset $\tau$ of the index set $\{1, 2, \ldots, r\}$, we write $m_{\tau}$ for the least common multiple of the set $\{m_i : i \in \tau\}$, and we set $e_{\tau} = \text{degree}(m_{\tau})$. This includes the empty set $\tau = \emptyset$, for which $m_{\emptyset} = 1$ and $e_{\emptyset} = 0$. The desired numerator polynomial (1.4) can be written as an alternating sum of $2^r$ powers of $z$:

$$\kappa_I(z) = \sum_{\tau \subseteq \{1, 2, \ldots, r\}} (-1)^{\vert \tau \vert} \cdot z^{e_{\tau}}.$$  

The cases $r = 0, 1, 2$ were seen above. The general case is inclusion-exclusion. Note that $\kappa_I \in \mathbb{Z}[z]$ with $\kappa_I(0) = 1$. By regrouping the terms of (1.4),

$$h_I(q) = \sum_{\tau \subseteq \{1, 2, \ldots, r\}} (-1)^{\vert \tau \vert} \binom{n + q - e_{\tau} - 1}{n-1}.$$  

This expression is a polynomial for $q \gg 0$. More precisely, the Hilbert function $h_I(q)$ coincides with the Hilbert polynomial $\text{HP}_I(q)$ for all $q$ that exceed $e_{\{1,2,\ldots,r\}}$. This number is the degree of the least common multiple of all generators of $I$. $\square$
Remark 1.26. The inclusion-exclusion principle carried out in the proof of Theorem 1.25 is a powerful idea, but it also hints at possible simplifications. We wrote the numerator polynomial $\kappa_I(z)$ and the Hilbert polynomial $HP_I(q)$ as an alternating sum of $2^r$ terms. However, in most applications $r$ is much larger than $n$, and the vast majorities of terms will cancel each other. Doing the correct bookkeeping leads us to the topic of minimal free resolutions of monomial ideals. This is a main theme in a subject area known as combinatorial commutative algebra. Yes, please google this.

Example 1.27. Let $n = 2$ and consider the monomial ideal 

$$I = \langle x \rangle \cap \langle y \rangle \cap \langle x, y \rangle^{r+1} = \langle x^r y, x^{r-1} y^2, x^{r-2} y^3, \ldots, x^2 y^{r-1}, xy^r \rangle.$$ 

Our formula for $\kappa_I$ involves $2^r$ terms. After cancellations, only $2r$ remain:

$$\kappa_I(z) = 1 - rz^{r+1} + (r-1)z^{r+2}.$$ 

The Hilbert polynomial is the constant $HP(q) \equiv 2$. This is also the value of the Hilbert function $h_I(q)$ for $q > r$. Note that $h_I(q) = q + 1$ for $q \leq r$.

Definition 1.28 (Dimension, Degree). Let $I$ be a monomial ideal and write

$$HP_I(q) = \frac{g}{(d-1)!} q^{d-1} + \text{lower order terms in } q.$$ 

If the Hilbert polynomial $HP$ is nonzero the dimension of $I$ is $d$ and the degree of $I$ is $g$. Here $g$ is a positive integer. If $HP_I(q) \equiv 0$ then we say that $I$ is zero dimensional. In this case, $K[x]/I$ is a finite dimensional $K$ vector space. We define the degree of $I$ to be the dimension of that vector space.

Remark 1.29. The fact that $g$ is a positive integer is a result in combinatorics. This proof is omitted here, but we revisit this theme in Chapter 13. From the inclusion-exclusion formulas above, one can show that the numerator of the Hilbert series factors as $\kappa_I(z) = \lambda_I(z) \cdot (1-z)^{n-d}$, where $\lambda_I(z)$ is also a polynomial with integer coefficients. The degree of $I$ equals $g = \lambda_I(1)$.

Remark 1.30. It may seem artificial to define dimension and degree by distinguishing the cases where the Hilbert polynomial is zero or not. However, if the zero polynomial has degree $-1$ then the two definitions are compatible. Furthermore, given any ideal $I$ in $K[x]$, the degree of $\tilde{I}$ in $K[x, z]$, with one extra variable $z$ but the same generators as $I$, remains the same. Hence, we could equivalently define the degree of $I$ by adding $z$ and extracting the leading coefficient of $HP_{\tilde{I}}(q)$. This operation increases the dimension by one.

Example 1.31. Let $I$ be a principal ideal as in Example 1.23, generated by a monomial of degree $e > 0$. Then the dimension of $I$ is $n - 1$ and the degree of $I$ is $e$. This follows from the formula we gave for the Hilbert series.
Example 1.32. Let \( n = 2m \) be even and consider the monomial ideal
\[
I = \langle x_1x_2, x_3x_4, x_5x_6, \ldots, x_{2m-3}x_{2m-2}, x_{2m-1}x_{2m} \rangle.
\]
The dimension of \( I \) equals \( m \) and the degree of \( I \) equals \( 2^m \). It is instructive to work out the Hilbert series and the Hilbert polynomial of \( I \) for \( m = 3, 4 \).

We now consider an arbitrary ideal \( I \) in \( K[x] \). We no longer assume that \( I \) is generated by monomials. Let \( \prec \) be any degree-compatible monomial order. This means that \( |a| < |b| \) implies \( a \prec b \) for all \( a, b \).

Lemma 1.33. The number of standard monomials of \( I \) in degree \( q \) is independent of the choice of monomial order \( \prec \), provided \( \prec \) is degree-compatible.

Proof. Let \( K[x]_{\leq q} \) denote the vector space of polynomials of degree \( \leq q \). We write \( I_{\leq q} := I \cap K[x]_{\leq q} \) for the subspace of polynomials that lie in the ideal \( I \). Also, consider the set of standard monomials of degree at most \( q \):
\[
S_{\prec}(I)_{\leq q} = S_{\prec}(I) \cap K[x]_{\leq q}.
\]
We claim that \( S_{\prec}(I)_{\leq q} \) is a \( K \)-vector space basis for the quotient space \( K[x]_{\leq q}/I_{\leq q} \). It is linearly independent since no \( K \)-linear combination of \( S_{\prec}(I) \) lies in \( I \). But, given that \( \prec \) is degree compatible, it also spans because taking the normal form of a polynomial modulo the Gröbner basis can only decrease the total degree.

Definition 1.34. Let \( I \) be an arbitrary ideal in a polynomial ring \( K[x] \). The function from \( \mathbb{N} \) to \( \mathbb{N} \) that associates to \( q \) the dimension of the quotient space \( \dim K[x]_{\leq q}/I_{\leq q} \) is known as the affine Hilbert function. We also define the Hilbert function \( h_I \) of \( I \) to be the Hilbert function of the initial ideal \( \text{in}_{\prec}(I) \), where \( \prec \) is any degree-compatible term order. For all \( q \in \mathbb{N} \) we have
\[
h_I(q) = h_{\text{in}_{\prec}(I)}(q) = |S_{\prec}(I)_{\leq q}| - |S_{\prec}(I)_{\leq q-1}| = \dim(K[x]_{\leq q}/I_{\leq q}) - \dim(K[x]_{\leq q-1}/I_{\leq q-1}).
\]
This is the number of standard monomials whose degree equals \( q \). This number is independent of \( \prec \), thanks to Lemma 1.33. Thus the affine Hilbert function \( q \mapsto \sum_{j=0}^{q} h_I(j) \) is determined by the Hilbert function and vice versa.

We also define the Hilbert series and the Hilbert polynomial to be that of any degree-compatible initial monomial ideal. Namely, we set
\[
(1.6) \quad \text{HS}_I(z) := \text{HS}_{\text{in}_{\prec}(I)}(z) \quad \text{and} \quad \text{HP}_I(q) := \text{HP}_{\text{in}_{\prec}(I)}(q).
\]
We similarly define the affine Hilbert series and the affine Hilbert polynomial of \( I \). These can also be defined by the formulas in (1.6), assuming we take \( \text{in}_{\prec}(I) \) in a polynomial ring \( K[x, y] \) that has one more dummy variable \( y \).

We define the dimension and degree of \( I \) as the dimension and degree of \( \text{in}_{\prec}(I) \). These concepts are now well-defined, thanks to Lemma 1.33.
Example 1.35. Let $I$ be a principal ideal generated by a polynomial $f$ of degree $e$ in $n \geq 1$ variables. The Hilbert series of $I$ equals $HP_I(q) = \frac{1-q^e}{(1-q)^n}$. The affine Hilbert series equals $1 - q^e (1-q)^{n+1}$. The dimension of $I$ is $n-1$. The degree of $I$ is $e$. This follows from Example 1.31 because the singleton $\{ f \}$ is a Gröbner basis, and its initial monomial $\text{in}_<(f)$ has degree $e$ in any degree-compatible monomial order $\prec$.

Remark 1.36. The prefix "affine" for the Hilbert function is important in order to distinguish affine varieties from projective varieties. We will discuss these geometric concepts in Chapter 2. Later on in the book, and elsewhere in algebraic geometry, it will usually be clear from the context whether the affine version or the projective version is meant. However, in the present first chapter we want to be precise and make that distinction.

What we have accomplished in this section is to give a purely combinatorial definition of dimension and degree of an ideal $I$. In Chapter 2 we shall see that notion of dimension agrees with the intuitive one for the associated algebraic variety $V(I)$. Namely, a variety has dimension 0 if and only if it consists of finitely many points. The number of these points is counted by the degree of the corresponding radical ideal. Likewise, the ideal of a curve has dimension 1, the ideal of a surface has dimension 2 etc. The degree is a measure for how curvy these shapes are. One can show that a prime ideal has degree 1 if and only if it is generated by linear polynomials.

Example 1.37. Fix the polynomial ring $K[x, y, z]$ and let $f = xyz - x^2 - y^2 - z^2 + 1$ as in (1.1). The ideal $\langle f \rangle$ has dimension 2 and degree 3. Let $I$ be the ideal generated by its partial derivatives, as in Example 1.7. Then $I$ has dimension 0 and degree 5. The ideal $I + \langle f \rangle$, whose zeros are the four singular points of the surface in Figure 1, has dimension 0 and degree 4.

Exercises

(1) Prove that an ideal in a polynomial ring $K[x]$ is principal if and only if its reduced Gröbner basis is a singleton.

(2) Draw the plane curve $\{ f = 0 \}$ that is defined by polynomial $f = 5x^3 - 25x^2y + 25y^3 + 15xy - 50y^2 - 5x + 25y - 1$. What do you observe?

(3) For $n = 2$, define a monomial order $\prec$ such that $(2, 3) \prec (4, 2) \prec (1, 4)$.

(4) Let $n = 2$ and fix the monomial ideals $I = \langle x, y^2 \rangle$ and $J = \langle x^2, y \rangle$. Compute the ideals $I + J$, $I \cap J$, $IJ$ and $I^3J^4 = I_{123}J_{234}J_{345}$. How many minimal generators does the ideal $I_{123}J_{234}$ have?
(5) The radical \( \text{rad} I \) of an ideal \( I \) in a ring \( R \) is the smallest radical ideal containing \( I \). Prove that the radical of a primary ideal is prime. For ideals in a polynomial ring \( K[x] \), prove that

- the radical of a principal ideal is principal;
- the radical of a monomial ideal is a monomial ideal.

(6) Show that the following inclusions always hold and are strict in general:

\[
\text{rad}(I) \cap \text{rad}(J) \subseteq \text{rad}(IJ) \quad \text{and} \quad \text{in}_<(\text{rad}(I)) \subseteq \text{rad}(\text{in}_<(I)).
\]

(7) Using Gröbner bases, find the minimal polynomials of \( \sqrt[3]{6} + \sqrt[5]{8} \) and \( \sqrt[3]{6} - \sqrt[5]{8} \). This is analogous to the fifth item in Example 1.18.

(8) Find the implicit equation of the curve \( \{(x^5 - 6, x^7 - 8) \in \mathbb{R}^2 : x \in \mathbb{R}\} \).

(9) Study the ideal \( I = \langle x^3 - yz, y^3 - zx, z^3 - xy \rangle \). Is it radical? If not, find \( \text{rad} I \). Regarding \( I \) as a triple of equations, what are its solutions in \( \mathbb{R}^3 \)?

(10) For the ideals \( I \) and \( \text{rad} I \) in the previous exercise, determine the Hilbert function, the Hilbert series, Hilbert polynomial, dimension, and degree. Redo the exercise with the adjective “affine” prepended where possible.

(11) Find an ideal in \( \mathbb{Q}[x, y] \) whose reduced Gröbner basis (in lexicographic order) has cardinality 5 and there are precisely 19 standard monomials.

(12) Let \( I \) be the ideal generated by the \( n \) elementary symmetric polynomials in \( x_1, \ldots, x_n \). Pick a monomial order and find the initial ideal \( \text{in}_<(I) \).

(13) Let \( X \) be a \( 2 \times 2 \)-matrix whose entries are variables. Let \( I_s \) be the ideal generated by the entries of the matrix power \( X^s \) for \( s = 2, 3, 4, \ldots \). Investigate these ideals. What are the dimension and the degree of \( I_s \)?

(14) A symmetric \( 3 \times 3 \)-matrix with unknown entries has seven principal minors: three of size \( 1 \times 1 \), three of size \( 2 \times 2 \), and one of size \( 3 \times 3 \). Does there exist an algebraic relation among these minors? Hint: lexicographic Gröbner basis.

(15) Prove that if \( \text{in}_<(I) \) is radical then \( I \) is radical. Does the converse hold?

(16) Determine all straight lines that lie on the cubic surface in Figure 1.1.

(17) Identify maximal, prime, radical and primary ideals in the ring \( R = \mathbb{Z} \).

(18) Let \( I \) be the ideal generated by all \( 2 \times 2 \) minors of a \( 2 \times n \) matrix filled with \( 2n \) variables. What is the degree and dimension of \( I \) for \( n = 2, 3, 4 \)?

(19) Find a prime ideal of degree three and dimension one in \( n \) variables for

- \( n = 2 \),
- \( n = 3 \),
- \( n = 3 \) with further assumption that \( h_I(1) = 4 \).
(20) Compute the dimension and degree of the ideal generated by two random homogeneous polynomials of degree four in \( n = 4 \) variables, as in Example $1.19$. Next drop the hypothesis “homogeneous” and redo.
Chapter 2

Varieties

“Geometry is but drawn algebra”, Sophie Germain

A variety is the set of solutions to a system of polynomial equations in several unknowns. These are the main objects of study in algebraic geometry. Varieties are the geometric counterparts to ideals in a polynomial ring. The latter live on the algebraic side. We distinguish between affine varieties and projective varieties. The former arise from arbitrary polynomials, while the latter are the zero sets of systems of homogeneous polynomials. Geometers prefer projective varieties because of their nice properties, explained in some of the results we present, like Theorem 2.22. But, for starters, our readers are invited to peruse the pictures shown in this chapter.

2.1. Affine Varieties

Algebraic varieties represent solutions of a system of polynomial equations. Fix a field $K$ and consider polynomials $f_1, \ldots, f_k$ in $K[x] = K[x_1, \ldots, x_n]$. The variety defined by these polynomials is the set of their common zeros:

$$V(f_1, \ldots, f_k) := \{ p = (p_1, \ldots, p_n) \in K^n : f_1(p) = \cdots = f_k(p) = 0 \}.$$

Different sets of polynomials can define the same variety. For instance,

$$V(f_1, f_2) = V(f_1^2, f_2^5) = V(f_1, f_1 + f_2).$$

Instead of thinking about the polynomials themselves, we consider the ideal they generate, $I = \langle f_1, \ldots, f_k \rangle$, and we define $V(I) := V(f_1, \ldots, f_k)$. A subset of $K^n$ is a variety if it has the form $V(I)$ for some ideal $I \subset K[x]$. Given any ideal $I \subset K[x]$, by Hilbert’s Basis Theorem [1.14] we can always
find a finite set of generators. By Exercise 1, the definition of \( \mathcal{V}(I) \) does not depend on the choice of generators of \( I \).

**Remark 2.1.** Two distinct ideals may define the same variety. Consider two non-constant homogeneous polynomials \( f_1 \) and \( f_2 \). Then the ideal \( \langle f_1^2, f_2^3 \rangle \) is strictly contained in the ideal \( \langle f_1, f_2 \rangle = \langle f_1, f_1 + f_2 \rangle \). But these two distinct ideals define the same variety in (2.1). Chapter 6 on the Nullstellensätze deals with this issue for fields \( K \) that are either algebraically closed, like the complex numbers \( K = \mathbb{C} \), or real closed, like the reals \( K = \mathbb{R} \). In the latter case, our variety is unchanged if we pass to the principal ideal \( \langle f_1^2 + f_2^3 \rangle \).

Algebraic geometry is the study of the geometry of varieties. As in many branches of mathematics, one considers the basic, irreducible building blocks for the objects of study. A variety \( \mathcal{V}(I) \) is called irreducible if it cannot be written as a finite union of proper subvarieties in \( K^n \). In symbols, \( \mathcal{V}(I) \) is irreducible if and only if, for any ideals \( J \) and \( J' \) in \( K[x] \) we have

\[
\mathcal{V}(I) = \mathcal{V}(J) \cup \mathcal{V}(J') \implies \mathcal{V}(I) = \mathcal{V}(J) \text{ or } \mathcal{V}(I) = \mathcal{V}(J').
\]

Any variety can be decomposed into irreducible varieties. The relevant algebraic tool is primary decomposition. This is our topic in the next chapter.

**Example 2.2.** Consider the ideal \( I = \langle xy \rangle \subset \mathbb{R}[x, y] \). Its variety \( \mathcal{V}(I) = \mathcal{V}(x) \cup \mathcal{V}(y) \) is a union of two lines in the plane \( \mathbb{R}^2 \). Hence, this is a reducible variety. Algebraically, \( I \) is the intersection of two larger ideals \( \langle x \rangle \) and \( \langle y \rangle \). Their respective varieties \( \mathcal{V}(x) \) and \( \mathcal{V}(y) \) are irreducible. This follows from Proposition 2.3 because \( \langle x \rangle \) and \( \langle y \rangle \) are prime ideals.

For any field \( K \), we can turn \( K^n \) into a topological space, using the Zariski topology. In this topology, the closed sets are the varieties in \( K^n \). In this setting, the definition of an irreducible variety coincides with the definition of an irreducible topological space. If \( K = \mathbb{R} \) or \( K = \mathbb{C} \) then we also have the classical Euclidean topology on \( K^n \). The Euclidean topology is much finer than the Zariski topology because it has many more open sets.

Our aim is to relate geometric properties of the variety \( \mathcal{V}(I) \) to algebraic properties of the ideal \( I \). Consider a maximal ideal of the form \( M := \langle x_1 - p_1, \ldots, x_n - p_n \rangle \) in \( K[x] \). The point \((p_1,\ldots,p_n)\) lies in \( \mathcal{V}(I) \) if and only if \( I \subseteq M \). Given any subset \( W \subset K^n \), we consider the set of all polynomials that vanish on \( W \). This set is a radical ideal, denoted

\[
\mathcal{I}(W) := \{ f \in K[x] : f(p) = 0 \text{ for all } p \in W \}.
\]

The set \( W \) is a variety if and only if \( W = \mathcal{V}(\mathcal{I}(W)) \). Furthermore, given any two varieties \( V \) and \( W \) in \( K^n \), we have \( V \subseteq W \) if and only if \( \mathcal{I}(W) \subseteq \mathcal{I}(V) \).

**Proposition 2.3.** A variety \( W \subset K^n \) is irreducible if and only if its ideal \( \mathcal{I}(W) \) is prime.
Proof. Suppose \( \mathcal{I}(W) \) is prime and \( W = \mathcal{V}(J) \cup \mathcal{V}(J') \). If \( W \neq \mathcal{V}(J) \) then there exists \( f \in J \) and \( v \in W \) such that \( f(v) \neq 0 \). Therefore, \( f \notin \mathcal{I}(W) \). For any \( g \in J' \) we know that \( fg \) vanishes on \( \mathcal{V}(J) \) and \( \mathcal{V}(J') \), hence on \( W \). Thus \( fg \in \mathcal{I}(W) \). As \( \mathcal{I}(W) \) is prime, we have \( g \in \mathcal{I}(W) \). We conclude that \( J' \subseteq \mathcal{I}(W) \). By Exercise 2.3 this implies \( W = \mathcal{V}(\mathcal{I}(W)) \subseteq \mathcal{V}(J') \).

For the converse, suppose that \( W \) is irreducible and \( fg \in \mathcal{I}(W) \). Hence
\[
W = W \cap \mathcal{V}(fg) = W \cap (\mathcal{V}(f) \cup \mathcal{V}(g)) = (W \cap \mathcal{V}(f)) \cup (W \cap \mathcal{V}(g)).
\]
Without loss of generality, \( W = W \cap \mathcal{V}(f) \). This means that \( W \subseteq \mathcal{V}(f) \) and hence \( f \in \mathcal{I}(W) \). This argument proves that \( \mathcal{I}(W) \) is a prime ideal. \( \square \)

Remark 2.4. Proposition 2.3 relates geometry and number theory. Prime ideals appear in applications as the constraints satisfied by a generative model. Such models are common in statistics. One considers a vector \( \theta \) of real parameters and one expresses probabilities (or moments of densities) as functions in \( \theta \). These functions are often polynomials or rational functions in \( \theta \), and one is interested in all valid polynomial constraints among the probabilities in question. Geometrically, this corresponds to computing the closure (in the Zariski topology) of the image of a polynomial map. This closure is an irreducible variety, so its ideal is prime by Proposition 2.3. That prime ideal represents the image and hence the generative model. It is computed as the kernel of the ring map dual to the polynomial map.

Example 2.5. We give an illustration for the most basic generative model, namely the independence model for two random variables \( X \) and \( Y \), each with state space \( \{1, \ldots, m\} \). Possible probability distributions of \( X \) (resp. \( Y \)) are represented by vectors \( (p_1, \ldots, p_m) \) (resp. \( (q_1, \ldots, q_m) \)) in \( \mathbb{R}^m \) with nonnegative entries that sum to 1. The probability that \( X \) (resp. \( Y \)) is in state \( i \) equals \( p_i \) (resp. \( q_i \)). The joint random variable \( (X,Y) \) has \( m^2 \) states. Under the assumption that \( X \) and \( Y \) are independent, all possible probability distributions of \( (X,Y) \) belong to a variety in \( \mathbb{R}^{m^2} \).

Consider the map that takes a distribution of \( X \) and a distribution of \( Y \) to the joint distribution of \( (X,Y) \). This map extends to a polynomial map
\[
\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m^2}, \quad (p_1, \ldots, p_m, q_1, \ldots, q_m) \mapsto (p_1 q_1, p_1 q_2, \ldots, p_1 q_m, p_2 q_1, \ldots, p_m q_m).
\]
In statistics one incorporates the requirement \( \sum p_i = \sum q_i = 1 \). We do so by restricting the domain. We write the resulting map explicitly for \( m = 3 \):
\[
(p_1, p_2, q_1, q_2) \mapsto (p_1 q_1, p_1 q_2, p_1 (1-q_1-q_2), p_2 q_1, p_2 q_2, p_2 (1-q_1-q_2), (1-p_1-p_2) q_1, (1-p_1-p_2) q_2, (1-p_1-p_2)(1-q_1-q_2)).
\]
In Exercise 9, we ask for the variety and ideal given by the image of this map.

The algebraic study of the independence model was a point of departure for the development of algebraic statistics. In that subject it is now common to use prime ideals to represent statistical models. This allows for the use of algebraic invariants (like dimension and degree) and algebraic methods (like Gröbner bases) for data analysis and inference. Readers wishing to learn more about algebraic statistics should consult the textbooks [19, 46, 55].

We have argued that prime ideals are basic building blocks in algebraic geometry and its applications. This motivates the following definition on the algebra side. We now take $R$ to be any commutative ring with unity.

**Definition 2.6.** The spectrum of the ring $R$ is the set of proper prime ideals:

$$
\text{Spec}(R) := \{ p \subseteq R : p \text{ is a prime ideal} \}.
$$

The set $\text{Spec}(R)$ is a topological space with the Zariski topology. Its closed sets are the varieties $\mathcal{V}(I) = \{ p \in \text{Spec } R : I \subseteq p \}$, where $I$ is any ideal in $R$.

The spectrum of the ring remembers a lot of information: all prime ideals and how they are related geometrically. Our most basic example of a ring $R$ is the polynomial ring $K[x]$ in $n$ variables over a field $K$. Its spectrum is a topological space with a very rich structure. Among the points of the spectrum are the usual ones $(p_1, \ldots, p_n) \in K^n$. These correspond to maximal ideals of the form $(x_1 - p_1, \ldots, x_n - p_n)$. However, the spectrum $\text{Spec}(K[x])$ has points corresponding to all irreducible subvarieties of $K^n$, not just those of dimension 0. In this manner, $K^n$ is a subset of $\text{Spec}(K[x])$.

Exercise 4 asks you to prove that the Zariski topology on $K^n$ is the induced one from the Zariski topology on $\text{Spec } K[x]$. An example of a point in $\text{Spec } \mathbb{R}[x]$ that is not of the form $\langle x - a \rangle$ for $a \in \mathbb{R}$ is the maximal ideal $\langle x^2 + 1 \rangle$. Indeed, $\mathbb{R}[x]/\langle x^2 + 1 \rangle \simeq \mathbb{C}$ by identifying $x$ with $i$.

Our next example is the coordinate ring of a variety $R = K[W]$ of a subvariety $W \subset K^n$. By definition, this is the quotient ring $R = K[x]/J$ where $J = \mathcal{I}(W)$ is the radical ideal in the polynomial ring $K[x]$ that encodes the variety. We interpret elements of $K[W]$ as polynomial functions on $W$. Indeed, as elements of $J$ vanish on $W$, the evaluation of $f \in K[W]$ on a point of $W$ does not depend on the choice of the representative polynomial. The prime ideals in $K[W]$ are in natural bijection with the prime ideals in $K[x]$ that contain $J$. Geometrically, these correspond to irreducible subvarieties of $W$. Among these are the points $(p_1, \ldots, p_n) \in W$, which correspond to maximal ideals $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$ in $K[W]$, just like before. The Zariski topologies on $W$ and $\text{Spec}(K[W])$ are compatible, in the sense of Exercise 4.

**Example 2.7.** A paraboloid in $\mathbb{R}^3$ is defined by the equation $z = x^2 + y^2$. Its ideal equals $J = \langle z - x^2 - y^2 \rangle$. The coordinate ring of the paraboloid
equals \( \mathbb{R}[x, y, z]/J \). Elements of this ring represent polynomial functions on the paraboloid. What are the Gröbner bases of \( J \) and what are the standard monomials? How about the dimension and the degree?

The Zariski topology on \( \text{Spec} \mathbb{R}[x, y, z]/J \) has various types of points. First, there are the classical real points on the surface. Second, there are pairs of complex conjugate points satisfying the equation \( z = x^2 + y^2 \). And, next there are all irreducible curves lying on the surface, one for each non-maximal prime ideal of \( \mathbb{R}[x, y, z] \) that contains \( J \). This includes curves that lie on the surface but have no real points, like that for \( \langle z^2 + 1, x^2 + y^2 - z \rangle \).

**Remark 2.8.** We continue the analogy from Remark 2.4 in order to give geometric intuition for the Chinese remainder theorem. Fix \( n_1, \ldots, n_k \in \mathbb{Z} \) pairwise coprime. In the language of varieties, the fact that \( \langle n_i \rangle + \langle n_j \rangle = \mathbb{Z} \) is equivalent to the fact that the associated varieties do not intersect—recall that the ideal of the intersection is the union of ideals. For each \( n_i \) we are given a residue class \( a_i \in \mathbb{Z}/n_i \mathbb{Z} \), i.e. a function on the variety associated to \( n_i \). As these varieties do not intersect, we expect to obtain a unique function on their union that restricts to the given functions on each piece.

The union of varieties is given by the intersection of the ideals, which corresponds to the product \( N = \prod_{i=1}^{k} n_i \). This is precisely the Chinese remainder theorem: there exists a unique \( a \in \mathbb{Z}/N\mathbb{Z} \) such that \( a = a_i \mod n_i \). We can push the analogy further. If the varieties intersect (i.e. the numbers are not pairwise coprime), then we expect the global function to exist if and only if the functions associated to the varieties agree on intersections.

Consider two varieties \( W_1, W_2 \) and a map \( f : W_1 \to W_2 \) between them. Given a function on the target variety, say \( g : W_2 \to K \), we define its pullback to be the function \( f^*(g) := g \circ f \) from \( W_1 \) to \( K \). Of course, here we are interested in polynomial functions, so \( g \) is an element of the ring \( K[W_2] \). Likewise, we want the pullback \( f^*(g) \) to be an element in the ring \( K[W_1] \).

Hence, given any polynomial map \( f : W_1 \to W_2 \) between varieties, we would like the map \( f^* : K[W_2] \to K[W_1] \) to be a well-defined ring morphism. In Exercise 2.9 you will show that any ring morphism \( K[W_2] \to K[W_1] \) induces a continuous map of topological spaces \( \text{Spec} \ K[W_1] \to \text{Spec} \ K[W_2] \). Hence, we may think about maps between varieties as homomorphisms between their rings of functions in the opposite direction. In the language of category theory, our star \( * \) is a contravariant functor from varieties to rings. It furnishes an equivalence of categories between irreducible affine varieties (over \( K \)) and finitely generated \( K \)-algebras that are integral domains.

**Example 2.9.** The following ring homomorphism is an isomorphism:

\[
 f^* : \mathbb{R}[x, y]/\langle y - x^2 \rangle \to \mathbb{R}[z], \ x \mapsto z, \ y \mapsto z^2.
\]
It arises from a map of varieties that takes a line to a parabola in the plane:

\[ f : \mathbb{R} \to \mathcal{V}(y - x^2) \subset \mathbb{R}^2, \quad \lambda \mapsto (\lambda, \lambda^2). \]

Under this parametrization of the parabola, the coordinate functions \( x \) and \( y \) on \( \mathbb{R}^2 \) pull back to the functions \( z \) and \( z^2 \) on the line \( \mathbb{R} \). We use the letter \( \lambda \) in the parametrization for extra clarity. It can get confusing when you pass to the map of spectra, a continuous map in the Zariski topologies.

**Remark 2.10.** Textbooks in algebraic geometry usually define affine varieties to be \( \text{Spec} \mathcal{R} \), with its Zariski topology, for any (commutative, with unity) ring \( \mathcal{R} \). Here \( \mathcal{R} \) need not be a finitely generated algebra over a field \( K \). For us, in this book, affine varieties are zero sets of polynomials in \( K[\!x] \).

The dependence on the field is crucial for geometric properties of maps.

Consider the squaring map \( K^1 \to K^1, \lambda \mapsto \lambda^2 \) from the affine line to itself.

- If \( K = \mathbb{C} \) then the squaring map is surjective.
- If \( K = \mathbb{R} \) then its image is the set of nonnegative real numbers. In both cases, the Zariski closure of the image is the whole line.
- If \( K = \mathbb{F}_p \) and \( p \neq 2 \), then the image is a proper subset of \( K^1 \). It is Zariski closed. What if we replace \( K \) by its algebraic closure?
- In each case, is the map \( \text{Spec}(\mathcal{R}[\!x]) \to \text{Spec}(\mathcal{R}[\!x]) \) surjective?

From the perspective of spectra, it is instructive to study the ideal \( I = \langle x^2 + 1 \rangle \) in \( \mathcal{R}[\!x] \). Exercise 6 asks the reader to give a description of \( \mathcal{V}(I) \).

**Example 2.11.** Consider the three ideals \( I_1 = \langle x^2 - y^2 \rangle \), \( I_2 = \langle x^2 - 2y^2 \rangle \), and \( I_3 = \langle x^2 + y^2 \rangle \) in \( \mathcal{R}[\!x, y] \). The first one is not prime for any \( K \). The second one is not prime for \( K = \mathbb{R} \) or \( \mathbb{C} \), but it is a prime ideal when \( K = \mathbb{Q} \). The ideal \( I_3 \) is not prime for \( K = \mathbb{C} \), but it is a prime ideal for \( K = \mathbb{Q} \) or \( \mathbb{R} \).

We prove the last statement. Suppose \( fg \in I_3 \subset \mathcal{R}[\!x, y] \). This means that \( fg = (x^2 + y^2) \cdot h \), where \( f, g, h \in \mathcal{R}[\!x, y] \). By the Fundamental Theorem of Algebra, every homogeneous polynomial \( p \) in two variables has a unique (up to multiplication by constants) representation as a product of linear forms with complex coefficients \( p = \prod l_j \). If \( p \) has real coefficients, then the decomposition is stable under complex conjugation: for every \( j \), either \( l_j \) has real coefficients or \( \overline{l_j} \) must also appear in the decomposition. We have \( x^2 + y^2 = (x + iy)(x - iy) \). In the ring \( \mathbb{C}[\!x, y] \), without loss of generality, we may assume \( (x + iy)|f \). But then, by the above argument also \( (x - iy)|f \). Thus \( f = (x + iy)(x - iy) \prod_i \overline{l_i} \) for \( l_i \in \mathbb{C}[\!x, y] \). However, \( \prod_i \overline{l_i} \) is stable under conjugation, i.e. defines a real polynomial. Thus \( x^2 + y^2 \) divides \( f \) in \( \mathcal{R}[\!x, y] \).

Many models arise in applications as the image of a polynomial map. It is important to note that the image need not be closed if \( K = \mathbb{C} \). And, it
need not be dense in its Zariski closure if $K = \mathbb{R}$. This will be discussed in
detail in Chapter 4. The following definition plays an important role.

**Definition 2.12.** A subset $A \subset K^n$ is **constructible** if it can be described
as a finite union of differences of varieties. Over the real numbers, a subset
$B \subset \mathbb{R}^n$ is **semi-algebraic** if it can be described as the solutions of a finite
system of polynomial inequalities (both $\geq$ and $>$) or a finite union of such.

**Remark 2.13.** Every constructible subset of $\mathbb{R}^n$ is semi-algebraic, but the
converse is not true. See below. The complement of a constructible set is
constructible, and the complement of a semi-algebraic set is semi-algebraic.

**Example 2.14.** Take $n = 2$ and $K = \mathbb{R}$. The singleton $\mathcal{V}(x,y) = \{(0,0)\}$
is constructible and hence so is $\mathbb{R}^2 \setminus \{(0,0)\} = \mathcal{V}(0) \setminus \mathcal{V}(x,y)$. The orthant
$\mathbb{R}_+^2 = \{(u,v) \in \mathbb{R}^2 : u \geq 0 \text{ and } v \geq 0\}$ is semi-algebraic. But it is not con-
structible, because the Euclidean closure of a constructible set is a variety.
Its complement $B = \{(u,v) \in \mathbb{R}^2 : u < 0 \text{ or } v < 0\}$ is also semi-algebraic.
Can you write $B$ as set of solutions to a finite list of polynomial inequalities?

The two most important invariants of a variety $V$ in $K^n$ are its **dimension**
and its **degree**. We defined these in Section 1.3, via the ideal $\mathcal{I}(V) \subset \mathbb{K}[x]$.

**Example 2.15.** Let $V$ be a linear subspace of $K^n$. The dimension of $V$ as a
variety equals its dimension as a linear space. The degree of $V$ is one. Indeed,
we may assume $\mathcal{I}(V) = \langle x_1, \ldots, x_s \rangle$ where $s = n - \dim(V)$. The result
follows from Example 1.22 because $K[x]/\langle x_1, \ldots, x_s \rangle \simeq K[x_{s+1}, \ldots, x_n]$.

Here is an important fact: if $V_1 \subsetneq V_2$ then $\dim(V_1) \leq \dim(V_2)$. The
inequality is strict if $V_2$ is irreducible. The latter is not easy to prove from
the definition we gave. It helps to consult a textbook in commutative algebra
for alternative (but equivalent) definitions of dimension, e.g. [4, Chapter 11].

Here is a method for computing the dimension of a variety $\mathcal{V}(I)$. Note that $\mathcal{V}(\text{in}_\prec(I))$ is a union of linear spaces in $K^n$, for any monomial order $\prec$.

1. Compute a Gröbner basis of $I$ and hence the monomial ideal $\text{in}_\prec(I)$.
2. Let $m_1, \ldots, m_k$ be monomials that generate $\text{in}_\prec(I)$. Find the small-
est (with respect to cardinality) set of variables $S = \{x_{i_1}, \ldots, x_{i_d}\}$
such that every generator $m_j$ is divisible by some variable in $S$.
3. The difference $n - d$ is the dimension of both $\mathcal{V}(\text{in}_\prec(I))$ and $\mathcal{V}(I)$.

The second most important invariant of a variety $V = \mathcal{V}(I) \subset K^n$ is the
degree. We now provide its geometric interpretation. A general subspace
$L \subset K^n$ with $\dim(L) + \dim(V) = n$ intersects $V$ in finitely many points.
For $K$ algebraically closed, their number is the degree of $V$. Indeed, this
follows inductively by the fact that a general linear polynomial is not a zero-
divisor in $K[V]$. Hence, adding it to $I$ changes the Hilbert function in such
2. Varieties

a way that the dimension drops by one and the degree remains the same. If \( \dim(I) = 0 \) and \( I \) is radical then the degree is the cardinality of \( V = V(I) \).

The following theorem is a generalization of the above considerations to polynomials of arbitrary degree. Its first statement, for two polynomials in two variables, goes back to Isaac Newton.

**Theorem 2.16** (Bézout’s theorem). Let \( f_1, \ldots, f_k \) be general polynomials in \( n \) variables of degrees \( d_1, \ldots, d_k > 0 \). For \( I = \langle f_1, \ldots, f_k \rangle \) we have \( \dim(I) = n - k \) and degree(\( I \)) = \( d_1 \cdots d_k \).

As the reader may guess, the crucial point is that each \( f_i \) is not a zero divisor in the ring \( K[x]/\langle f_1, \ldots, f_{i-1} \rangle \). For the complete proof we refer to [51, Section IV.2]. Note that we always have \( \dim(\langle f_1, \ldots, f_k \rangle) \geq n - k \). If equality holds the ideal \( I \) is called a complete intersection.

Some points on a variety are singular, like the four nodes of the cubic surface in Figure 1. Our aim is now to discuss singularities in general. We start with the case of a hypersurface, defined by one polynomial \( f \in K[x] \).

A point \( p \in V(f) \) is singular if all partial derivatives vanish, i.e. \( \frac{\partial f}{\partial x_i}(p) = 0 \) for \( i = 1, \ldots, n \). Thus the singular locus of \( f \) is the variety of the ideal \( \langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle \). If this ideal has no zeros, we say that the hypersurface \( V(f) \) is smooth. Smoothness is a very important condition. It tells us that our variety can be locally approximated by a linear space - the tangent space.

Let \( I = \langle f_1, f_2, \ldots, f_k \rangle \subset K[x] \) be a prime ideal, defining an irreducible variety \( Y = V(I) \) in \( K^n \) of dimension \( d \). A point \( p \in Y \) is singular if and only if the rank of the Jacobian matrix at \( p \) is smaller than the codimension:

\[
\text{rank} \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n}
\end{pmatrix}(p) < n - d.
\]

This definition does not depend on the choice of ideal generators (Exercise [14]). A point that is not singular is called smooth. For a smooth point the inequality above turns into equality. The singular locus \( \text{Sing}(Y) \) is a variety in \( K^n \). It may be defined by the ideal that is the sum of \( I \) and the ideal generated by \( (n - d) \times (n - d) \) minors of the Jacobian matrix. The kernel of this matrix evaluated at the point \( p \) is, by definition, the vector space parallel to the tangent space to \( V \) at \( p \). The definition of smooth point assures that the tangent space and the variety have the same dimension.

If a variety \( X \subset K^n \) is reducible and \( p \) lies in more than one irreducible component of \( X \), then \( p \) is singular in \( X \). If \( p \) belongs to a unique irreducible component \( Y \) then \( p \) is singular in \( Y \) if and only if it is singular in \( Y \).
2.2. Projective Varieties

By Bézout’s theorem we know the dimension and degree of an ideal \( I \) that is generated by general polynomials. Bertini’s theorem tells us that such an ideal \( I \) is prime and its variety \( \mathcal{V}(I) \) is smooth, provided \( \dim(I) > 0 \).

2.2. Projective Varieties

The geometric objects we encountered so far are subsets of \( K^n \). We called them varieties, but more precisely we should refer to them as affine varieties. We now change our perspective by focusing on projective varieties.

We start by recalling the construction of the projective space \( \mathbb{P}(V) \) over a vector space \( V \) of dimension \( n+1 \). The points of \( \mathbb{P}(V) \) are the lines through the origin in \( V \). The symbol \( [a_0 : \cdots : a_n] \in \mathbb{P}(V) \) denotes the line that goes also through the point \((a_0,\ldots,a_n) \in V \). Here not all \( a_i \) are zero.

Formally, \( \mathbb{P}(V) \) is the set of equivalence classes \([v] \), for \( v \in V \setminus \{0\} \), modulo the relation \( v_1 \sim v_2 \) if and only if \( v_1 = \lambda v_2 \) for some \( \lambda \in K^* = K \setminus \{0\} \). For the topological construction over \( \mathbb{R} \) or \( \mathbb{C} \), we note that each line through the origin in \( V \) intersects the unit sphere precisely in two points. Thus \( \mathbb{P}(V) \) may be regarded as a quotient of the sphere, identifying antipodal points. In particular, \( \mathbb{P}(V) \) is compact in the classical topology. On the subset \( S_i = \{a_i \neq 0\} \) of \( \mathbb{P}(V) \) we rescale to get \( a_i = 1 \). We thus identify \( S_i \) with \( K^n \). The affine spaces \( S_i = K^n \) cover \( \mathbb{P}^n := \mathbb{P}(V) \), because every point has some nonzero coordinate. We obtain \( \mathbb{P}^n \) by gluing these \( n+1 \) charts.

As before, we are interested in functions on \( \mathbb{P}^n \). The first problem is that, for a polynomial \( f \), it does not make sense to evaluate \( f \) on \([a_0 : \cdots : a_n]\), as the result depends on the choice of representative. It may even happen that \( f \) vanishes for some representatives, but not for others. Thus, we focus on homogeneous polynomials, i.e. linear combinations of monomials of fixed degree. If \( f \) is a homogeneous polynomial of degree \( d \) in \( n+1 \) variables, then \( f(ta_0,\ldots,ta_n) = t^d f(a_0,\ldots,a_n) \). In particular, \( f \) vanishes on some representative of \([a_0 : \cdots : a_n]\) if and only if it vanishes on any representative.

Let \( f_1,\ldots,f_k \) be homogeneous polynomials in \( K[x] \). They are allowed to have distinct degrees. We define the associated projective variety:

\[
\mathcal{V}(f_1,\ldots,f_k) = \{[a_0 : \cdots : a_n] \in \mathbb{P}(V) : f_i(a_0,\ldots,a_n) = 0 \text{ for } i = 1,\ldots,k\}.
\]

An ideal \( I \) in \( K[x] \) is homogeneous if it is generated by homogeneous polynomials \( f_1,\ldots,f_k \). Just like in the affine case, we set \( \mathcal{V}(I) := \mathcal{V}(f_1,\ldots,f_k) \).

**Remark 2.17.** Homogeneous ideals contain (many) nonhomogeneous polynomials. For instance, \( \langle x + y^2, y \rangle \) is a homogeneous ideal. See Exercise [11]

For any projective variety \( X \subset \mathbb{P}^n \) one defines the affine cone \( \hat{X} \) over it, i.e. the variety defined by the same ideal, but in \( V = K^{n+1} \). The dimension
and degree of a projective variety can be defined via its affine cone:

$$\dim(X) := \dim(\hat{X}) - 1 \quad \text{and} \quad \deg(X) := \deg(\hat{X}).$$

It is usually preferable to work with projective varieties. Algebraic geometry is simpler in $\mathbb{P}^n$ than in $K^n$. For instance, parallel lines in $K^2$ do not intersect, but any two lines in $\mathbb{P}^2$ intersect. If $X$ is any projective variety of degree $\geq 2$ then the affine cone $\hat{X}$ is always singular at the point $0 \in V$. However, if this is the only singular point of $\hat{X}$ then $X \subset \mathbb{P}^n$ is smooth.

If $Y$ is any variety in $K^n$ then there is an associated projective variety $\bar{Y}$ in $\mathbb{P}^n$, called the projective closure of $Y$. This is defined via its ideal. If $I \subset K[x_1, \ldots, x_n]$ is the ideal of $Y$ then the ideal $\bar{I}$ of $\bar{Y}$ lives in $K[x_0, x_1, \ldots, x_n]$. It is generated by the following infinite set of homogeneous polynomials:

$$\left\{ x_0^{\deg(g)} \cdot g \left( \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right) : g \in I \right\}.$$

Here is an algorithm for computing the ideal $\bar{I}$ of the projective closure $\bar{Y}$.

**Proposition 2.18.** Let $I$ be an ideal in $K[x_1, \ldots, x_n]$ and let $\mathcal{G}$ be its reduced Gröbner basis for a degree-compatible monomial ordering. Then $\bar{I}$ is generated by the homogeneous polynomials in (2.5) where $g$ runs over $\mathcal{G}$.

**Proof.** Let $f = f(x_0, x_1, \ldots, x_n)$ be any homogeneous polynomial in $\bar{I}$. Suppose $\mathcal{G} = \{g_1, g_2, \ldots, g_s\}$. The dehomogenization $f(1, x_1, \ldots, x_n)$ lies in $I$ and hence its normal form modulo the Gröbner basis $\mathcal{G}$ is zero. This gives a representation

$$f(1, x_1, \ldots, x_n) = \sum_{i=1}^{s} h_i(x_1, \ldots, x_n) g_i(x_1, \ldots, x_n),$$

where $\deg(h_i g_i) \leq \deg(f)$ for all $i$. By homogenizing the summands in this identity we obtain:

$$f(x_0, x_1, \ldots, x_n) = \sum_{i=1}^{s} x_0^{\deg(f)} h_i \left( \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right) \cdot g_i \left( \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right).$$

Hence, $f$ lies in the ideal generated by the set (2.5) with $I$ replaced by $\mathcal{G}$. \Box

**Corollary 2.19.** The dimension and degree of an affine variety $Y \subset K^n$ are preserved when passing to its projective closure $\bar{Y} \subset \mathbb{P}^n$:

$$\dim(\bar{Y}) = \dim(Y) \quad \text{and} \quad \deg(\bar{Y}) = \deg(Y).$$

**Proof.** The initial ideal of $\bar{I}$ and the initial ideal of $I$ have the same generators. These monomials in $x_1, \ldots, x_n$ determine dimension and degree. Note that the ideal $\bar{I}$ belongs to a ring with one more variable which increases dimension by one and does not change the degree. But, by the definition
2.2. Projective Varieties

in (2.4), the dimension of the projective variety is one less than that of the affine cone over it. This implies the corollary. □

Example 2.20. Let $I$ be the ideal generated by $x_i - x_1$ for $i = 2, 3, \ldots, n$. Then $Y = \mathcal{V}(I)$ is a curve of degree $n$ in $K^n$. For the degree reverse lexicographic monomial order $\prec$, the reduced Gröbner basis has $\binom{n}{2}$ elements, and $\text{in}_{\prec}(I) = \langle x_1, x_2, \ldots, x_{n-1} \rangle^2$. The ideal $\bar{I}$ is minimally generated by the $2 \times 2$-minors of the $2 \times n$ matrix

\[
\begin{pmatrix}
  x_0 & x_1 & x_2 & \cdots & x_{n-1} \\
  x_1 & x_2 & x_3 & \cdots & x_n
\end{pmatrix}.
\]

The initial monomials of the $\binom{n}{2}$ minors are the antidiagonal products. The projective variety $\bar{Y} = \mathcal{V}(\bar{I})$ is the rational normal curve of degree $n$ in $\mathbb{P}^n$.

We now return to discussing desirable properties of projective varieties.

Remark 2.21. If $K = \mathbb{C}$ or $K = \mathbb{R}$ then every projective variety is compact in the classical topology. Indeed, the projective space $\mathbb{P}^n$ is compact, and every subvariety $X$ is closed in the classical topology. Hence $X$ is compact. If $X$ is also smooth of dimension $d$ then $X$ is a compact real manifold, of dimension $d$ if $K = \mathbb{R}$ and of dimension $2d$ if $K = \mathbb{C}$. Many interesting manifolds arise in this manner. Well-known examples are one dimensional complex varieties, which give rise to Riemann surfaces. The case of elliptic curves corresponding to tori is discussed in the next Chapter 2.3.

One nice property of projective varieties is that their images under polynomial maps are closed. This is known as Chevalley’s Theorem and we will see the precise statement in Theorem 4.20. Another nice fact about projective varieties is the following result about their intersections.

Theorem 2.22. Fix an algebraically closed field $K$. Let $X, Y$ be two projective varieties in the $n$-dimensional ambient space $\mathbb{P}^n$, where $d_1 = \dim(X)$ and $d_2 = \dim(Y)$. Then their intersection $X \cap Y$ has dimension at least $d_1 + d_2 - n$. In particular, if $d_1 + d_2 \geq n$ then $X \cap Y$ is always non-empty.

The hypotheses are needed in this theorem. Consider the intersection of two surfaces in $\mathbb{P}^3$, where $n = 3$ and $d_1 = d_2 = 2$. Its dimension is at least $d_1 + d_2 - n = 1$. The statement fails in affine space $\mathbb{C}^3$ where we can take two parallel planes. It also fails in $\mathbb{P}^3$ if the field is $K = \mathbb{R}$.

Example 2.23. Consider the two surfaces $X = \mathcal{V}(x_0^2 + x_1^2 - x_2^2 + x_3^2)$ and $Y = \mathcal{V}(x_0^2 + x_1^2 + x_2^2 - x_3^2)$ in $\mathbb{P}^3$. Over $\mathbb{C}$, their intersection is the union of four lines, so $\dim(X \cap Y) = 1$ as expected. However, over $\mathbb{R}$, the intersection consists of two points, so $\dim(X \cap Y) = 0$, which would violate Theorem 2.22.
Many models in the sciences and engineering are given by homogeneous polynomial equations. Typically, these constraints arise from a construction familiar from linear algebra. Whenever one encounters such a model then it makes much sense to regard it as a projective variety. We close this section with two examples.

**Example 2.24** (Nilpotent Matrices). An \(n \times n\)-matrix \(A = (a_{ij})\) is a point in a projective space \(\mathbb{P}^{n^2-1}\). The set of nilpotent matrices \(A\) is an irreducible projective variety \(X \subset \mathbb{P}^{n^2-1}\). We have \(\dim(X) = n^2 - n - 1\) and \(\text{degree}(X) = n!\). Indeed, \(X\) is a complete intersection. Its prime ideal \(\mathcal{I}(X)\) is generated by the coefficients of the characteristic polynomial of \(A\), except the leading one which is a unit. For instance, if \(n = 2\) then \(\mathcal{I}(X) = (\text{trace}(A), \det(A))\).

**Example 2.25** (Kalman Varieties). In control theory, one is interested in the set of \(n \times n\)-matrices \(A\) that have an eigenvector in a given linear subspace of \(K^n\). This set is a projective variety in \(\mathbb{P}^{n^2-1}\). For instance, let \(n = 4\) and consider \(4 \times 4\)-matrices that have an eigenvector with the last two coordinates zero. This Kalman variety has dimension 13 and degree 4 in \(\mathbb{P}^{15}\). It is defined by the \(2 \times 2\)-minors of
\[
\begin{pmatrix}
a_{31} & a_{41} & a_{11}a_{31} + a_{21}a_{32} + a_{31}a_{33} + a_{34}a_{41} & a_{11}a_{41} + a_{21}a_{42} + a_{31}a_{43} + a_{41}a_{44} \\
a_{32} & a_{42} & a_{12}a_{31} + a_{22}a_{32} + a_{32}a_{33} + a_{34}a_{42} & a_{12}a_{41} + a_{22}a_{42} + a_{32}a_{43} + a_{42}a_{44}
\end{pmatrix}.
\]

2.3. Geometry in Low Dimensions

Smooth projective varieties in low dimensions furnish interesting manifolds. Studying the geometry and topology of these manifolds leads to valuable insights that are useful also for understanding higher-dimensional scenarios.

We work in projective spaces over the real numbers \(\mathbb{R}\) and over the complex numbers \(\mathbb{C}\). To distinguish these, we use the notations \(\mathbb{P}^n_\mathbb{R}\) and \(\mathbb{P}^n_\mathbb{C}\). We regard both of these as compact real manifolds, of dimension \(n\) and \(2n\) respectively. Students of topology are encouraged to review the homology groups of the manifolds \(\mathbb{P}^n_\mathbb{R}\) and \(\mathbb{P}^n_\mathbb{C}\).

Let us start with \(n = 1\). The real projective line \(\mathbb{P}^1_\mathbb{R}\) is a circle. The complex projective line \(\mathbb{P}^1_\mathbb{C}\) is a sphere, known as the *Riemann sphere*. Every proper subvariety of \(\mathbb{P}^1_\mathbb{R}\) or \(\mathbb{P}^1_\mathbb{C}\) is a finite collection of points, defined by a *binary form* \(f(x, y)\), i.e. a homogeneous polynomial in two variables. For instance, let \(f = x^{11}y - 11x^6y^6 - xy^{11}\). The variety \(\mathcal{V}(f)\) has dimension 0 and degree 12 in \(\mathbb{P}^1_\mathbb{C}\). These 12 points on the Riemann sphere are famous in the history of geometry and arithmetic. They serve as the vertices of the icosahedron in Felix Klein’s *Lectures on the Icosahedron*. Out of these 12 complex solutions four are real. The remaining eight come in four conjugate pairs.
We now move on to $n = 2$. The real projective plane $\mathbb{P}_R^2$ is a surface. However, it cannot be embedded homeomorphically in $\mathbb{R}^3$ (only in $\mathbb{R}^4$), thus it is impossible to make a good picture. The simplest curve in the projective plane $\mathbb{P}_K^2$ is a line $L$, defined by one linear form in three variables. Of course, $L$ is a projective line $L \simeq \mathbb{P}_K^1$, so the discussion in the previous paragraph applies. The complement $\mathbb{P}_K^2 \setminus L$ is the affine plane $K^2$. In particular, this complement is connected when $K = \mathbb{R}$. The decomposition into $L$ and $K^2$ may be used to give a schematic picture of $\mathbb{P}_R^2$. We identify $\mathbb{R}^2$ with the interior of the square. The boundary of the square should represent the line $L$. However, we need to identify opposite points of the boundary - this is often represented by putting directed arrows on the boundary as in Figure 1.

For any curve $C$ in $\mathbb{P}_C^2$, the complement $\mathbb{P}_C^2 \setminus C$ is connected because $C$ is a surface in the 4-dimensional manifold $\mathbb{P}_C^4$. By contrast, consider a smooth conic, i.e. a curve of degree two, $C \subset \mathbb{P}_R^2$. Then $\mathbb{P}_R^2 \setminus C$ has two connected components. One is a disk and the other is a Möbius strip (cf. Figure 2). The former is the inside of $C$ and the latter is the outside of $C$. A curve $D$ in $\mathbb{P}_R^2$ is called a pseudoline if $\mathbb{P}_R^2 \setminus D$ is connected and it is an oval otherwise. Every oval behaves like a conic in $\mathbb{P}_R^2$: it has an inside and an outside.

**Theorem 2.26.** Let $C$ be a smooth curve of degree $d$ in the projective plane $\mathbb{P}_K^2$. If $K = \mathbb{C}$ then $C$ is an orientable surface of genus $g = \frac{(d-1)(d-2)}{2}$. If $K = \mathbb{R}$ then $C$ is a curve with at most $g + 1$ connected components. If $d$ is even then all components are ovals. If $d$ is odd then one component is a pseudoline but all others are ovals.

**Proof.** The first part of the theorem is standard and may be found in many classical books [28, I Ex. 7.2], [32]. For the statements about real curves we refer the reader to [6 Section 6, Chapter 12].
Let us illustrate the above theorem for $d = 3$, i.e. $g = 1$. For example, consider a smooth cubic curve given in its Weierstrass form, such as
\[ f(x, y, z) = zy^2 - x^3 + xz^2. \]
Such curves are also known as elliptic curves. We decompose the projective plane $\mathbb{P}_R^2$ into a line $L$ given by the equation $z = 0$ and its complement: the affine plane $A = \mathbb{R}^2$. The cubic has two components: an oval and a pseudoline. We can see both of them by intersecting $C$ with $A$, as depicted in Figures 3 and 4.

There is an additional point $P$ of the curve we do not see in Figure 4, that belongs to the line $L$. It is given by $z = x = 0$ and $y = 1$. We may consider the surface in $\mathbb{R}^3$ that is the affine cone over our curve, as in Figure 4.

How can we imagine the complex elliptic curve? This is not simple, as the correct picture just of the affine part would be in $\mathbb{C}^2 \simeq \mathbb{R}^4$. However, there exists a homeomorphism (but not a polynomial map!) of the complex curve $C$ with the real topological torus, i.e. the product of two circles $S^1 \times S^1$. It can be described as follows. We fix a point $p \in C$. For any point $q \in C$ consider a path $\gamma$ from $p$ to $q$. This is always possible since $C$ is connected over the complex numbers $C$. To a point $q$ we associate the complex number $\int_{\gamma} \frac{dx}{y}$. Identifying the complex plane with $\mathbb{R}^2$ we obtain a map $f : C \rightarrow \mathbb{R}^2$. It turns out that $f(q)$ depends on our choice of $\gamma$. Indeed, let us choose $p$ given by $z = 1$, $y = 0$ and $x = -1$. We may choose $q = p$ and $\gamma$ equal to the oval depicted in Figure 3. The integral $\int_{\gamma} \frac{dx}{y}$ will be a nonzero real number $\lambda$. Thus $f(p)$ may be equal to any integral multiple of $\lambda$. Further, on the
2.3. Geometry in Low Dimensions

Figure 3. Real elliptic curve $y^2 = x^3 - x$. The connected component on the left is an oval. The connected component on the right is a pseudoline. For a more complete picture see Figures 4 and 5.

Figure 4. The cone $zy^2 - x^3 - xz^2 = 0$ over an elliptic curve. The irreducible variety is depicted in orange and blue according to two connected components in the real projective space. We intersect the cone with the grey plane given by $z = 1$. This corresponds to the affine chart, and the blue curve $C$ we obtain is exactly the same as in Figure 5.
curve $C$ there exists another loop $\gamma'$ giving rise to the integral $\int_{\gamma'} \frac{dx}{y}$ that is a complex number $\tau$. We consider the lattice $M$ in $\mathbb{C} \simeq \mathbb{R}^2$ whose elements are the integral combinations $a\lambda + b\tau$ for $a, b \in \mathbb{Z}$. We know that $f(p)$ may be any point in $M$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/M$ be the natural projection. The map $\pi \circ f : C \to \mathbb{R}^2/M$ is now well-defined!

As $\mathbb{R}^2/M$ may be identified with the torus, we indeed obtain a homeomorphism $C \simeq \mathbb{R}^2/M$. The real part of the curve $C$ is mapped to two disjoint circles, as shown in Figure 6. Indeed, both the oval and the pseudoline are circles - they are only distinguished by their embedding into $\mathbb{P}^2_{\mathbb{R}}$. 

Figure 5. The cone $zy^2 - x^3 - xz^2 = 0$ over an elliptic curve, as in Figure 4. The grey sphere represents the projective space $\mathbb{P}^2_{\mathbb{R}}$, where we have to identify the antipodal points. The intersection of the surface with the sphere has three connected components. Two of them are identified, when we identify the antipodal points. These two components correspond to the oval – indeed cutting it out of the sphere, separates it into two pieces, even after identifying antipodal points. The other component corresponds to a pseudoline. It does not separate the sphere after identifying antipodal points. The points on the blue curve $C$ in Figure 4 correspond to pairs of antipodal points on the blue curve in this picture with one exception. This curve has one more pair of antipodal points - these are represented by a thickened blue point. Indeed, the line through that point is parallel to the grey plane in Figure 4. This point corresponds to the unique point of the projective curve that does not belong to the affine chart given by $z = 1$. It is $z = 0, x = 0, y = 1$. 

2. Varieties
2.3. Geometry in Low Dimensions

Figure 6. Two pictures showing a real torus, that is homeomorphic to the elliptic curve. The left picture presents the torus as $\mathbb{R}^2/\mathbb{Z}^2$. The right one is the familiar figure we know from topology. The two thickened circles, on both pictures, correspond to the real part of the curve.

Remark 2.27. We contrast the topological torus mentioned here with the algebraic torus $(\mathbb{C}^\ast)^n$ playing a central role in Chapters 8 and 10. Indeed, a variety with a dense algebraic torus action will be called toric. The elliptic curve $C$ is a basic example of a smooth, projective variety that is not toric.

Remark 2.28. Elliptic curves make probably their first appearance in third century. Diophantus of Alexandria, in modern terms, asked for a (positive) rational point on a specific elliptic curve $y(6 - y) = x^3 - x$. As we argued above, an elliptic curve has a structure of a group (torus). The geometric interpretation of this was already well-known in 19th century. Since early 20th century, elliptic curves play a central role in (modern) number theory (studied mainly over fields of finite characteristic or rational numbers). By the end of the 20th century the group structure (over fields with finite characteristic!) started to be intensively used in applied cryptography.

Example 2.29. Let us consider a cuspidal curve defined by $x^3 - y^2$. Over $\mathbb{R}$ it is presented in Figure 7 on the left. How can we draw it over $\mathbb{C}$? If we identify $\mathbb{C}$ with $\mathbb{R}^2$ we obtain a surface in $\mathbb{R}^4$. Indeed:

$$(x_1 + ix_2)^3 - (y_1 + iy_2)^2 = 0 \iff x_1^3 - 3x_1x_2^2 = y_1^2 - y_2^2 \text{ and } 3x_1^2x_2 - x_2^3 = 2y_1y_2.$$ 

Hence, interpreted as a surface in $\mathbb{R}^4$, the variety is cut out by two polynomials. Although we may not make a picture in $\mathbb{R}^4$ we can project the given surface to $\mathbb{R}^3$. The result is the surface presented in Figure 7 on the right, together with the black line, that is the real part.

The surface seems more singular - this is the result of projection. The original surface in $\mathbb{R}^4$ has just one singular point. In Chapter 4 methods allowing the computation of projections of algebraic varieties are presented.
Exercises

(1) Prove that the definition of \( \mathcal{V}(I) \) does not depend on the choice of the generators of \( I \).

(2) (a) Show that \( J \subseteq I \) implies \( \mathcal{V}(I) \subseteq \mathcal{V}(J) \).
   
   (b) Show that for any subsets \( A, B \subseteq K^n \) if \( A \subseteq B \) then \( \mathcal{I}(B) \subseteq \mathcal{I}(A) \).
   
   (c) Give counterexamples to both opposite implications.

(3) Prove that varieties (in \( K^n \)) satisfy the axioms of closed sets.

(4) By identifying the point \((p_1, \ldots, p_n) \in K^n\) with the prime ideal \( \langle x_1 - p_1, \ldots, x_n - p_n \rangle \), we consider \( K^n \) as a subset of \( \text{Spec} \, K[x] \). Show that the Zariski topology induced from \( \text{Spec} \, K[x] \) is the Zariski topology on \( K^n \).

(5) Show that a morphism of rings \( f: R_1 \to R_2 \) induces a map \( f^*: \text{Spec} \, R_2 \to \text{Spec} \, R_1 \), by proving that the preimage of a prime ideal is prime. Show that the induced map is continuous with respect to the Zariski topology.

(6) Describe the variety \( \mathcal{V}(I) \) in the affine line \( K^1 \) for \( I = \langle x^2 + 1 \rangle \) when \( K = \mathbb{C}, \mathbb{R}, \mathbb{Q} \). Also, describe \( \mathcal{V}(I) \subset \text{Spec}(K[x]) \) for each of these fields.

(7) Realize the set of \( n \times n \) nilpotent matrices as an affine variety. What is its dimension? What is its degree?
(8) (a) Consider a polynomial \( f \in K[x] \) (e.g. \( f = x \)). Let \( D \) be the (open) set \( D_f = \{ p \in K^n : f(p) \neq 0 \} \). Construct an affine variety \( V \) and a polynomial map inducing a bijection \( V \to D \).
(b) Realize nondegenerate \( n \times n \) matrices as an affine variety.

(9) (a) Use (or not) your favorite computer algebra system to determine the ideal of the image of the map given by formula (2.3). What is the meaning of the lowest degree polynomial in this ideal?
(b) Describe the ideal of the image of the map given by formula (2.2).
(c) Generalize the previous point to more (independent) variables possibly with different (but finite) number of states.

(10) Determine the set of all prime numbers \( p \) such that \( I = \langle x^2 - 2y^2 \rangle \) is a prime ideal in the polynomial ring \( F_p[x,y] \) over the finite field \( F_p \).

(11) For a polynomial \( f = \sum_a c_a x^a \) we call the degree \( k \) part of \( f \) the homogeneous polynomial \( \sum_{a: |a| = k} c_a x^a \).
(a) Provide an example of a homogeneous ideal generated by nonhomogeneous polynomials.
(b) Prove that an ideal \( I = \langle f_1, \ldots, f_j \rangle \) is homogeneous if and only if for any \( i \) and \( k \) the degree \( k \) part of \( f_i \) belongs to \( I \).
(c) Propose an algorithm that, given a set of generators of \( I \subset K[x] \), decides if \( I \) is a homogeneous ideal.

(12) (a) Let \( I \subset K[x] \) be a monomial ideal. Prove that \( \mathcal{V}(I) \) is a union of (some) vector subspaces of \( K^n \) spanned by basis vectors.
(b) How do you characterize the sets of basis vectors that span a subspace belonging to \( \mathcal{V}(I) \)?

(13) Draw various pseudolines in \( \mathbb{P}_C^2 \), in analogy to Figure 2. Topologically, what is the complement of a pseudoline?

(14) Prove that the rank of the Jacobian matrix does not depend on the choice of the generators of the ideal.

(15) Can you solve the problem of Diophantus of Alexandria in Remark 2.28?
Hint: Consider a tangent line to the elliptic curve at the point \((-1,0)\).

(16) Extending Example 2.25, determine the Kalman variety of all \( 6 \times 6 \) matrices that have an eigenvector with the last three coordinates zero.

(17) Let \( X \subset \mathbb{C}^4 \) be the variety defined by two random polynomials of degree three in four variables. What do we learn about \( X \) from Bézout’s Theorem, and what do we learn about \( X \) from Bertini’s Theorem?
Solving systems of polynomial equations is a key task in nonlinear algebra. But, what does it mean to solve such a system? How should the solutions be presented? The answer to this question depends on the dimension of the variety of solutions. If the variety is zero-dimensional then it consists of finitely many points in $K^n$ and we aim to list each point explicitly. If $K = \mathbb{R}$ or $K = \mathbb{C}$ then this is usually done by displaying a floating point approximation to each of the $n$ coordinates of a solution.

If the solution variety has positive dimension then it has infinitely many points and we cannot list them all. In that case, the answer consists of a description of each irreducible component. Algebraically, this leads us to the topic of primary decomposition. If the given ideal is not radical then its constituents are primary ideals and we distinguish between minimal primes and embedded primes. To some readers, these objects may seem unnatural at first. However, they become quite natural when we interpret multivariate polynomials as linear partial differential equations with constant coefficients.

### 3.1. Zero-dimensional Ideals

Let $K$ be a field and consider the polynomial ring $K[x]$ in one variable $x$. Every ideal in $K[x]$ is principal, so it has the form $I = (f)$. The variety $\mathcal{V}(I)$ consists of the zeros of $f$ and is zero-dimensional (unless $f = 0$). The polynomial $f$ has a unique factorization $f = \prod_{i=1}^{k} g_i^{a_i}$, where each $g_i$ is irreducible.
and the \( a_i \) are positive integers. The set of solutions decomposes as

\[
\mathcal{V}(I) = \mathcal{V}(g_1) \cup \cdots \cup \mathcal{V}(g_k).
\]

On the level of ideals we have the following decomposition as an intersection:

\[
I = \langle g_1 \rangle^{a_1} \cap \cdots \cap \langle g_k \rangle^{a_k}.
\]

This primary decomposition remembers the multiplicity \( a_i \) of each factor \( g_i \), so it contains more information than the irreducible decomposition of \( \mathcal{V}(I) \).

The decomposition depends on the field \( K \). If \( K \) is algebraically closed, such as \( K = \mathbb{C} \), then each factor \( g_i \) is a linear polynomial \( g_i(x) = x - u_i \), where \( u_1, \ldots, u_k \) are the zeros of \( f \). If \( K = \mathbb{R} \) then each \( g_i \) is either linear or quadratic. If \( K = \mathbb{Q} \) then \( g_i \) can have arbitrarily high degree. In each case, the quotient ring \( K[x]/\langle g_i \rangle \) is a field. It is an algebraic extension of \( K \).

**Example 3.1.** The polynomial \( f = x^3 - 2x^2 + x - 2 \in \mathbb{R}[x] \) satisfies

\[
\langle f \rangle = \langle x - 2 \rangle \cap \langle x^2 + 1 \rangle.
\]

The first ideal in the intersection corresponds to the real zero 2, while the second to the pair of complex zeros \( i \) and \( -i \). Such factorizations are easy to find for not too large polynomials in a computer algebra system. What if we now replace the given polynomial by \( g = x^3 - 2x^2 + x - 1 \)? How does the ideal \( \langle g \rangle \) decompose in \( \mathbb{R}[x] \)? And in \( \mathbb{C}[x] \)?

Polynomials of degree \( m \) in one variable can have up to \( m \) zeros. The number \( m \) can be large. Often we are not interested in all the zeros, but only in specific ones. For instance, we might only be interested in solutions that are real and positive. This restriction is very important for many applications, e.g. in statistics where the solutions represent probabilities.

**Example 3.2.** Let \( I = \langle x^m - x - 1 \rangle \), where \( m \geq 2 \). The variety \( \mathcal{V}(I) \) consists of \( m \) complex points but only one of them is real and positive. Thus, \( \mathcal{V}(I) \cap \mathbb{R}_{>0} \) is a singleton. This follows from *Descartes’ Rule of Signs*, which states that the number of positive real solutions is bounded above by the number of sign alternations in the coefficient sequence. If \( m \) is even then there is also one negative solution.

In many applications one encounters polynomials whose coefficients depend on parameters. For instance, let \( \epsilon \) be an unknown that represents a small positive real number. Let \( \mathbb{Q}(\epsilon) \) be the field of rational functions in that unknown and \( K = \overline{\mathbb{Q}(\epsilon)} \) its algebraic closure. Elements in \( K \) can be expressed as series in \( \epsilon \) with rational exponents. These are known as *Puiseux series*. This is analogous to the floating point expansion of numbers in \( \mathbb{R} \).
3.1. Zero-dimensional Ideals

Example 3.3. The polynomial \( f = \epsilon^2x^3 + x^2 + x - \epsilon \) is irreducible in \( \mathbb{Q}(\epsilon)[x] \). It factors into three linear factors \( f = \epsilon^2(x - u_1)(x - u_2)(x - u_3) \) in \( K[x] \):

\[
\begin{align*}
u_1 &= -\epsilon^{-2} + 1 + \epsilon^2 + \epsilon^3 + 2\epsilon^4 + 3\epsilon^5 + 5\epsilon^6 + 10\epsilon^7 + \cdots \\
u_2 &= -1 - \epsilon - 3\epsilon^3 + 3\epsilon^4 - 16\epsilon^5 + 32\epsilon^6 - 121\epsilon^7 + \cdots \\
u_3 &= \epsilon - \epsilon^2 + 2\epsilon^3 - 5\epsilon^4 + 13\epsilon^5 - 37\epsilon^6 + 111\epsilon^7 + \cdots
\end{align*}
\]

Each of these three roots is an algebraic number over \( \mathbb{Q}(\epsilon) \). We wrote them as Puiseux series. If we think of \( \epsilon \) as a very small positive quantity then \( u_1 \sim -\epsilon^{-2}, u_2 \sim -\epsilon^0 \) and \( u_3 \sim \epsilon^1 \). The exponents \(-2, 0 \) and \( 1 \) tell us the asymptotic behavior. They are known as tropical solutions; cf. Chapter 7.

We have seen that solving a polynomial equation \( f = 0 \) amounts to decomposing the principal ideal \( I = \langle f \rangle \), i.e. presenting it as an intersection of simpler ideals. The situation is analogous for systems of polynomials in \( n \geq 2 \) variables, i.e. ideals \( I \subset K[x] \), where \( x = (x_1, \ldots, x_n) \). Suppose now that \( K \) is algebraically closed and assume that \( \mathcal{V}(I) \) is zero-dimensional. This means that the quotient ring \( K[x]/I \) is a finite-dimensional vector space over \( K \). By Theorem 1.17, a basis is given by the standard monomials for a given monomial order. The number of standard monomials is an upper bound for the cardinality of \( \mathcal{V}(I) \). Equality holds if and only if \( I \) is radical.

In the next section we will decompose our zero-dimensional ideal \( I \) as

\[
I = \bigcap_{i=1}^{k} q_i,
\]

where \( \text{rad}(q_i) \) is a prime ideal. Every prime ideal of dimension 0 in \( K[x] \) is a maximal ideal, so each \( \text{rad}(q_i) \) is a maximal ideal. Since \( K \) is algebraically closed, \( \mathcal{V}(q_i) \) is a point in \( K^n \). These points are the solutions to our system.

Example 3.4. Let \( n = 2 \) and \( I = \langle xy, x^2 - x, y^2 - y \rangle \). This ideal is radical:

\[
I = \langle x, y \rangle \cap \langle x - 1, y \rangle \cap \langle x, y - 1 \rangle.
\]

The variety of this ideal consists of three points: \( \mathcal{V}(I) = \{(0, 0), (1, 0), (0, 1)\} \).

If the given ideal is not radical then we cannot express it as an intersection of maximal ideals. This should not be surprising: already in the case of one variable, if a root has a multiplicity then we need a power of its ideal.

Example 3.5. Let \( I = \langle xy, y^2 - y, x^2y - x^2 \rangle \). We have the decomposition

\[
I = \langle x, y - 1 \rangle \cap \langle x^2, y \rangle.
\]

The varieties of both ideals are points: \( (0, 1) \) and \( (0, 0) \) respectively. However, the second ideal remembers additional data. It is not just \( \langle x, y \rangle \), but indicates a ‘multiplicity’ of the solution \( (0, 0) \). We are now equipped to measure this multiplicity! The degree of \( I \) equals 3. The first ideal in the decomposition contributes with degree one, while the second with degree 2.
We now discuss an example that was seen in Exercise 9 of Chapter 1.

**Example 3.6.** Fix the rationals $K = \mathbb{Q}$ and $I = \langle x^3 - yz, y^3 - xz, z^3 - xy \rangle$ in $K[x, y, z]$. This ideal is an irredundant intersection of 11 distinct ideals:

$I = Q \cap \langle x - 1, y - 1, z - 1 \rangle \cap \langle x + 1, y + 1, z - 1 \rangle \cap \langle x - 1, y + 1, z + 1 \rangle \cap \langle x + 1, y + 1, z + 1 \rangle \cap \langle x - 1, y + z, z^2 + 1 \rangle \cap \langle x + 1, y - z, z^2 + 1 \rangle \cap \langle y - 1, x + z, z^2 + 1 \rangle \cap \langle y + 1, x - z, z^2 + 1 \rangle.$

The first intersectand is a primary ideal with radical $\text{rad}(Q) = (x, y, z)$:

$Q = \langle x^2 y, x^2 z, xy^2, xz^2, y^2 z, yz^2, x^3 - yz, y^3 - xz, z^3 - xy \rangle.$

Each of the other 10 intersectands is a prime ideal. If we were to replace $K$ by the complex numbers $\mathbb{C}$ then six of the prime ideals decompose further:

$\langle x - 1, y + z, z^2 + 1 \rangle = \langle x - 1, y - i, z + i \rangle \cap \langle x - 1, y + i, z - i \rangle.$

We learn that $V(I)$ consists of 17 complex points. Only five are real.

The idea of decomposing a mathematical object into simpler pieces is important. In the next section we present a theory of decomposing ideals. We shall express them as intersections of simpler ideals. Our point of departure is the following proposition. It shows how algebraic varieties may be decomposed. This proposition applies to varieties of any dimension.

**Proposition 3.7.** Any variety in $K^n$ can be uniquely represented as a finite union of irreducible varieties (pairwise not contained in each other).

**Proof.** We start by proving the existence of such a decomposition. Any variety $W$ is either irreducible or it is a union $W_1 \cup W_2$. In the latter case we apply the same reasoning to the pieces. For instance, we may write $W_1$ as a union $W_2 \cup W_3$ etc. We obtain an ascending chain of ideals, $\mathcal{I}(W_1) \subseteq \mathcal{I}(W_2) \subseteq \ldots$. This chain stabilizes by Hilbert’s Basis Theorem. Thus the decomposition procedure finishes with finitely many irreducible varieties.

Suppose we have two irreducible decompositions of the same variety:

$V_1 \cup \cdots \cup V_k = W_1 \cup \cdots \cup W_s.$

Fix any index $i_0$ in $\{1, \ldots, s\}$. The identity implies $W_{i_0} = \bigcup_{j=1}^k (V_j \cap W_{i_0})$. Since $W_{i_0}$ is irreducible, there exists an index $j_0$ such that $W_{i_0} = V_{j_0} \cap W_{i_0}$, and hence $W_{i_0} \subseteq V_{j_0}$. But, we similarly find that $V_{j_0} \subseteq W_{i_1}$ for some $i_1 \in \{1, \ldots, k\}$. We cannot have $W_{i_0} \subsetneq W_{i_1}$, so that $W_{i_0} = V_{j_0}$. Hence, for every component $W_{i_0}$ on the right there is a unique component $V_{j_0}$ on the left that is equal to $W_{i_0}$. The uniqueness of the decomposition follows. $\square$
3.2. Primary Decomposition

This section develops a vast generalization of the following two basic facts:

1. Every integer \( n > 1 \) can be uniquely decomposed as a product of powers of prime numbers:
   \[
   n = p_1^{a_1} \cdots p_k^{a_k}.
   \]

2. Any variety can be uniquely decomposed as a union of irreducible varieties - Proposition 3.7.

The algebraic notion of an ideal connects the first (number-theoretic) fact and the second (geometric) fact. Indeed, any integer \( n \) can be identified with the ideal \( \langle n \rangle \) in the ring \( \mathbb{Z} \). The elements of \( \langle n \rangle \) are the integer multiples of \( n \). The ideal \( \langle n \rangle \) is prime in \( \mathbb{Z} \) if and only if \( n \) is a prime number. We can restate fact (1) in terms of intersections of powers of prime ideals as follows:

\( 1' \) Every nonzero ideal \( I \subset \mathbb{Z} \) has a unique decomposition
   \[
   I = (I_1)^{a_1} \cap \cdots \cap (I_k)^{a_k},
   \]
   where the \( I_i \) are prime ideals.

Over an algebraically closed field, we have an identification of varieties with radical ideals (cf. Chapter 6). This yields the following restatement of (2):

\( 2' \) Every radical ideal \( I \subset \mathbb{C}[x] \) has a unique decomposition as an intersection of prime ideals, pairwise not contained in each other:
   \[
   I = p_1 \cap \cdots \cap p_k.
   \]

These examples suggest that our aim should be to decompose ideals \( I \) in a ring \( R \). Here, a decomposition of \( I \) is a presentation as an intersection of other ideals. At this point, we need to answer the following questions:

1. What kind of ideals should be allowed in the intersection?
2. What restrictions should be put on the ring \( R \)?
3. Can we expect the decomposition to be unique?

We start with the first question. The number-theoretic example suggests all ideals might be intersections of powers of prime ideals. But this is not true.

**Example 3.8.** The ideal \( I = \langle x^2, y \rangle \) is not an intersection of powers of prime ideals in \( \mathbb{C}[x,y] \). Indeed, suppose \( I = \bigcap_i p_i^{a_i} \). For all \( i \), we have \( p_i \supset I \). Hence \( p_i = \langle x, y \rangle \), as this is the only prime ideal containing \( I \). The ideal \( \bigcap_i \langle x, y \rangle^{a_i} \) would be a power of \( \langle x, y \rangle \), whereas \( I \) is no such power.

The right constituents are primary ideals. Recall that \( I \) is primary if and only if \( ab \in I \) and \( a \notin I \) implies \( b^n \in I \) for some \( n \), given any \( a, b \in R \).
Next consider question (2): which rings $R$ to take? Clearly, $\mathbb{Z}$ and $K[x]$ share a lot of nice properties. But there is a larger class of rings that works.

**Definition 3.9.** A ring $R$ is *Noetherian* if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$$

stabilizes, i.e. there exists $k$ such that $I_k = I_{k+1} = I_{k+2} = \ldots$.

Noetherian rings are named after the German algebraist Emmy Noether. A hint how important they are is given in Exercise 4. Note that $\mathbb{Z}$ and $K[x]$ are Noetherian rings because their ideals are finitely generated. Before stating our main existence theorem, let us introduce a technical definition.

**Definition 3.10.** An ideal $I$ in a ring $R$ is *irreducible* if and only if whenever $I = J_1 \cap J_2$ for some ideals $J_1, J_2$ in $R$ then $I = J_1$ or $I = J_2$.

**Theorem 3.11.** Let $I$ be an ideal in a Noetherian ring $R$. Then there exist primary ideals $q_1, q_2, \ldots, q_k$ in $R$ such that

$$I = q_1 \cap q_2 \cap \cdots \cap q_k.$$

**Proof.** First we show that every ideal in $R$ is a finite intersection of irreducible ideals. Suppose not, and let $I_1$ be an ideal that cannot be presented in this way. In particular, it is not irreducible. Thus, $I_1 = J_1 \cap J_2$ and each $J_i$ strictly contains $I_1$. If $J_1$ and $J_2$ are finite intersections of irreducible ideals, then so is $I_1$. Hence, we may assume $J_1$ cannot be presented as such a finite intersection. Let $I_2 := J_1$. We have $I_1 \subseteq I_2$. We repeat the construction starting with $I_2$ and get an ideal $I_3$ with $I_1 \subseteq I_2 \subseteq I_3$, where $I_3$ is not a finite intersection of irreducible ideals. Continuing, we get a chain of strictly ascending ideals. However, this is not possible in a Noetherian ring.

We next prove that every irreducible ideal $q$ is primary. By replacing the ring $R$ with $R/q$, we may assume $q = \{0\}$. Suppose $ab = 0$ and $a \neq 0$. We must prove that $b$ is nilpotent. Consider the following ascending chain:

$$\{x \in R : bx = 0\} =: \text{Ann}(b) \subseteq \text{Ann}(b^2) \subseteq \text{Ann}(b^3) \subseteq \ldots$$

Since $R$ is Noetherian, ascending chains of ideals become stationary. Hence $\text{Ann}(b^n) = \text{Ann}(b^{n+1})$ for some $n$. We claim that $\langle a \rangle \cap \langle b^n \rangle = \{0\}$. Indeed, suppose $\lambda a = \mu b^n \in \langle a \rangle \cap \langle b^n \rangle$ for some $\lambda, \mu \in R$. Clearly,

$$0 = \lambda ab = \mu b^{n+1}.$$

Hence, $\mu \in \text{Ann}(b^{n+1}) = \text{Ann}(b^n)$. Thus, $\mu b^n = 0$. As $\{0\}$ was assumed to be irreducible, and $\langle a \rangle \supseteq \{0\}$, we have $b^n = 0$. This completes the proof. $\square$

We now pass to the third question, concerning uniqueness. We need not assume that $R$ is Noetherian, as long as the ideal $I$ in question is an intersection of finitely many primary ideals: $I = \bigcap_{i=1}^k q_i$. Here it is assumed
that each $q_i$ is necessary, i.e. $\bigcap_{j \neq i_0} q_j \not\subset q_{i_0}$ for all $1 \leq i_0 \leq k$. The next two lemmas suggests grouping the primary ideals $q_i$ by their radical.

**Lemma 3.12.** The radical of a primary ideal $q$ is the unique smallest prime ideal containing it.

The proof is left as Exercise 5 for the reader. A primary ideal $q$ whose radical equals a given prime ideal $p$ is called $p$-primary. In many cases, the powers of a prime ideal $p$ are primary. This is the case for polynomial rings in one variable, cf. Exercise 2. However, it is not true in general that the power of a prime ideal is a primary ideal, even in the polynomial ring $\mathbb{C}[x]$.

**Example 3.13.** Let $P$ be the ideal generated by the nine $2 \times 2$ minors of a $3 \times 3$ matrix $X = (x_{ij})$ of unknowns. This ideal is prime and it contains none of the $x_{ij}$. We claim that the ideal $P^2$ is not primary. To see this, we verify (using Gröbner bases) that $x_{ij} \cdot \det(X)$ lies in $P^2$ for all $1 \leq i,j \leq 3$. However, $P^2$ is generated by quartics and contains no cubics. Thus, neither $\det(P)$ nor any power of $x_{ij}$ is in $P^2$. We conclude that $P^2$ is not primary.

We next focus on $p$-primary ideals for a fixed prime ideal $p$.

**Lemma 3.14.** If $q_1, \ldots, q_k$ are $p$-primary ideals, then so is $q_1 \cap \cdots \cap q_k$.

**Proof.** The following shows that the radical of $I := \bigcap_{i=1}^k q_i$ equals $p$:

$$a \in \text{rad}(I) \iff \exists n : a^n \in I \iff \exists n \forall i : a^n \in q_i \iff \forall i : a \in \text{rad}(q_i) = p \iff a \in p.$$  

To see that $I$ is primary, we assume that $ab \in I$ and $a \not\in I$. Then $a \not\in q_{i_0}$ for some $i_0$. Since $ab \in q_{i_0}$ and $q_{i_0}$ is primary, $b \in \text{rad}(q_{i_0}) = p = \text{rad}(I)$. Hence $b^n \in I$ for some $n$. □

Lemma 3.14 suggests that, given any primary decomposition $I = \bigcap_{i=1}^k q_i$, we aggregate the $q_i$’s with the same radical and replace them by their intersection. The result is still a primary decomposition of $I$. This motivates the following definition. A *minimal primary decomposition* is a representation

$$I = q_1 \cap q_2 \cap \cdots \cap q_k,$$

where the $q_i$’s are primary ideals that have pairwise distinct radicals and the intersection is irredundant, meaning $\bigcap_{j \neq i_0} q_j \not\subset q_{i_0}$ for all $1 \leq i_0 \leq k$.

To sum up, we have proved the following result for a Noetherian ring $R$:

1. every ideal has a (finite) primary decomposition, and
2. every (finite) primary decomposition of an ideal can be changed to a minimal one (apply Lemma 3.14 and remove unnecessary ideals).

We next show that minimal primary decompositions may still be not unique.
Example 3.15. The following are two minimal primary decompositions:

\[(3.2) \quad \langle x^2, xy \rangle = \langle x \rangle \cap \langle x, y \rangle^2 = \langle x \rangle \cap \langle x^2, y \rangle \subset \mathbb{C}[x, y].\]

It turns out that, while the primary ideals \(q_i\) in the decomposition (3.1) need not be unique, their radicals are unique. Recall that the quotient of an ideal \(I\) by a ring element \(a\) is the ideal \((I : a) = \{b \in R : ab \in I\}\).

Theorem 3.16. For any ideal \(I\) in a ring \(R\) with a minimal primary decomposition (3.1), the \(k\) prime ideals \(\text{rad}(q_i)\) do not depend on the choice of that decomposition. They are precisely the prime ideals of the form \(\text{rad}(I : a)\) for some \(a \in R\). If \(R\) is Noetherian then the last radical is not needed: they are precisely the prime ideals \((I : a)\) for some \(a \in R\).

Proof. Fix a minimal primary decomposition \(I = \bigcap_{i=1}^{k} q_i\). Intersection commutes with ideal quotients, so \((I : a) = \bigcap_{i=1}^{k} (q_i : a) = \bigcap_{a \notin q_i} (q_j : a)\). It also commutes with radicals: \(\text{rad}(I : a) = \bigcap_{a \notin q_i} \text{rad}(q_j : a)\). We next argue that \(a \notin q_i\) implies \(\text{rad}(q_i : a) = \text{rad}(q_i)\). Suppose \(b \in \text{rad}(q_i : a)\), i.e. \(b^n a \in q_i\). As \(q_i\) is primary and \(a \notin q_i\), we have \((b^n)^m \in q_i\), i.e. \(b \in \text{rad}(q_i)\). Hence, \(\text{rad}(q_i : a) \subset \text{rad}(q_i)\) and the other inclusion is obvious. We conclude that \(\text{rad}(I : a) = \text{rad}(q_i)\) equals the intersection of the prime ideals \(\text{rad}(q_j)\) satisfying \(a \notin q_j\).

By Exercise 7, if \(\text{rad}(I : a)\) is prime then it equals \(\text{rad}(q_j)\) for some \(j\). Next, consider any \(\text{rad}(q_{i_0})\). As the primary decomposition is minimal, there exists \(a \in \bigcap_{j \neq i_0} q_j \setminus q_{i_0}\). The conclusion above shows \(\text{rad}(I : a) = \text{rad}(q_{i_0})\).

It remains to prove the last assertion. If \((I : a)\) is prime, then it equals its radical. Thus, we must consider a prime ideal of the form \(\text{rad}(I : a)\) and show that it equals \((I : a')\) for some \(a' \in I\). We already know that \(\text{rad}(I : a) = \text{rad}(q_{i_0})\) for some \(i_0\). By Exercise 8, \(\text{rad}(q_{i_0})^n \subset q_{i_0}\) for some positive integer \(n\). Hence, there exists \(n\) such that \((\bigcap_{j \neq i_0} q_j) \cdot (\text{rad}(q_{i_0}))^n \subset I\). We fix the smallest \(n\) with this property. Then we pick

\[a' \in \left( (\bigcap_{j \neq i_0} q_j) \cdot (\text{rad}(q_{i_0}))^{n-1} \right) \setminus I.\]

(Here, if \(n = 1\) then \(\text{rad}(q_{i_0})^{n-1}\) is the ring \(R\).) By definition, \(a' \cdot \text{rad}(q_i) \subset I\), and thus \(\text{rad}(q_{i_0}) \subset (I : a')\). However, \(a' \in (\bigcap_{j \neq i_0} q_j) \setminus I\), thus \(a' \notin q_{i_0}\). We have the inclusions \(\text{rad}(q_{i_0}) \subset (I : a') \subset (I : a') = \text{rad}(q_{i_0})\), which are in fact equalities. The last equation follows from the previous paragraph. \(\square\)

Definition 3.17. The associated primes of an ideal \(I\) are the radicals of the primary ideals appearing in a minimal primary decomposition. Equivalently, these are the prime ideals of the form \(\text{rad}(I : a)\) for some element \(a\) of the ring. If the ring is Noetherian, these are the prime ideals of the form \((I : a)\).
Before going further, let us discuss the geometric meaning of the associated primes. If $I = \bigcap_{i=1}^{k} q_i$, then $\text{rad}(I) = \bigcap_{i=1}^{k} \text{rad}(q_i)$. Thus, every component in the irreducible decomposition of the variety $\mathcal{V}(I)$ corresponds to one of the associated primes of $I$. However, the converse is not true.

**Example 3.18.** Let $I = \langle x^2, xy \rangle$ as in Example 3.15. We have $\text{rad}(I) = \langle x \rangle$. The variety $\mathcal{V}(I)$ is irreducible. It is a line in a plane. However, the minimal primary decompositions reveal that $I$ has two associated primes. The expected prime $\langle x \rangle$ and the unexpected prime $\langle x, y \rangle$ - a point on the line. Thus, the associated primes remember more information than just the variety. There is a point corresponding to the ideal $\langle x, y \rangle$ on the line defined by $\langle x \rangle$. This point is ‘hidden’ or embedded inside the line.

The formal replacement of varieties (corresponding to radical ideals) by arbitrary ideals allowed a tremendous advance of 20th century algebraic geometry. One is now able to work with 'functions’ that are nonzero, but their square is zero, using basic, well-understood algebra. This advance should be compared to the introduction of complex numbers in 18th and 19th century, where (basically in the same way) instead of answering the question ‘does there exist a square root of $-1$?’ one introduces imaginary numbers and shows how to use them in an efficient way. Still, we should not forget the classical geometry we started from. The line from Example 3.18 is of a different nature than the point, and these two should be distinguished.

**Definition 3.19.** For an ideal $I$, let $\text{Ass}(I)$ be the set of associated primes. The minimal (with respect to inclusion) elements of $\text{Ass}(I)$ are the **minimal primes** of $I$. Associated primes that are not minimal are called **embedded**.

In what follows, our standing assumption is that $R$ is a Noetherian ring. An embedded prime $p$ of an ideal $I$ must contain a minimal prime $p'$. This means that the irreducible component $\mathcal{V}(p')$ of $\mathcal{V}(I)$ strictly contains the irreducible variety $\mathcal{V}(p)$. We do not see $\mathcal{V}(p)$ geometrically inside $\mathcal{V}(I)$: it is **embedded** in $\mathcal{V}(p')$. Further the minimal primes correspond exactly to irreducible components of $\mathcal{V}(I)$. They are the irredundant intersectands in

$$\text{rad}(I) = \bigcap_{i=1}^{k} \text{rad}(q_i).$$

The next lemma offers another explanation for the name minimal primes.

**Lemma 3.20.** A prime ideal is a minimal prime of $I$ if and only if it is a minimal element (with respect to inclusion) among the primes that contain $I$.

**Proof.** It is enough to prove that every prime $p$ containing $I$ contains also a prime in $\text{Ass}(I)$. Then $p$ also contains a minimal prime. They are equal if $p$
is minimal with respect to inclusion. Thus, suppose \( p \) contains \( I = \bigcap_{i=1}^{k} q_i \).
By Exercise 7 \( p \supseteq q_{i_0} \) for some some \( i_0 \). Hence, \( p = \text{rad}(p) \supseteq \text{rad}(q_{i_0}) \).

The geometry that distinguishes embedded and minimal primes suggests an idea how to get additional uniqueness properties in primary decompositions. Indeed, in Example 3.15 it is the ideal corresponding to the embedded component that changes, while the minimal prime remains the same.

**Theorem 3.21.** Let \( I = \bigcap_{i=1}^{k} q_i \) be a minimal primary decomposition. The primary ideals \( q_i \) corresponding to minimal primes are determined by \( I \).

**Proof.** Let \( q_{i_0} \) be such that \( \text{rad}(q_{i_0}) \) is a minimal prime. We claim that
\[
q_{i_0} = \{ a : ab \in I \text{ for some } b \not\in \text{rad}(q_{i_0}) \}.
\]
We already saw that the right hand side does not depend on the decomposition of \( I \). Thus the equation implies the theorem. We prove both inclusions.

Let \( a \in q_{i_0} \). For every \( i \neq i_0 \) we have \( q_i \not\subseteq \text{rad}(q_{i_0}) \). Otherwise, \( \text{rad}(q_i) \subseteq \text{rad}(q_{i_0}) \), which would contradict the hypothesis that \( \text{rad}(q_{i_0}) \) is minimal. Hence, there exists \( b_i \in q_i \setminus \text{rad}(q_{i_0}) \). We define \( b := \prod_{j \neq i_0} b_j \). As \( \text{rad}(q_{i_0}) \) is prime we have \( b \not\in \text{rad}(q_{i_0}) \). However, \( ab \in q_j \) for \( j \neq i_0 \), as \( b \in q_j \). Furthermore, \( ab \in q_{i_0} \), as \( a \in q_{i_0} \). This implies \( ab \in I = \bigcap_{i=1}^{k} q_i \). This means that \( a \) is contained in the right hand side.

Now we pick \( a \) and \( b \not\in \text{rad}(q_{i_0}) \) such that \( ab \in I \). In particular, \( ab \in q_{i_0} \).
If \( a \not\in q_{i_0} \) we get a contradiction to the fact that \( q_{i_0} \) is primary. This shows that the right hand side is contained in the left hand side.

Let \( I \) and \( J \) be ideals in a Noetherian ring \( R \). Using the definition in Proposition 1.3 we consider the following chain of ideal quotients in \( R \):
\[
(I : J) \subseteq (I : J^2) \subseteq (I : J^3) \subseteq \cdots
\]
This chain stabilizes and we set \((I : J^\infty) = (I : J^m)\) for \( m \in \mathbb{N} \) sufficiently large. The ideal \((I : J^\infty)\) is called the **saturation** of \( I \) with respect to \( J \). If \( J = \langle f \rangle \) is a principal ideal then also the notation \((I : f^\infty)\) is used for this.

**Corollary 3.22.** Let \( I \) be the ideal in Theorem 3.21. The primary ideal \( q_i \) corresponding to a minimal prime \( p_i = \text{rad}(q_i) \) of \( I \) can be computed as
\[
q_i = (I : (a_1 \cdots a_{i-1}a_{i+1} \cdots a_k)^\infty),
\]
where \( a_j \) is an element of \( p_j \setminus p_i \) for \( j \in \{1, \ldots, k\}\setminus\{i\} \).

**Proof.** This is equivalent to \((3.3)\) when \( i = i_0 \), since \( p_i = \text{rad}(q_i) \). By definition, the saturation on the right hand side consists of all ring elements \( b \) such that \( b(a_1 \cdots a_{i-1}a_{i+1} \cdots a_k)^m \subseteq I \) for some positive integer \( m \).
Primary decomposition for monomial ideals is easier than for general polynomial ideals. The associated primes are generated by subsets of the variables and they can be characterized combinatorially. We here just show this for one example. For more information we refer to the textbook \[44\].

**Example 3.23.** Let \( n = 3 \) and \( I = \langle xy^2 z^3, x^2 yz^3, xy^3 z^2, x^3 yz^2, x^2 y^3 z, x^3 y^2 z \rangle \). This has seven associated primes. A minimal primary decomposition equals

\[
I = \langle x \rangle \cap \langle y \rangle \cap \langle z \rangle \cap \langle x^2, y^2 \rangle \cap \langle x^2, z^2 \rangle \cap \langle y^2, z^2 \rangle \cap \langle x^3, y^3, z^3 \rangle.
\]

This example generalizes to \( n \geq 4 \) as follows. The ideal \( I \) is generated by the \( n! \) monomials \( \prod_{i=1}^{n} x_i^{\pi_i} \), indexed by permutations \( \pi \in S_n \), and \( \text{Ass}(I) \) consists of all \( 2^n - 1 \) ideals generated by nonempty subsets of \( \{x_1, \ldots, x_n\} \).

There are many algorithms and implementations for computing primary decompositions. The input is an ideal \( I \) in a polynomial ring \( K[x] \) and the output is the set \( \text{Ass}(I) \) and primary ideals \( q_1, \ldots, q_k \) satisfying (3.1). Traditionally, these are symbolic methods built upon Gröbner bases. In recent years, numerical tools for decomposing ideals and varieties have received much attention. Solving polynomial systems means running such software.

### 3.3. Linear PDE with Constant Coefficients

In this section we offer an alternative perspective on the problem of solving systems of polynomial equations. This is aimed at highlighting the role of embedded primes and primary ideals in a context of practical importance.

Every polynomial with real or complex coefficients can be interpreted as a linear differential operator with constant coefficients. This operator is obtained by simply replacing \( x_i \) by the differential operator \( \frac{\partial}{\partial x_i} \). Every ideal \( I \) in \( \mathbb{R}[x_1, x_2, \ldots, x_n] \) can thus be interpreted as a system of linear partial differential equations (PDE) with constant coefficients. Suppose we are interested in the solutions to these PDE within some nice class of functions, like polynomial functions, real analytic functions \( \mathbb{R}^n \to \mathbb{R} \), or complex holomorphic functions \( \mathbb{C}^n \to \mathbb{C} \). Then the set of solutions to our PDE is a linear space over \( \mathbb{R} \) or \( \mathbb{C} \). We are interested in computing a basis for that solution space. This computation rests on the primary decomposition of the ideal \( I \). Both minimal primes and embedded primes will play a role, and all primary components will contribute to our basis for the solution space. But, first of all, let us start by interpreting the usual points of \( \mathcal{V}(I) \) in terms of PDE.

**Lemma 3.24.** Let \( I \) be an ideal in \( \mathbb{C}[x] \). A point \( (a_1, \ldots, a_n) \in \mathbb{C}^n \) lies in the variety \( \mathcal{V}(I) \) if and only if the exponential function \( \exp(a_1 x_1 + \cdots + a_n x_n) \) is a solution of the partial differential equations given by \( I \).
Proof. Let \( f(x) = \exp(a_1 x_1 + \cdots + a_n x_n) \). Then \( \frac{\partial f}{\partial x_i} = a_i f \) for all \( i \). Let \( g \) be any polynomial in \( n \) variables and \( g(\frac{\partial}{\partial x_i}) \) the corresponding differential operator. By induction on the degree of \( g \), with degree one as base case, we find that applying the operator \( g(\frac{\partial}{\partial x_i}) \) to the function \( f(x) \) yields \( g(a_1, \ldots, a_n) \) times \( f(x) \). This is zero for all \( g \in I \) if and only if \( (a_1, \ldots, a_n) \in \mathcal{V}(I) \). \( \square \)

Lemma 3.24 embeds the classical solutions of a polynomial system into the solution space of the associated linear PDE. But, if the ideal is not radical then there are more solutions, governed by the primary decomposition. We shall explain this for the ideal in Example 3.6 and in Exercise 9 of Chapter 1.

Example 3.25. We revisit Example 3.6 and Exercise 9 of Chapter 1. Let \( n = 3 \) and \( I = \langle x^3 - yz, y^3 - xz, z^3 - xy \rangle \). The corresponding system of PDE asks for all functions \( f = f(x, y, z) \) that satisfy

\[
\frac{\partial^3 f}{\partial x^3} = \frac{\partial^2 f}{\partial y \partial z} \quad \text{and} \quad \frac{\partial^3 f}{\partial y^3} = \frac{\partial^2 f}{\partial x \partial z} \quad \text{and} \quad \frac{\partial^3 f}{\partial z^3} = \frac{\partial^2 f}{\partial x \partial y}.
\]

To make this problem precise, we would need to specify the class of functions \( f \) that are allowed. For instance, we might take all holomorphic functions \( f : \mathbb{C}^3 \to \mathbb{C} \). Or we might seek real analytic solutions \( f : \mathbb{R}^3 \to \mathbb{R} \), or, among these, all polynomial solutions. Let’s leave this unspecified for now.

The degree of our ideal \( I \) is 27 = \( 3 \times 3 \times 3 \), which comes from the degrees of the three generators of \( I \). The number 27 is also the dimension of the space of holomorphic solutions \( f \) to (3.4). A basis of that solution space is

\[
1, x, y, z, x^2, y^2, z^2, x^3 + 6yz, y^3 + 6xz, z^3 + 6xy, x^4 + y^4 + z^4 + 24xyz, \\
\exp(x - y - z), \exp(x + y + z), \exp(-x - y + z), \exp(-x + y - z), \\
\exp(x - iy + iz), \exp(x + iy - iz), \exp(-x - iy - iz), \exp(-x + iy + iz), \\
\exp(ix - y + iz), \exp(ix + y - iz), \exp(ix - iy + z), \exp(ix + iy - z), \\
\exp(-ix - y - iz), \exp(-ix + y + iz), \exp(-ix - iy - z), \exp(-ix + iy + z).
\]

The subspace of polynomial solutions has dimension 11 and is spanned by the first row. The larger subspace of real analytic solutions has dimension 15 and is spanned by the first two rows. All other basis functions are exponential of linear forms that have \( i = \sqrt{-1} \) among its coefficients. The 16 basis solutions in the last four rows, along with the solution \( 1 = \exp(0x + 0y + 0z) \), are explained by Lemma 3.24. They are the exponentials corresponding to the 17 distinct points in \( \mathcal{V}(I) \subset \mathbb{C}^3 \).

This basis in (3.5) was derived from the minimal primary decomposition

\[
I = Q \cap \bigcap_{a+b+c \equiv 0 \mod 4} \langle x - ia, y - ib, z - ic \rangle \quad \text{in} \quad \mathbb{C}[x, y, z].
\]

This decomposition is obtained by refining the primary decomposition over the rational numbers \( \mathbb{Q} \) shown in Example 3.6. The 16 ideals in the intersection on the right hand side are maximal and hence prime. They correspond
to the 16 exponential solutions in (3.5). The ideal $Q$ is primary to the maximal ideal $\text{rad}(Q) = \langle x, y, z \rangle$. Since all associated primes are minimal, by Theorem 3.21, this primary ideal is uniquely determined from $I$:

$$Q = \langle x^2y, x^2z, xy^2, xz^2, yz^2, x^3 - yz, y^3 - xz, z^3 - xy \rangle.$$ 

This zero-dimensional primary ideal has degree 11. It contributes the 11 polynomial solutions to the three partial differential equations in (3.4).

Here is a general result explaining our observations from Example 3.25.

**Theorem 3.26.** Let $I$ be a zero-dimensional ideal in $\mathbb{C}[x_1, \ldots, x_n]$, interpreted as a system of linear PDE. The space of holomorphic solutions has dimension equal to the degree of $I$. There exist non-zero polynomial solutions if and only if the maximal ideal $M = \langle x_1, \ldots, x_n \rangle$ is associated to $I$. In that case, the polynomial solutions are precisely the solutions to the system of PDE given by the $M$-primary component $(I : (I : M^\infty))$.

**Proof.** Fix a degree compatible monomial order and let in($I$) be the initial monomial ideal of $I$ for that order. The set $S$ of standard monomials is finite. It consists of monomials of degree $< D$ for some $D$. These form a basis for the solution space of the PDE associated with in($I$). For each $s \in S$ we will construct explicitly a power series solution to the system given by $I$. We will also show that these solutions for a linear basis of all solutions that are given by a power series. These are the holomorphic solutions.

Let $d \geq D$. The pairing between monomials in the $\partial/\partial x_i$ and the monomials in the $x_j$ defines a nondegenerate inner product on the finite-dimensional vector space $\mathbb{C}[x]_{\leq d}$. If we interpret each element in $I_{\leq d}$ as a differential operator, then the orthogonal complement with respect to that inner product is the space of polynomials in $\mathbb{C}[x]_{\leq d}$ that, when differentiated by elements in $I_{\leq d}$, do not have a constant term. As a linear space $I_{\leq d}$ has a basis given by operators of the type $x^v + \sum_{u \in S} \lambda_u x^u$, where $x^v \not\in S$.

By solving the resulting triangular system of linear equations, we construct a unique solution basis whose elements have the form

$$p_u(x_1, \ldots, x_n) = x^u + \text{higher order terms}, \quad \text{where } x^u \text{ runs over } S.$$ 

By increasing the value of $d$, we obtain a formal series (3.6) that solves our PDE. This formal series terminates as a polynomial if and only if it is annihilated by the operators $(\partial/\partial x_i)^d$ for $i = 1, 2, \ldots, n$. This is equivalent to saying that the series is a solution to any $M$-primary ideal containing $I$. If $I$ is $M$-primary then all solutions $p_u$ are polynomials. The solution space is then spanned by polynomials and its dimension equals $|S| = \text{degree}(I)$.

Suppose now that $I$ is primary in $\mathbb{C}[x]$. Since $I$ is zero-dimensional, its radical is the maximal ideal $\text{rad}(I) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$, where
\( \mathcal{V}(I) = \{(a_1, \ldots, a_n)\} \subset \mathbb{C}^n \). By translating \((a_1, \ldots, a_n)\) to the origin \((0, \ldots, 0)\), we can apply the analysis in the previous paragraph. From this and Lemma 3.24, we obtain degree(I) many polynomials \( p_u \), each with its lowest term a standard monomial \( x^u \in S \), such that

\[
(3.7) \quad p_u(x_1, \ldots, x_n) \cdot \exp(a_1 x_1 + \cdots + a_n x_n)
\]
solves the PDE given by \( I \). These functions form a basis for the holomorphic solutions to \( I \). None of them is a polynomial unless \((a_1, \ldots, a_n) = (0, \ldots, 0)\).

Next, let \( I \) be an arbitrary zero-dimensional ideal. Its minimal primary decomposition (3.1) is unique, by Theorem 3.21. The solution space to \( I \), regarded as PDE, contains the solution spaces of its primary components \( q_1, q_2, \ldots, q_k \). For each of these, we constructed a basis of holomorphic functions (3.7). The union of these bases is a basis for the solution space of \( I \), and its cardinality equals degree(I).

Finally, we argue that if \( M \in \text{Ass}(I) \) then the \( M \)-primary component of \( I \) is the double quotient \((I : (I : M^\infty))\). Actually, this is a special case of Corollary 3.22 since every maximal ideal is prime. In the primary decomposition (3.1), suppose that \( q_1 \) is \( M \)-primary. Then \((I : M^\infty) = q_2 \cap \cdots \cap q_k\). Taking the ideal quotient of \( I \) by \( q_2 \cap \cdots \cap q_k \) recovers the ideal \( q_1 \). Hence \( q_1 = (I : (I : M^\infty)) \) as desired. This completes the proof. \( \square \)

In the preceding discussion, we studied the solutions to zero-dimensional polynomial systems in the guise of linear PDE with constant coefficients. We saw that the solution space of such an ideal \( I \) is always a vector space of the same dimension, namely the degree of \( I \). This is different from the situation for solving polynomial equations. The variety \( \mathcal{V}(I) \) of classical solutions in \( \mathbb{C}^n \) changes its cardinality depending on whether \( I \) is radical or not. The solution space to the PDE, on the other hand, always has the expected dimension degree(I), independently of whether the ideal \( I \) is radical or not.

The solution spaces to our PDE vary gracefully under parameter changes. This underscores the utility of primary decompositions in the context of solving equations. We demonstrate this perspective in a simple example.

**Example 3.27** \((n = 2)\). Consider the ideal \( I = \langle x^2 - \delta^2, y^2 - \epsilon^2 \rangle \subset \mathbb{R}[x, y] \), where \( \delta, \epsilon \) are small real parameters. As before, we view \( I \) as a PDE system:

\[
\frac{\partial^2 f}{\partial x^2} = \delta^2 f \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = \epsilon^2 f.
\]

For \( \delta, \epsilon \neq 0 \), the solution space is spanned by the four exponential functions

\[
f_{ij} := \exp\left((-1)^i \delta x + (-1)^j \epsilon y\right) \quad \text{where} \quad i, j \in \{0, 1\}.
\]
However these four functions become linearly dependent when \( \delta \epsilon = 0 \). We therefore change the basis of our four-dimensional solution space as follows:

\[
\begin{align*}
g_{00} &= \frac{1}{4} (f_{00} + f_{01} + f_{10} + f_{11}) = 1 + \frac{\delta}{2} x^2 + \frac{\epsilon}{2} y^2 + \cdots \\
g_{01} &= \frac{1}{4 \epsilon} (f_{00} - f_{01} + f_{10} - f_{11}) = y + \frac{\delta}{2} x^2 y + \frac{\epsilon}{6} y^3 + \cdots \\
g_{10} &= \frac{1}{4 \delta} (f_{00} + f_{01} - f_{10} - f_{11}) = x + \frac{\delta}{6} x^3 + \frac{\epsilon}{2} xy^2 + \cdots \\
g_{11} &= \frac{1}{4 \delta \epsilon} (f_{00} - f_{01} - f_{10} + f_{11}) = xy + \frac{\delta}{6} x^3 y + \frac{\epsilon}{6} xy^3 + \cdots
\end{align*}
\]

This family remains linearly independent for all values of \( \delta \) and \( \epsilon \). In particular, for \( \delta = \epsilon = 0 \), we obtain the standard basis \( S = \{1, x, y, xy\} \) modulo the ideal \( \text{in}(I) = \langle x^2, y^2 \rangle \). This is a basis for the solutions to \( \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0 \).

We next discuss briefly the PDE from polynomial ideals \( I \) that are not zero-dimensional. It is still true that the primary decomposition of \( I \) reveals the solution space of these PDE. The precise statement is an important result in analysis known as Ehrenpreis’ Fundamental Principle or as Palamodov-Ehrenpreis Theorem. The details of this theorem are outside our scope. For the statement of this result see [58 §10.5] and the references given there.

We here illustrate the role of primary decomposition in one example. The key observation is that embedded primes reveal spurious solution spaces.

**Example 3.28.** Let \( n = 4 \) and consider the ideal

\[ J = \langle xw, xz + yw, yz \rangle. \]

Somewhat surprisingly, this is not radical. Its radical is the monomial ideal

\[ \sqrt{J} = \langle x, y \rangle \cap \langle z, w \rangle = \langle xw, xz, yw, yz \rangle. \]

The given ideal \( J \) has three associated primes. The primes \( \langle x, y \rangle \) and \( \langle z, w \rangle \) are minimal primes, and the maximal ideal \( \langle x, y, z, w \rangle \) is an embedded prime. A minimal primary decomposition of the given ideal equals

\[ J = \langle x, y \rangle \cap \langle z, w \rangle \cap (J + \langle x, y, z, w \rangle^3). \]

The third primary ideal is not unique. If we replace the third power of the maximal ideal by any higher power, then the intersection remains the same.

As before, we interpret the generators of \( J \) as a system of linear PDE:

\[ \frac{\partial^2 f}{\partial x \partial w} = \frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial y \partial w} = \frac{\partial^2 f}{\partial y \partial z} = 0. \]

The linear space of solutions \( f(x, y, z, w) \) is infinite-dimensional. It is spanned by all functions of the form \( g(w, z) \) and \( h(x, y) \), together with the one special function \( xz - yw \). The former correspond to the two minimal primes. The latter spurious solution arises from the embedded primary component.
Whenever one encounters a system of polynomial equations with special structure, and one is curious about the variety of solutions, it pays to explore the primary decomposition and to ponder the solutions to the associated PDE. Students who struggle with schemes in an algebraic geometry class will find our PDE interpretation a useful way to understand their structure.

Given a system of polynomial equations, the primary decomposition often reveals interesting structures. Most importantly, it tells us how to break up the solutions into meaningful pieces. As an illustration, we examine the following question from linear algebra: Let $A, B, C$ be $2 \times 2$-matrices. In which ways is it possible that the triple product $ABC$ is the zero matrix?

We approach this problem as follows. We set $n = 12$ and we fix the polynomial ring $\mathbb{R}[a_{ij}, b_{ij}, c_{ij}]$ whose variables are the 12 entries of the matrices $A, B, C$. Let $I$ be the ideal in $\mathbb{R}[a_{ij}, b_{ij}, c_{ij}]$ that is generated by the four entries of the matrix product $ABC$. For example, one of the four generators of $I$ is the upper left entry of $ABC$. This is the trilinear form

$$ a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21}. $$

In the back of our minds, we think of this as a partial differential equation:

$$ \frac{\partial^3 f}{\partial a_{11}\partial b_{11}\partial c_{11}} + \frac{\partial^3 f}{\partial a_{12}\partial b_{21}\partial c_{11}} + \frac{\partial^3 f}{\partial a_{11}\partial b_{12}\partial c_{21}} + \frac{\partial^3 f}{\partial a_{12}\partial b_{22}\partial c_{21}} = 0. $$

The scheme-theoretic version of our linear algebra question is this: Which functions on matrix triples satisfy these four partial differential equations?

A computation with a computer algebra system reveals that the ideal $I$ is radical. It is the intersection of six prime ideals. Three of them are ideals generated by the entries of $A$ or $B$ or $C$ respectively. The next two associated primes are generated respectively by the $2 \times 2$ minors of the matrices

$$ \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}. $$

Finally, the last associated prime of $I$ is the ideal $I + \langle \det(A), \det(C) \rangle$. Thus $\text{Ass}(I)$ consists of six primes, and all are minimal. Using the computer algebra system of their choice, the reader now ought to check that our ideal $I$ is indeed equal to the intersection of these six prime ideals.

Geometrically, we are studying a variety $\mathcal{V}(I)$ in the affine space $\mathbb{C}^{12}$. It is the solution set of four cubic equations. We found that $\mathcal{V}(I)$ is the union of six irreducible components. Three of them are linear spaces of dimension 8. The other three irreducible components have dimension 9 and they are not linear spaces. Their degrees are 4, 4 and 8 respectively. In response to the original linear algebra question, the six irreducible components of $\mathcal{V}(I)$
correspond to the following six scenarios for a triple of $2 \times 2$-matrices:

$$\text{rank}(A) = 0 \quad \text{or} \quad \text{rank}(B) = 0 \quad \text{or} \quad \text{rank}(C) = 0 \quad \text{or}$$

$$\text{rank}(A) = \text{rank}(B) = 1 \quad \text{or} \quad \text{rank}(B) = \text{rank}(C) = 1$$

or $\text{rank}(A) = \text{rank}(C) = 1$.

Each of the six irreducible components $V(I)$ is a rational variety, and it admits a nice polynomial parametrization. Using Lemma 3.24, we can then write down all exponential solutions to the four partial differential equations, like (3.9), that are given by $I$. The solutions come in six families.

The solutions contributed by the first irreducible component $\{\text{rank}(A) = 0\}$ are the functions $f(B,C)$ that do not depend on the matrix $A$. The solutions contributed by the last irreducible component have the form

$$f(A,B,C) = \exp \left[ r_1 s_1 a_{11} + r_1 s_2 a_{12} + r_2 s_1 a_{21} + r_2 s_2 a_{22} + (t_{11} u_2 - s_2 t_{12}) b_{11} + (s_2 t_{21} - t_{11} u_1) b_{12} + (s_1 t_{12} - t_{22} u_2) b_{21} + (t_{22} u_1 - s_1 t_{21}) b_{22} + u_1 v_1 c_{11} + u_1 v_2 c_{12} + u_2 v_1 c_{21} + u_2 v_2 c_{22} \right],$$

where $r_i, s_j, t_{ij}, u_i, v_j$ are arbitrary complex numbers. The functions $f$ above satisfy the PDE because the coefficients of $a_{11}, a_{12}, \ldots, c_{22}$ furnish a parametrization of the irreducible variety $\{ABC = 0, \text{rank}(A) = \text{rank}(C) = 1\}$.

Here is our conclusion for this example, valid for the entire book: taking a fresh look at linear algebra offers a point of entry to nonlinear algebra.

---

**Exercises**

1. Let $R = \mathbb{C}[x, y]/\langle x^2, xy, y^2 \rangle$. Is $\{0\}$ an irreducible ideal? Is it primary?
2. Let $n = 1$. Prove that an ideal $I$ in the univariate polynomial ring $\mathbb{Q}[x]$ is a power of a prime ideal if and only if $I$ is primary.
3. Prove that a ring is Noetherian if and only if every ideal is finitely generated.
4. (a) Prove that if $R$ is Noetherian, then so is $R/I$ for any ideal $I$.
   (b) Prove Hilbert Basis Theorem: If $R$ is Noetherian, then so is $R[x]$.
5. Prove Lemma 3.12
6. Check that Example 3.15 provides two distinct minimal primary decompositions. What are all primary decompositions of this ideal?
7. (a) Prove that a prime ideal $p$ cannot be equal to an intersection of (finitely many, more than one, incomparable) ideals.
   (b) More generally prove that if a prime ideal contains an intersection of finitely many ideals, then it contains one of them.
3. Solving and Decomposing

(8) Prove that in a Noetherian ring every ideal contains a power of its radical. Give a counterexample in case of a non-Noetherian ring.

(9) Find three polynomials in three unknowns, each having degree precisely five, whose variety in \( \mathbb{C}^3 \) consists of precisely 37 complex solutions.

(10) Find all solutions \((x, y)\) of the two equations \(x^2 + y = \epsilon\) and \(y^2 + x = \epsilon\) over the algebraic closure of the field \(\mathbb{Q}(\epsilon)\). Write down series solutions.

(11) Let \(n = 10\), \(K = \mathbb{R}\), and consider the ideal generated by the ten polynomials \(x_i y_j + x_j y_i\) where \(1 \leq i < j \leq 5\). These are the \(2 \times 2\)-subpermanents of a \(2 \times 5\)-matrix of unknowns. Find a minimal primary decomposition of \(I\). Interpret your result in terms of solving partial differential equations.

(12) Which \(2 \times 3\) matrices \(A\) and \(B\) satisfy \(AB^T = BA^T\)? How about \(3 \times 2\)?

(13) Let \(K = \mathbb{F}_2\) be the field with two elements. Find an ideal \(I\) in \(K[x, y]\) that has precisely ten associated primes, of which five are embedded.

(14) Consider the ideal \(I = \langle x + y + z, xy + xy + yz, xyz \rangle\) generated by the elementary symmetric polynomials in \(\mathbb{Q}[x, y, z]\). Interpret these as linear PDE and determine the solution space of these PDE.

(15) Interpret the ideal \(P^2\) in Example 3.13 as a system of linear PDE and determine the solution space. How do we see that \(P^2\) is not primary?

(16) Let \(A, B, C\) be \(3 \times 3\)-matrices. In which ways is it possible that the triple product \(ABC\) is the zero matrix? In other words, write down the nine trilinear equations like (3.8) and describe the variety of solutions.
Chapter 4

Mapping and Projecting

“A technique is a trick that works”, Gian-Carlo Rota

A frequently encountered challenge is to compute the image of a polynomial map. Such an image need not be an algebraic variety. However, a natural outer approximation of the image is given by its Zariski closure. The Zariski closure of the image is a variety, described by the polynomials that vanish on it. In this chapter we show how this variety can be found by eliminating variables. Gröbner bases and resultants serve as our primary tools. Further, we provide theorems that allow us to understand the difference between the image and its closure. The answer we obtain depends heavily on the setting, whether we work over the complex numbers \( \mathbb{C} \) or over the real numbers \( \mathbb{R} \), and whether the given polynomials are homogeneous or nonhomogeneous.

4.1. Elimination

In this section we introduce elimination of variables for polynomial ideals. This is our main tool for computing the closure of the image of a polynomial map. We show how to carry it out in practice using Gröbner bases.

We fix an algebraically closed field \( K \) and the polynomial ring \( K[x] = K[x_1, \ldots, x_n] \). Every ideal \( I \subset K[x] \) has an associated affine variety:

\[
\mathcal{V}(I) = \{ p \in K^n : f(p) = 0 \text{ for all } f \in I \}.
\]
We consider the projection from $K^n$ onto the linear subspace $K^m$ that is given by the first $m$ coordinates:

$$\pi : K^n \to K^m, (p_1, \ldots, p_m, p_{m+1}, \ldots, p_n) \mapsto (p_1, \ldots, p_m).$$

If $V$ is a variety in $K^n$ then its image $\pi(V)$ need not be a variety.

**Example 4.1** ($n = 2, m = 1$). The image of the hyperbola $V = \mathcal{V}(xy - 1)$ under the projection $K^2 \to K^1$ from the plane to the $x$-axis equals $\pi(V) = K^1 \setminus \{0\}$. This is not a variety in $K^1$. Note that the image becomes closed if, prior to projecting, we first perform a change of coordinates. For instance, if we replace $V$ by the hyperbola $V' = \mathcal{V}((x+y)(x-y) - 1)$ then $\pi(V') = K^1$.

By definition, the Zariski closure $\overline{\pi(V)}$ of the image is a variety in $K^m$. It is the smallest variety containing $\pi(V)$. We call the variety $\overline{\pi(V)}$ the closed image of $V$ under the map $\pi$. A more general case, when $\pi$ is an arbitrary polynomial map will be discussed in the next section. The next theorem characterizes the ideal of the closed image $\overline{\pi(V)}$.

**Theorem 4.2.** Let $I \subset K[x]$ be an ideal and $V = \mathcal{V}(I)$ its variety in $K^n$, where $K$ is an algebraically closed field. Then its closed image in $K^m$ is the variety $\overline{\pi(V)} = \mathcal{V}(J)$ defined by the elimination ideal

$$J = I \cap K[x_1, \ldots, x_m].$$

If $I$ is radical or prime then the elimination ideal $J$ has the same property.

**Proof.** If $J$ is not a prime ideal then there exist polynomials $f$ and $g$ in $K[x_1, \ldots, x_m]$ such that $fg \in J$ but $f, g \notin J$. The same polynomials show that $I$ is not prime. Similarly, if $J$ is not radical then there exists $f$ in $K[x_1, \ldots, x_m]$ and $r \geq 2$ such that $f^r \in J$ but $f \notin J$. The same $f$ shows that $J$ is not radical. Similar reasoning shows that all ideals $I \subset K[x]$ satisfy

$$\text{Rad}(I) \cap K[x_1, \ldots, x_m] = \text{Rad}(I \cap K[x_1, \ldots, x_m]).$$

Since passing to the radical does not change the variety of a given ideal, we may assume that $I$ and $J$ are radical ideals. We shall now make a forward reference and use the Nullstellensatz (Chapter 6). A polynomial belongs to $I$ if and only if it vanishes on $V = \mathcal{V}(I)$. This holds, in particular, for polynomials $f$ in the subring $K[x_1, \ldots, x_m]$. Such an $f$ belongs to $J$ if and only if it vanishes on $\pi(V)$ if and only if it vanishes on $\pi(V)$. The latter condition means that $f$ lies in the radical ideal of $\overline{\pi(V)}$. We conclude that the radical ideal of the closed image $\overline{\pi(V)}$ is precisely the elimination ideal $J$. For further details we refer to [11], §4.4, Theorem 3. □

**Theorem 4.2** says that the algebraic operation of elimination corresponds to the geometric operation of projection. This holds in many settings, not
just in algebraic geometry. For instance, Gaussian elimination in linear algebra corresponds to projection of linear subspaces, and Fourier-Motzkin elimination in convex geometry corresponds to projection of polyhedra. Alternatively, from the perspective of logic, we can think of our projection as quantifier elimination. We are eliminating the \( n - m \) existentially quantified variables from the first-order logic statement \( \exists x_{m+1}, \ldots, x_n : x \in V \).

Elimination and projection are fundamental operations in many applications. One good example is the problem of matrix completion or tensor completion which arises frequently in data science. Here is an illustration.

**Example 4.3** (Matrix Completion). Fix \( n = 15 \) and let \( V \) be the irreducible variety of symmetric \( 5 \times 5 \)-matrices \( X = (x_{ij}) \) of rank \( \leq 2 \). Its prime ideal \( I = \mathcal{I}(V) \) is minimally generated by 50 homogeneous cubic polynomials, namely the \( 3 \times 3 \)-minors of the matrix \( X \). In fact, these 50 cubics form a Gröbner basis for the degree reverse lexicographic order.

Now let \( m = 10 \) and order the variables so that the five diagonal entries \( x_{11}, x_{22}, x_{33}, x_{44}, x_{55} \) come last. Then the elimination ideal is principal:

\[
J = \langle x_{14}x_{15}x_{23}x_{25}x_{34} - x_{13}x_{15}x_{24}x_{25}x_{34} - x_{14}x_{15}x_{23}x_{24}x_{35} \\
+ x_{13}x_{14}x_{24}x_{25}x_{35} + x_{12}x_{15}x_{24}x_{34}x_{35} - x_{12}x_{14}x_{25}x_{34}x_{35} \\
+ x_{13}x_{15}x_{23}x_{24}x_{45} - x_{13}x_{14}x_{23}x_{25}x_{45} - x_{12}x_{15}x_{23}x_{34}x_{45} \\
+ x_{12}x_{13}x_{25}x_{34}x_{45} + x_{12}x_{14}x_{23}x_{35}x_{45} - x_{12}x_{13}x_{24}x_{35}x_{45} \rangle.
\]

The ideal generator of degree 5 is known as the *pentad* in algebraic statistics [19, Example 4.2.8]. Its 12 terms correspond to the 12 Hamiltonian cycles in the complete graph \( K_5 \). The hypersurface \( V(J) \) equals the closed image \( \pi(V) \) of the determinantal variety \( V \) under the projection from \( K_{15} \) onto the subspace \( K_{10} \) whose coordinates are the off-diagonal entries.

Our result has the following interpretation in terms of matrix completion. If the 10 off-diagonal entries of a symmetric \( 5 \times 5 \)-matrix are given then this can be completed to a matrix of rank \( \leq 2 \) only if the pentad vanishes. This constraint appears in the statistical theory of factor analysis [19].

Our next example shows how to find algebraic relations via elimination.

**Example 4.4.** The first four power sums in three variables \( x, y, z \) are the polynomials \( x^i + y^i + z^i \) for \( i = 1, 2, 3, 4 \). These four polynomials must be algebraically dependent since they involve only three variables. But, what is the algebraic relation satisfied by these four power sums?

We approach this question by setting \( n = 7, m = 4 \) and fixing the ideal

\[
I = \langle x + y + z - p_1, \ x^2 + y^2 + z^2 - p_2, \ x^3 + y^3 + z^3 - p_3, \ x^4 + y^4 + z^4 - p_4 \rangle.
\]
This ideal lives in a polynomial ring in seven variables. We wish to eliminate the three original variables $x, y, z$. Thus, we ask for the elimination ideal

$$J = I \cap K[p_1, p_2, p_3, p_4].$$

This is a principal prime ideal. Its generator is a polynomial of degree four:

$$J = \langle p_4^4 - 6p_1^2p_2 + 3p_2^2 + 8p_1p_3 - 6p_4 \rangle.$$

This is the desired relation. Please check by plugging in the power sums.

The computations in our two examples were carried out using Gröbner bases. Here is how this works. We first fix the lexicographic monomial order $\prec$ on $K[x]$ with $x_1 \prec x_2 \prec \cdots \prec x_n$. We then compute the reduced Gröbner basis for the ideal generated by the given polynomials. And, finally, we select those polynomials from the output that use only the first $m$ variables.

**Theorem 4.5.** If $G$ is a lexicographic Gröbner basis for an ideal $I$ in the polynomial ring $K[x]$ then its elimination ideal $J$ in (4.7) has the Gröbner basis $G' = G \cap K[x_1, \ldots, x_m]$. If $G$ is the reduced Gröbner basis of $I$ then $G'$ is the reduced Gröbner basis of $J$.

**Proof.** Clearly, the set $G'$ is contained in $J = I \cap K[x_1, \ldots, x_m]$. Consider any nonzero polynomial $f \in J$. The initial monomial $\text{in}_\prec(f)$ is divisible by $\text{in}_\prec(g)$ for some $g \in G$. None of the variables $x_{m+1}, \ldots, x_n$ appears in the monomial $\text{in}_\prec(g)$. Every trailing term of $g$ is lexicographically smaller, so it cannot use any of the last $n - m$ variables. Hence, $g$ lies in $G'$.

We have shown that some initial monomial from $G'$ divides $\text{in}_\prec(f)$. Since the polynomial $f$ was chosen arbitrarily from $J \setminus \{0\}$, this means that $G'$ is a Gröbner basis for $J$. If the given Gröbner basis $G$ is reduced then $G'$ also satisfies the two requirements for being a reduced Gröbner basis. \qed

This result shows that the lexicographic Gröbner basis $G$ solves the elimination problem simultaneously for all $m$. Thus computing $G$ means triangularizing a given system of polynomial equations. We saw in Example 1.19 that it can be quite costly to compute a lexicographic Gröbner basis. One therefore often uses different strategies to carry out the elimination process. But Theorem 4.5 represents the main idea that underlies these strategies. Lexicographic elimination is a key tool for solving systems of polynomial equations. It is instructive to try this for some zero-dimensional varieties.

**Example 4.6.** Here is a simple question: can you find three real numbers $x, y, z$ whose $i$-th power sum equals $i$ for $i = 1, 2, 3$? To answer this question, we compute the lexicographic Gröbner basis of the following ideal:

$$I = \langle x + y + z - 1, x^2 + y^2 + z^2 - 2, x^3 + y^3 + z^3 - 3 \rangle.$$
This Gröbner basis equals
\[ \mathcal{G} = \{ 6z^3 - 6z^2 - 3z - 1, 2y^2 + 2yz - 2y + 2z^2 - 2z - 1, x + y + z - 1 \}. \]

Theorem 4.2 says that we can solve our equations by back-substitution. Indeed, the equations have six complex zeros. We first compute the three roots of the cubic in \( z \), we substitute them into the second equation and solve for \( y \), and then we set \( x = 1 - y - z \). The cubic has one real root and two complex conjugate roots:
\[ z \in \{ 1.4308, -0.21542 - 0.26471i, -0.21542 + 0.26471i \}. \]

By symmetry, the zeros of \( I \) are precisely the six points in \( \mathbb{C}^3 \) whose coordinates are permutations of the three complex numbers above. Hence, the answer to our question is “no”. The variety \( \mathcal{V}(I) \) has no real points.

### 4.2. Implicitization

Implicitization is a special instance of elimination. Here, the problem is to compute the image of a polynomial map between two affine spaces. This can be done by forming the graph of the map and then projecting onto the image coordinates. To be precise, we consider a map of the form
\[ (4.2) \quad f : K^m \rightarrow K^n, \quad p = (p_1, \ldots, p_m) \mapsto (f_1(p), \ldots, f_n(p)), \]
where \( f_1, \ldots, f_n \) are polynomials in \( K[z_1, \ldots, z_m] \), and \( K \) is an algebraically closed field. We write \( \text{image}(f) \) for the image of \( K^m \) under this map. This need not be a variety, as the following example shows:

**Example 4.7.** Let \( m = 2, n = 3 \) and consider the map \( f = (z_1, z_1z_2, z_1z_2^2) \) from the plane \( K^2 \) into 3-space \( K^3 \). The Zariski closure of the image is the surface \( \mathcal{V} = \mathcal{V}(x_1x_3 - x_2^3) \). The point \((0,0,1)\) is in the surface but not in \( \text{image}(f) \). For \( K = \mathbb{C} \) we can approximate \((0,0,1)\) by a sequence of points that do lie in the image, e.g. by taking \( z_1 = \epsilon^2 \) and \( z_2 = \epsilon^{-1} \) for \( \epsilon \to 0 \).

The closed image of the map \( f : K^m \rightarrow K^n \) is the Zariski closure of the set-theoretic image \( \text{image}(f) \). The closed image is denoted \( \overline{\text{image}(f)} \subset K^n \).

**Corollary 4.8.** Given the map \( f \) in (4.2), let \( I \) be the ideal in the polynomial ring \( K[x,z] \) in \( n + m \) variables which is generated by \( f_i(z_1, \ldots, z_m) - x_i \) for \( i = 1, 2, \ldots, n \). The closed image of \( f : K^m \rightarrow K^n \) is the variety defined by the elimination ideal \( J = I \cap K[x] \). In symbols, \( \overline{\text{image}(f)} = \mathcal{V}(J) \).

**Proof.** Allowing \( X \) to be the whole space, it is enough to prove the last statement. The graph of \( f \) restricted to \( X \) is Zariski closed in \( K^{n+m} \), and...
$I + I_X$ is the ideal that defines it. The image of $X$ is the projection of the graph onto $K^n$. With this, the claim follows from Theorem 4.2 \hfill \Box

Example 4.9 (Plücker relations). What are the algebraic relations among the $2 \times 2$-minors of a $2 \times 5$-matrix? We answer this question by setting $m = n = 10$ and by considering the map $f : K^{10} \rightarrow K^{10}$ that takes a matrix 
\[
\begin{pmatrix}
z_{11} & z_{12} & z_{13} & z_{14} & z_{15} 
z_{21} & z_{22} & z_{23} & z_{24} & z_{25}
\end{pmatrix}
\]

with the $2 \times 2$-minors of a $2 \times 2$-matrix, with basis elements indexed by three numbers $0 \leq i < j \leq 5$. The graph of $f$ is described by an ideal $I$ in the polynomial ring $K[x, z]$ in 20 variables. Note that $I$ is generated by 10 polynomials. The desired elimination ideal equals

\[
I \cap K[x] = \langle x_{12}x_{34} - x_{13}x_{24} - x_{14}x_{23}, x_{12}x_{35} - x_{13}x_{25} - x_{15}x_{23}, x_{12}x_{45} - x_{14}x_{25} - x_{15}x_{24}, x_{13}x_{45} - x_{14}x_{35} - x_{15}x_{34}, x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} \rangle.
\]

These five quadrics are the Plücker relations among the maximal minors. They play a key role in our study of Grassmannians in Chapter 5. The ten variables in $K[x]$ can be written as the entries of a skew-symmetric matrix

\[
X = \begin{pmatrix}
0 & x_{12} & x_{13} & x_{14} & x_{15} \\
x_{12} & 0 & x_{23} & x_{24} & x_{25} \\
x_{13} & x_{23} & 0 & x_{34} & x_{35} \\
x_{14} & x_{24} & x_{34} & 0 & x_{45} \\
x_{15} & x_{25} & x_{35} & x_{45} & 0
\end{pmatrix}.
\]

The Plücker relations are the *pfaffians* of size $4 \times 4$, that is, the square roots of the principal $4 \times 4$ minors of $X$. Thus $\mathcal{V}(I \cap K[x])$ is the variety of skew-symmetric $5 \times 5$ matrices of rank $\leq 2$. We shall see in Chapter 5 that, as a projective variety in $\mathbb{P}^9$, this is the Grassmannian of lines in $\mathbb{P}^4$. Each such line is written in Plücker coordinates as the image of the rank 2 matrix $X$.

The notion of the determinant is central to linear algebra. In nonlinear algebra, there is an analogous notion of a hyperdeterminant for tensors.

Example 4.10 (Hyperdeterminant). Let $X = (x_{ijk})$ be a tensor of format $2 \times 2 \times 2$, where the $n = 8$ tensor entries are variables. Tensors will be discussed in Chapter 9. We here just view $X$ as an element of an 8-dimensional linear space, with basis elements indexed by three numbers $0 \leq i, j, k \leq 1$.

The tensor represents an affine-trilinear polynomial in $m = 3$ variables:

\[
f = x_{000} + x_{100}z_1 + x_{010}z_2 + x_{001}z_3 + x_{110}z_1z_2 + x_{101}z_1z_3 + x_{011}z_2z_3 + x_{111}z_1z_2z_3.
\]

For any fixed $X$, this polynomial defines a surface $\mathcal{V}(f)$ in $K^3$. We are interested in the condition under which this surface is singular. It is singular at the point $z$ if and only if the pair $(X, z) \in K^{11}$ lies in the variety of

\[
I = \langle f, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3} \rangle.
\]
The elimination ideal $I \cap K[x]$ is principal. We find that its generator is
\[x_{110}^2 x_{011}^2 + x_{100}^2 x_{001}^2 + x_{000} x_{101}^2 + x_{000} x_{111}^2 + 4 x_{000} x_{110} x_{011} x_{101} + 4 x_{010} x_{100} x_{001} x_{111} - 2 x_{100} x_{110} x_{001} x_{011} - 2 x_{010} x_{110} x_{001} x_{101} - 2 x_{010} x_{100} x_{011} x_{101} - 2 x_{000} x_{110} x_{001} x_{111} - 2 x_{000} x_{100} x_{111} x_{111} - 2 x_{000} x_{100} x_{011} x_{111} - 2 x_{000} x_{100} x_{111} x_{111} .\]

This quartic is the $2 \times 2 \times 2$ hyperdeterminant. It vanishes when the surface $V(f)$ fails to be smooth in $K^3$. Hyperdeterminants exist for tensors of many larger formats. Their study is a fascinating topic in nonlinear algebra. A standard reference is the book by Gel’fand, Kapranov and Zelevinsky [25].

The most basic scenario in elimination arises when $m$ variables are eliminated from a system of $m + 1$ equations. One expects the result to be a single equation in the coefficients of that system. We saw this for $m = 3$ in Examples 4.4 and 4.10. The theory of resultants is custom-taylored to predict the eliminant in such cases. We set this up over the field $\mathbb{Q}$ as follows.

Let $i \in \{1, 2, \ldots, m + 1\}$ and fix a general inhomogeneous polynomial $f_i$ of degree $d_i$ in $z_1, \ldots, z_m$. This polynomial has $\binom{d_i + m}{m}$ unknown coefficients $x_i,u$, one for each monomial $z^u$ of degree $\leq d_i$. The total number of unknown coefficients equals $n = \sum_{i=1}^{m+1} \binom{d_i + m}{m}$. We write $\mathbb{Q}[x,z]$ for the resulting polynomial ring in $n + m$ variables. Inside this ring we consider the ideal

$$I = \langle f_1, f_2, \ldots, f_m, f_{m+1} \rangle \subset \mathbb{Q}[x,z].$$

We are interested in the ideal in $\mathbb{Q}[x]$ found by eliminating the $m$ variables $z_i$. This ideal describes the polynomial conditions on the coefficients of $m + 1$ polynomials in $m$ variables which represents the statement that they have a common root over the algebraic closure $\overline{\mathbb{Q}}$ of our field $\mathbb{Q}$. To be very precise, in the affine space $\mathbb{P}^m$, the condition of having a common root is not closed, thus the geometric interpretation is best seen after projectivization.

**Theorem 4.11.** The elimination ideal $I \cap \mathbb{Q}[x]$ is principal. Its generator is an irreducible polynomial in the entries of the coefficient vector $x$. This is denoted $\text{Res}(f_1, \ldots, f_{m+1})$ and called the resultant. The degree of the resultant in the coefficients of $f_i$ equals $d_1 \cdots d_{i-1} d_{i+1} \cdots d_{m+1}$ for $i = 1, 2, \ldots, m+1$.

**Proof.** We refer to [12, Chapter 3] for the proof. In that source, and in many others, the $f_i$ are taken to be homogeneous polynomials in $m + 1$ variables. We here prefer the inhomogeneous case, which allows for a simpler formulation as an elimination ideal. The two versions are equivalent.

Here, we just briefly explain where the formula for the degree comes from. To obtain the degree $D_i$ of the resultant in the coefficients of $f_i$, let us fix general coefficients of $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{m+1}$. The degree $D_i$ is the degree of the polynomial $P$ we obtain from the resultant after substituting all of these coefficients. By Bézout’s Theorem 2.16 the $m$ fixed polynomials define $D'_i := d_1 \cdots d_{i-1} d_{i+1} \cdots d_{m+1}$ many points in the affine space $\mathbb{P}^m$. 
4. Mapping and Projecting

The condition that \( f_i \) passes through one of these points is a linear condition in the coefficients of \( f_i \). Hence, \( P \) vanishes if and only if its variables satisfy one of \( D_i' \) many linear conditions. Thus the degree \( D_i \) of \( P \) equals \( D_i' \). □

**Example 4.12** (Determinants). Let \( d_1 = \cdots = d_{m+1} = 1 \). The \( m+1 \) polynomials \( f_i \) are affine-linear. They can be written as a matrix-vector product

\[
\begin{pmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_m \\
  f_{m+1}
\end{pmatrix} =
\begin{pmatrix}
  x_{1,1} & x_{1,2} & \cdots & x_{1,m} & x_{1,m+1} \\
  x_{2,1} & x_{2,2} & \cdots & x_{2,m} & x_{2,m+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_{m,1} & x_{m,2} & \cdots & x_{m,m} & x_{m,m+1} \\
  x_{m+1,1} & x_{m+1,2} & \cdots & x_{m+1,m} & x_{m+1,m+1}
\end{pmatrix} 
\begin{pmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_m \\
  1
\end{pmatrix}.
\]

The resultant \( \det(f_1, \ldots, f_{m+1}) \) is the determinant of the coefficient matrix \( (x_{i,j}) \). This determinant is a homogeneous polynomial of degree \( m + 1 \) in \( n = (m + 1)^2 \) unknowns with \( (m + 1)! \) terms. It has degree one in the coefficients of each \( f_i \). Note that, if we set all but last column to zero, then the determinant clearly vanishes. The polynomials in this case become constants and do not have common zero. To better understand this, one could pass to the projective space, i.e. homogenize the affine-linear forms \( f_i \).

**Example 4.13** (Eliminating one variable from two quadratic polynomials). Let \( m = 1 \) and \( d_1 = d_2 = 2 \) and abbreviate \( z = z_1 \). Our system consists of two univariate polynomials of degree two with six unspecified coefficients:

\[
f_1 = x_{11}z^2 + x_{12}z + x_{13} \quad \text{and} \quad f_2 = x_{21}z^2 + x_{22}z + x_{23}.
\]

The generator of the elimination ideal \( \langle f_1, f_2 \rangle \cap \mathbb{Q}[x] \) is the *Sylvester resultant* \( \text{(4.3)} \)

\[
\text{Res}(f_1, f_2) = \det \begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 \\ 0 & x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} & 0 \\ 0 & x_{21} & x_{22} & x_{23} \end{pmatrix}.
\]

This is a bihomogeneous polynomial of bidegree \((d_1, d_2) = (2, 2)\). Its expansion has 7 terms. It vanishes if the two quadrics have a common zero.

The formula \( \text{(4.3)} \) generalizes to two polynomials in \( z \) of arbitrary degrees \( d_1, d_2 \). The resultant is the determinant of the *Sylvester matrix* of format \((d_1+d_2) \times (d_1+d_2)\). This matrix is denoted \( \text{Syl}_{d_1,d_2} \) and it is displayed below. The first \( d_2 \) rows are formed by the coefficients of the first polynomial \( f_1 \), suitably shifted and padded with zeros. The last \( d_1 \) rows are similarly formed by the coefficients of the second polynomial \( f_2 \). The matrix is as follows:
4.2. Implicitization

For \( d_1 = d_2 = 2 \) this is the \( 4 \times 4 \) matrix seen in (4.3).

**Theorem 4.14.** The determinant of the Sylvester matrix \( \text{Syl}_{d_1, d_2} \) is equal to the resultant \( \text{Res}(f_1, f_2) \) of the two univariate polynomials

\[
\begin{align*}
  f_1(z) &= x_{11}z^{d_1} + \cdots + x_{1,d_1+1}z + x_{1,1+1} \\
  f_2(z) &= x_{21}z^{d_2} + \cdots + x_{2,d_2+1}z + x_{2,1+1}.
\end{align*}
\]

**Proof.** We first note that \( \det(\text{Syl}_{d_1, d_2}) \) is a non-zero polynomial. We can see this by specializing \( f_1 = z^{d_1} \) and \( f_2 = 1 \). Here the Sylvester matrix \( \text{Syl}_{d_1, d_2} \) specializes to the identity matrix, so its determinant is non-zero.

Let \( Z \) denote the column vector with entries \( z^{d_1+1}, z^{d_1+d_2-2}, \ldots, z, 1 \), and \( F \) the column vector with entries \( z^{d_2-1}f_1, \ldots, zf_1, f_1, z^{d_1-1}f_2, \ldots, zf_2, f_2 \). Both vectors have length \( d_1 + d_2 \). They are related by the Sylvester matrix:

\[
\text{Syl}_{d_1, d_2} \cdot Z = F.
\]

Multiplying on the left by the adjoint of the Sylvester matrix, we obtain

\[
\det(\text{Syl}_{d_1, d_2}) \cdot Z = \text{adj}(\text{Syl}_{d_1, d_2}) \cdot F.
\]

The last coordinate of the column vector \( Z \) equals 1. Hence, the last coordinate in this equation shows that \( \det(\text{Syl}_{d_1, d_2}) \) is a polynomial linear combination of the entries of \( F \), and hence it lies in the ideal \( \langle f_1, f_2 \rangle \). The Sylvester determinant is a non-zero homogeneous polynomial of degree \( d_1 + d_2 \) that lies in the ideal \( \langle f_1, f_2 \rangle \cap \mathbb{Q}[x] \). We know from Theorem 4.11 that this ideal is principal, and its generator \( \text{Res}(f_1, f_2) \) also has degree \( d_1 + d_2 \). This implies that the resultant \( \text{Res}(f_1, f_2) \) is equal to the Sylvester determinant \( \det(\text{Syl}_{d_1, d_2}) \), up to a non-zero multiplicative constant. \( \square \)

**Example 4.15.** Let \( f_1(z) \) and \( f_2(z) \) be univariate polynomials of degree \( d_1 \) and \( d_2 \) in \( \mathbb{Q}[z] \). This defines a map \( f : \mathbb{C} \rightarrow \mathbb{C}^2 \) whose closed image is an algebraic curve in the plane \( \mathbb{C}^2 \) with coordinates \( x_1, x_2 \). The implicit...
Mapping and Projecting

4. Mapping and Projecting

Equation of this curve is the resultant $\text{Res}_z(x_1 - f_1(z), x_2 - f_2(z))$, taken with respect to the variable $z$. For a concrete example consider the plane cubic curve given parametrically by $f = (z^3 + 4z, z^2 - 3)$. Its equation equals

$$\det \begin{bmatrix} -1 & 0 & -4 & x_1 & 0 \\ 0 & -1 & 0 & -4 & x_1 \\ -1 & 0 & x_2 + 3 & 0 & 0 \\ 0 & -1 & 0 & x_2 + 3 & 0 \\ 0 & 0 & -1 & 0 & x_2 + 3 \end{bmatrix} = x_2^3 - x_1^2 + 17x_2^2 + 91x_2 + 147.$$

If $m \geq 2$ then the resultant $\text{Res}(f_1, f_2, \ldots, f_{m+1})$ is more difficult to compute, and there does not always exists a formula as the determinant whose entries are linear expressions in the coefficients of $f_1, f_2, \ldots, f_{m+1}$. In some cases, however, such formulas are available in the literature. For instance, Sylvester already gave such a formula for $m = 2$ and $d_1 = d_2 = d_3$.

A considerable body of information on matrix formulas for resultants can be found in the excellent book by Gel’fand, Kapranov and Zelevinsky [25].

4.3. The Image of a Polynomial Map

We discussed methods for computing the Zariski closure of the image of a polynomial map. Can we say something about the image itself? The answer is ’yes’, but the situation very much depends on the field $K$ and whether we are in the projective case or in the affine case. In this section we discuss tools for computing such images. We begin by highlighting the difference between the real numbers and complex numbers with regard to this problem.

We start with a real, affine variety $X \subset \mathbb{R}^m$. We would like to understand the image of $X$ under a polynomial map:

$$f = (f_1, \ldots, f_n) : \mathbb{R}^m \to \mathbb{R}^n.$$  

Easy examples show that the Zariski closure of the image and the image itself can differ a lot. For instance, this happens for $n = m = 1$, $X = \mathbb{R}$ and $f(z) = z^2$. Is there a chance in general of describing the image using polynomials? The following theorem provides a positive answer. It refers to the concept of semi-algebraic sets, which was introduced in Definition 2.12.

**Theorem 4.16** (Tarski-Seidenberg). Working over the field of real numbers, the image of a variety in $\mathbb{R}^m$ under the map $f$ is a semi-algebraic set in $\mathbb{R}^n$.

**Proof.** See [6] Section 1.4].

Thus to provide a description of the image over $\mathbb{R}$ we need two ingredients: polynomial equations and polynomial inequalities, suitably combined.
Example 4.17. Let \( m = 6 \), \( n = 9 \) and \( f \) the map that multiplies a \( 3 \times 2 \)-matrix \( Z \) with its transpose to get a \( 3 \times 3 \) matrix \( X = (x_{ij}) \):

\[
Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{bmatrix} \mapsto X = \begin{bmatrix} z_{11}^2 + z_{12}^2 & z_{11}z_{21} + z_{12}z_{22} & z_{11}z_{31} + z_{12}z_{32} \\ z_{11}z_{21} + z_{12}z_{22} & z_{21}^2 + z_{22}^2 & z_{21}z_{31} + z_{22}z_{32} \\ z_{11}z_{31} + z_{12}z_{32} & z_{21}z_{31} + z_{22}z_{32} & z_{31}^2 + z_{32}^2 \end{bmatrix}
\]

The image of \( f \) is the set of positive semidefinite symmetric \( 3 \times 3 \) matrices of rank \( \leq 2 \). This is a 5-dimensional semialgebraic set in the space \( \mathbb{R}^9 \) of \( 3 \times 3 \)-matrices. Its polynomial description consists of the four equations

\[
(4.4) \quad x_{12} = x_{21}, \ x_{13} = x_{31}, \ x_{23} = x_{32} \quad \text{and} \quad \det(X) = 0,
\]

and the six inequalities

\[
x_{11} \geq 0, \ x_{22} \geq 0, \ x_{33} \geq 0, \ x_{11}x_{22} \geq x_{12}^2, \ x_{11}x_{33} \geq x_{13}^2, \ x_{22}x_{33} \geq x_{23}^2.
\]

If we pass to an algebraically closed field, then the situation is much simpler. The image of \( f : \mathbb{C}^6 \to \mathbb{C}^9 \) in Example 4.17 is closed: it is precisely the subvariety defined by (4.4). In general, we have the following result.

**Theorem 4.18** (Chevalley). If \( K \) is algebraically closed, then the image of a variety under a polynomial map is a constructible set. Hence, if \( K = \mathbb{C} \) then the Euclidean closure and the Zariski closure of the image coincide.

**Proof sketch.** The image is a projection of the graph of the map. One can apply the Nullstellensatz - see Chapter 6 - to turn the problem into one from linear algebra. Resultants and their matrices play a central role. Details can be found in e.g. [64, Sections 7.4.6–7.4.8].

Suppose we want to check if a random (in a reasonable sense) point belongs to the image and we work over \( \mathbb{C} \). It is enough to check polynomial equations, which can be obtained as described in previous two subsections.

Obtaining the whole description of the image is slightly more complicated. We may proceed as follows:

- Compute the closed image \( X_0 \).
- Subtract from it a proper subvariety \( X_1 \).
- Add back \( X_2 \) - a proper subvariety of \( X_1 \), etc.

This procedure must finish in a finite number of steps, by Hilbert’s Basis Theorem. For a recent algorithm and its implementation we refer to [27].

**Example 4.19.** Let \( m = n = 3 \) and consider the map

\[
f : \mathbb{C}^3 \to \mathbb{C}^3, \ (z_1, z_2, z_3) \mapsto (z_1z_2, z_1z_3, z_2z_3).
\]

Its image is Zariski dense in \( \mathbb{C}^3 \). It equals

\[
\text{image}(f) = (\mathbb{C}^3 \setminus \mathcal{V}(x_1x_2x_2)) \cup \mathcal{V}(x_1x_2, x_1x_3, x_2x_3).
\]
Over the real numbers, we would also need the inequality $x_1x_2x_3 \geq 0$.

The images are nicest when we work with projective varieties over a field like $\mathbb{C}$. The following theorem may be regarded as an algebraic analog of the fact that images of compact sets under continuous maps are compact.

**Theorem 4.20.** Let $X$ be a projective variety over an algebraically closed field. Then the image of $X$ under a regular map is (Zariski) closed.

**Proof.** We refer to [51, Section 5.2]. If we work over $\mathbb{C}$ then the projective variety $X$ is compact in the Euclidean topology. Hence, the image of $X$ is also compact and hence closed. However, by Chevalley’s Theorem 4.18 the Zariski closure and the Euclidean closures of the image coincide. We conclude that the image is also Zariski closed. □

This is a very powerful theorem. For instance, it implies the following. Consider a map $f = (f_1, \ldots, f_n)$ given by homogeneous polynomials of the same degree. Assume that the affine variety $\mathcal{V}(f_1, \ldots, f_n)$ equals $\{0\}$. Then the image of $f$ is Zariski closed - we can compute it using elimination. For an application see Exercise 18. In general, the following theorem holds.

**Theorem 4.21.** Consider a map $f = (f_1, \ldots, f_n) : \mathbb{C}^{m+1} \to \mathbb{C}^n$ given by homogeneous polynomials of the same degree. Suppose that $\dim \mathcal{V}(f_1, \ldots, f_n) = b + 1$ and the closed image of $f$ has affine dimension $d + 1$, i.e. projective dimension $d$. If $d + b < m$ then the image of $f$ is closed.

**Proof.** We consider $f$ as a map from $\mathbb{P}^m \setminus \mathcal{V}(f_1, \ldots, f_n)$. Let $\mathbb{P}^d \subset \mathbb{P}^m$ be a generic projective subspace of dimension $d < m - b$. It is disjoint from $\mathcal{V}(f_1, \ldots, f_n)$. Thus we may assume that $f$ is well-defined on $\mathbb{P}^d$. The image of $\mathbb{P}^d$ under $f$ is closed by Theorem 4.20. It is contained in the image of $\mathbb{P}^m$ under $f$. Both have the same dimension and are irreducible. So, the images coincide, and we conclude that the image of $f$ is closed. □

---

**Exercises**

1. Eliminate the variable $z$ from the equations $x^3y^3z^3 - x - y - z = 1$ and $x^5 + y^5 + z^5 = 2$. Discuss the resulting curve in the $(x,y)$-plane.

2. Prove: If an ideal $I$ in a polynomial ring is prime then so are its elimination ideals, and same for radical. Find examples showing that the converse does not hold. What is the geometric meaning of these facts?
(3) Compute the determinants of the Sylvester matrices $Syl_{1,5}$, $Syl_{2,4}$ and $Syl_{3,3}$. Each of them is a polynomial of degree 6 in 8 unknowns. Which of these three polynomials has the most terms?

(4) A plane curve has the parametrization $z \mapsto (f(z), g(z))$ where $f$ and $g$ are polynomials of degree 10. At most how many terms do you expect the implicit equation of that plane curve to have?

(5) Can you find an invertible $5 \times 5$-matrix that is skewsymmetric?

(6) You are given all entries of a skewsymmetric $5 \times 5$ matrix $X = (x_{ij})$ except for $x_{12}$ and $x_{45}$. Under which condition on the 8 visible entries can you complete with $\text{rank}(X) \leq 2$?

(7) Let $\pi$ be the linear map from $\mathbb{C}^3$ to $\mathbb{C}^2$ given by the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

Given an algebraic curve $V$ in $\mathbb{C}^3$, explain how one can compute the planar curve $\pi(V) \subset \mathbb{C}^2$.

(8) Consider the Fermat curve $V = V(x^3 + y^3 + z^3)$ in the projective plane $\mathbb{P}^2$. Compute the ideal in 6 variables whose variety is the image of $V$ under the Veronese map

$$\mathbb{P}^2 \to \mathbb{P}^5, (x : y : z) \mapsto (x^2 : xy : xz : y^2 : yz : z^2).$$

(9) Determine the prime ideal of relations among the $3 \times 3$-minors of a $3 \times 6$-matrix. Where in the next chapter does this prime ideal appear?

(10) Let $V_1$ and $V_2$ be algebraic curves in $\mathbb{C}^3$ and let $V_1 + V_2$ be their pointwise sum. The Zariski closure $\overline{V_1 + V_2}$ is an algebraic variety in $\mathbb{C}^3$. Explain how one can compute the vanishing ideal of this variety.

(11) Compute the hyperdeterminant of a $2 \times 2 \times 3$ tensor whose 12 entries are unknowns. How many terms does this hyperdeterminant have?

(12) Apply the resultant method in Example 4.15 to compute the implicit equation of the plane cubic curve that has the parametrization

$$z \mapsto \left( 2z^3 + 3z^2 + 5z + 7, 11z^3 + 13z^2 + 17z + 19 \right).$$

(13) Let $m = 2$, $d_1 = 1$, $d_2 = d_3 = 2$. The total number of coefficients is $n = 15 = 3+6+6$. Compute the resultant $\text{Res}(f_1, f_2, f_3)$ explicitly, as a polynomial in all 15 unknowns. How many terms does it have?

(14) Which constraints hold for off-diagonal entries of a rank one $3 \times 3$-matrix?

(15) Which constraints hold for the off-diagonal entries of a nilpotent $3 \times 3$-matrix? Answer this question over the complex numbers $\mathbb{C}$.

(16) Which constraints hold for the off-diagonal entries of an orthogonal $3 \times 3$-matrix? Answer this question over the real numbers $\mathbb{R}$.

(17) Let $m = 2$ and $d_1 = d_2 = d_3 = 2$. Then $\text{Res}(f_1, f_2, f_3)$ is the resultant of three quadrics in the plane. This is a polynomial in $18 = 6+6+6$
variables of degree $12 = 4+4+4$. How many terms does it have? Find an explicit matrix formula for $\text{Res}(f_1, f_2, f_3)$.

(18) Let $V$ be the complex vector space of homogeneous polynomials in $n$ variables of degree $d$. The $d$-th powers of linear forms form a subset of $V$. Is it Zariski closed for any $n$ and $d$? What happens if we change the field to the real numbers?

(19) Consider a degree two polynomial $ax^2 + bx + c$, where $a$, $b$, $c$ are unknown. When does it have a double root and how is this related to resultants? What happens for higher degree polynomials?
Chapter 5

Linear Spaces and Grassmannians

“Geometry is not true, it is advantageous”, Henri Poincaré

In previous chapters we saw the construction of projective space. We argued that in many applications projective varieties are preferable to affine varieties. Points in a projective space correspond to lines through the origin in the underlying vector space. In this chapter we replace lines with higher-dimensional linear subspaces. The role of the projective space is now played by a Grassmannian. This is a smooth projective variety whose points correspond to linear subspaces of a fixed dimension. For instance, the Grassmannian of lines in projective 3-space is a 4-dimensional variety. Its subvarieties represent families of lines. Counting lines that satisfy a certain property (e.g. lying on a cubic surface) leads us to enumerative algebraic geometry, a subject in which Grassmannians play a fundamental role.

5.1. Coordinates for Linear Spaces

Let $V$ be a vector space of dimension $n$ over a field $K$. In Chapter 2 we constructed the projective space $\mathbb{P}(V)$. Its points are the 1-dimensional subspaces of $V$. We note that $\mathbb{P}(V)$ is the key example of a compact algebraic variety when $K = \mathbb{C}$. Our aim is to generalize this construction from lines to subspaces of arbitrary dimension $k$ with $0 < k \leq \dim V$. We will construct a projective variety $G(k, V)$ whose points correspond bijectively to $k$-dimensional subspaces of $V$. This variety is called the Grassmannian, after the 19th century mathematician Hermann Grassmann. If $V = K^n$ then
5. Linear Spaces and Grassmannians

we use the notation \( \mathbb{P}^{n-1} \) for the projective space \( \mathbb{P}(V) \), and we write \( G(k, n) \) for the Grassmannian \( G(k, V) \).

We start with an explicit construction in coordinates, by fixing a basis \( e_1, \ldots, e_n \) of \( V \). Consider any \( k \) linearly independent vectors \( v_1, \ldots, v_k \in V \). We represent them in the form of a \( k \times n \) matrix \( M \) of rank \( k \). To these vectors, or equivalently to a full rank \( k \times n \) matrix, we associate the linear subspace \( W := \langle v_1, \ldots, v_k \rangle \) in \( V \). This association is surjective, but not injective, as we may replace the \( v_i \)'s by linear combinations. In other words, the group \( GL(k) \) of invertible \( k \times k \) matrices acts on the set of \( k \times n \) matrices by left multiplication, and this does not change the linear span of the rows.

We know some polynomial functions that do not change (up to rescaling) under taking linear combinations of the rows: these are the \( k \times k \) minors of the \( k \times n \) matrix. Suppose that \( W \) is a \( k \)-dimensional subspace of \( V \). Pick any basis and express \( W \) as the row space of a \( k \times n \)-matrix. We then write \( i(W) \) for the vector of all \( k \times k \)-minors of that matrix, up to scale.

Example 5.1. Let \( k = 2, n = 5 \) and let \( W \) be the linear subspace of \( V = K^5 \) that is spanned by \((1, 1, 1, 1, 1)\) and \((a_1, a_2, a_3, a_4, a_5)\) for some scalars \( a_i \) that are not all identical. Then \( i(W) \) is the point in \( \mathbb{P}^9 = \mathbb{P}(K^{(2)}) \) whose ten homogeneous coordinates are \( a_i - a_j \) for \( 1 \leq i < j \leq 5 \).

This construction defines a map

\[
i : \{ \text{\( k \)-dimensional subspaces of \( V \) } \} \rightarrow \mathbb{P}(K^{(k)}).\]

This map is well-defined since \( i(W) \) does not depend on chosen basis of \( W \).

Lemma 5.2. The map \( i \) is injective.

Proof. Consider two \( k \)-dimensional subspaces \( W_1, W_2 \subset V \). Assume \( i(W_1) = i(W_2) \). The matrices \( \tilde{M}_{W_1} \) and \( \tilde{M}_{W_2} \) that represent \( W_1 \) and \( W_2 \) have rank \( k \). Without loss of generality we may assume that the first \( k \) columns are linearly independent. By performing linear operations on the rows of both matrices, we transform \( \tilde{M}_{W_1} \) to a matrix \( \tilde{M}_W \) whose left-most \( k \times k \) submatrix is the identity. We observe that any entry of \( \tilde{M}_{W_1} \) not in the first \( k \) columns, is equal to some maximal minor or its negative. Thus, if \( i(W_1) = i(W_2) \) then the two matrices \( \tilde{M}_{W_1} \) and \( \tilde{M}_{W_2} \) must be equal. This implies \( W_1 = W_2 \). \( \square \)

Example 5.3. The reasoning in the proof above is illustrated by the matrix

\[
\tilde{M}_W = \begin{pmatrix}
1 & 0 & 0 & p_{234} & p_{235} & p_{236} \\
0 & 1 & 0 & -p_{134} & -p_{135} & -p_{136} \\
0 & 0 & 1 & p_{124} & p_{125} & p_{126}
\end{pmatrix}.
\]

Let \( k = 3 \) and \( n = 6 \). Each entry \( p_{ijk} \) in the \( 3 \times 3 \) block on the right equals the \( 3 \times 3 \)-minor of the matrix \( \tilde{M}_W \) given by the column indices \( i, j, k \).
The image of \( i \) is the Grassmannian \( G(k,n) \). Its inclusion in \( \mathbb{P}(K(k)) \) is the Plücker embedding. For readers familiar with the exterior power of a vector space, here is a more invariant description of the Grassmannian:

\[
G(k,n) = \{ [v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\wedge^k V) : v_1, \ldots, v_k \in V \text{ are linearly independent} \}.
\]

Indeed, first we may identify \( \mathbb{P}(K(n^k)) \) with \( \mathbb{P}(\wedge^k V) \) by fixing a basis of \( V \) and the induced basis of \( \wedge^k V \). Expanding \( v_1 \wedge \cdots \wedge v_k \) in that basis, we indeed obtain the \( k \times k \) minors of the \( n \times k \) matrix \([v_1, \ldots, v_k]\). The group \( GL(V) \) acts naturally on \( V \), taking subspaces to subspaces. This induces an action on \( \mathbb{P}(\wedge^k V) \) which restricts to the Grassmannian. A matrix \( g \in GL(V) \) transforms \( v_1 \wedge \cdots \wedge v_k \) to \( g(v_1) \wedge \cdots \wedge g(v_k) \). We note that the action is transitive: for any \( p_1, p_2 \in G(k,V) \) there exists a (non-unique) automorphism \( g \in GL(V) \) such that \( g(p_1) = p_2 \). This holds because any set of \( k \) linearly independent vectors may be transformed by an invertible linear map to any other such set. Hence, \( G(k,V) \) is an orbit under the action of \( GL(V) \) on \( \mathbb{P}(\wedge^k V) \). In fact, \( G(k,V) \) is the unique closed orbit in this space.

Projective algebraic varieties that are orbits of linear algebraic groups are called homogeneous, the Grassmannians being prominent examples. Homogeneous varieties are always smooth. Indeed, any algebraic variety always contains a smooth point and an action of a group must take a smooth point to a smooth point - a version of this statement is given in Exercise 2.

Our next aim is to show that the Grassmannian is a projective variety. Equivalently, we need to express the property that \( \binom{n}{k} \) given numbers are the minors of a \( k \times n \) matrix by the vanishing of (homogeneous) polynomials.

**Theorem 5.4.** The Grassmannian \( G(k,n) \) is Zariski closed and irreducible.

**Proof.** Lemma 5.2 gives us an idea how to proceed. Namely, first let us assume that the matrix \( M_W \) representing \( W \) has the form seen in (5.1),

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix}
A,
\]

where \( A \) is a \( k \times (n-k) \) matrix. Each maximal minor of \( M_W \) is now, up to sign, a minor of \( A \) of some size. Further, by Laplace expansion, a \( q \times q \) minor of \( A \) for \( q > 1 \) may be expressed, as a quadratic polynomial, in terms of smaller minors. This provides with \( \sum_{q=2}^{\min(k,n-k)} \binom{k}{q} \binom{n-k}{q} \) inhomogeneous quadratic equations in the minors of the \( k \times (n-k) \)-matrix \( A \). These equations define the part of the image of our map \( i \) that lies in the affine open set \( K(k)^{-1} \subset \mathbb{P}(K(k)) \) given by the nonvanishing of the first Plücker coordinate.
If \( i(W) \) has its first coordinate zero then some other coordinate will be nonzero, as the matrix \( M_W \) must have some invertible \( k \times k \) submatrix. If we multiply \( M_W \) on the left by the inverse of that matrix then we obtain a matrix that looks like (5.2) but with its columns permuted. The same construction as before gives us a system of

\[
\sum_{q=2}^{\min(k,n-k)} \binom{k}{q} \binom{n-k}{q} \text{ inhomogeneous quadratic equations in the } k(n-k) \text{ entries of the new matrix } A.
\]

Each of the quadratic equations in \( k(n-k) \) variables obtained above can be written as a homogeneous quadric in the \( \binom{n}{k} \) coordinates on \( \mathbb{P}(K(n)) \). Namely, a minor of \( A \) is replaced by the corresponding maximal minor of \( M_W \), and then the quadric is homogenized by the special minor that corresponds to the identity matrix in (5.2). Each of the homogenized polynomials vanishes on the Grassmannian. Indeed, each set of polynomials, by construction, vanishes on the part of the Grassmannian that corresponds to matrices with a fixed nonzero minor. However, any full rank matrix may be approximated by full rank matrices with all minors nonvanishing.

We also know that our equations define the Grassmannian on each open affine chart of the projective space \( \mathbb{P}(K(n)) \). Hence, the collection of all the constructed homogeneous quadratic equations gives a full polynomial description of \( G(k,n) \) as a set.

The affine cone over the Grassmannian \( G(k,n) \) is the image of a polynomial map \( W \mapsto i(W) \) from the matrix space \( K^{k \times n} \) to the Plücker space \( \mathbb{P}(K(n))^{n-k-1} \) defined by taking all maximal minors of a \( k \times n \) matrix \( W \). In particular, the image is irreducible and hence so is the Grassmannian \( G(k,n) \).

\[ \square \]

### 5.2. Plücker Relations

We have proved that the set \( G(k,n) \) is cut out by quadratic equations. In fact, with slightly more effort one can show that \( I(G(k,n)) \) may be generated by quadratic polynomials. These are known as Plücker relations [42 Chapter 3]. Further below we will discuss the Plücker relations for \( k = 2 \).

From a more algebraic perspective the equations vanishing on \( G(k,n) \) are exactly the polynomial relations among maximal minors. We point out that finding the ideal of polynomial relations among the nonmaximal minors of a fixed size is an open problem in commutative algebra.

Our argument shows that the intersection \( G(k,n) \cap K(n)_{n-k}^{-1} \) of the Grassmannian with the open affine set \( K(n)_{n-k}^{-1} \) is the affine space \( K^{k \times (n-k)} \) indicated in (5.2). Just as the usual projective space, the Grassmannian \( G(k,n) \) is glued together from affine spaces. In the same way, as a point in \( \mathbb{P}^{n-1} \) has a nonunique representation as an \( n \)-tuple of numbers, a point in \( G(k,n) \) may be represented by a \( k \times n \) matrix. To make the representation unique,
for projective space one chooses $a_i \neq 0$ and sets $a_i = 1$. This corresponds to choosing a nonzero maximal minor in the matrix and setting the submatrix to be the identity. This, our constructions are generalizations of those known for the projective space $\mathbb{P}^{n-1} = G(1, n)$ to arbitrary Grassmannians.

**Corollary 5.5.** The dimension of the Grassmannian $G(k, n)$ equals $k(n-k)$.

**Remark 5.6.** The Grassmannian $G(k, n)$ parametrizes $k$-dimensional vector subspaces of an $n$-dimensional vector space, or equivalently $k-1$ dimensional projective subspaces of an $n-1$ dimensional projective space.

**Example 5.7** ($k=2, n=4$). The Grassmannian $G(2, 4)$ is the image of the map

$$
\begin{bmatrix}
  a & b & c & d \\
  e & f & g & h
\end{bmatrix} \mapsto (af-be : ag-ce : ah-de : bg-cf : bh-df : ch-dg) \in \mathbb{P}^5.
$$

Alternatively, after fixing a basis $(v_1, v_2, v_3, v_4)$ of $V \simeq K^4$, the following identity holds in the second exterior power $\bigwedge^2 V$:

$$(av_1 + bv_2 + cv_3 + dv_4) \wedge (ev_1 + fv_2 + gv_3 + hv_4) =$$

$$(af-be)v_1 \wedge v_2 + (ag-ce)v_1 \wedge v_3 + (ah-de)v_1 \wedge v_4$$

$$+ (bg-cf)v_2 \wedge v_3 + (bh-df)v_2 \wedge v_4 + (ch-dg)v_3 \wedge v_4.$$ 

The Grassmannian $G(2, 4)$ has dimension 4. It is a hypersurface in $\mathbb{P}^5$. We write the coordinates on $\mathbb{P}^5$ as $(p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34})$. The indices refer to the minors of a $2 \times 4$-matrix. Following the proof of Theorem 5.4, we look at matrices (5.2). They take the form

$$
\begin{bmatrix}
  1 & 0 & c & d \\
  0 & 1 & g & h
\end{bmatrix}.
$$

The expansion of the rightmost $2 \times 2$-minor yields the inhomogeneous quadratic equation $p_{34} = ch - dg = (-p_{23})p_{14} - (-p_{24})p_{13}$. We homogenize this equation with the extra variable $p_{12}$. We conclude that the Grassmannian $G(2, 4)$ is the hypersurface in $\mathbb{P}^5$ that is defined by the \textit{Plücker quadric}

\begin{equation}
(5.3)
 p_{23}p_{14} - p_{13}p_{24} + p_{12}p_{34}.
\end{equation}

We now discuss the homogeneous prime ideal $I(G(k, n))$ of the Grassmannian $G(k, n)$. A complete description is known, in terms of certain quadratic relations that form a Gröbner basis. These are known as \textit{straightening relations}. For a derivation and explanation see e.g. [56, Chapter 3].

We here present the answer in the special case $k = 2$. The corresponding Grassmannian $G(2, n)$ is the space of lines in $\mathbb{P}^{n-1}$. The ambient space in this case is $\mathbb{P}(\bigwedge^2 \mathbb{C}^n)$. The vector space $\bigwedge^2 \mathbb{C}^n$ is simply the space of skew-symmetric $n \times n$ matrices. Hence, it is convenient to write the $\binom{n}{2}$ Plücker coordinates as the entries of a skew-symmetric $n \times n$-matrix $P = (p_{ij})$. 
The (affine cone over the) Grassmannian $G(2,n)$ consists of special skew-symmetric $n \times n$ matrices. Below we prove that these are exactly matrices of rank two. As a skew-symmetric matrix cannot have odd rank, we consider the submatrices of $P$ having size $4 \times 4$. To bound the rank of a skew-symmetric matrix it is enough to consider principal submatrices, i.e. those that have the same indexing set of rows and columns. One such submatrix is given by taking the first four rows and first four columns. The determinant of that matrix is the square of the Plücker quadric (5.3). One refers to the square root of the determinant of a skew-symmetric matrix of even order as its pfaffian. Thus the $4 \times 4$ pfaffians of our matrix $P$ are the $\binom{n}{4}$ quadrics

\[ p_{il}p_{jk} - p_{ik}p_{jl} + p_{ij}p_{kl} \quad \text{for} \quad 1 \leq i < j < k < l \leq n. \]

**Theorem 5.8.** The $\binom{n}{4}$ quadrics in (5.4) form the reduced Gröbner basis of the Plücker ideal $I(G(2,n))$, for any monomial ordering on the polynomial ring in the $\binom{n}{2}$ variables $p_{ij}$ that selects the underlined leading terms.

**Proof.** The argument in the proof of Theorem 5.4 shows that the quadrics (5.4) cut out $G(2,n)$ as a subset in $\mathbb{P}^{\binom{n}{2}-1}$. In other words, our Grassmannian is given as the set of skew-symmetric $n \times n$-matrices whose $4 \times 4$ pfaffians vanish. These are the skew-symmetric matrices of rank 2.

By Hilbert’s Nullstellensatz (proved in Chapter 6), we can conclude that the ideal $I(G(2,n))$ is the radical of the ideal generated by (5.4). We need to argue that the latter ideal is radical. However, this follows from the assertion that (5.4) is a Gröbner basis. Indeed, the leading monomials $p_{il}p_{jk}$ are square-free, so they generate a radical monomial ideal. However, if the initial ideal in($J$) of an ideal $J$ is radical then also $J$ itself is radical. So, all we need to do is to verify the Gröbner basis property for our quadrics. That Gröbner basis is then automatically a reduced Gröbner basis because none of the two trailing terms in (5.4) is a multiple of some other leading term.

To verify the Gröbner basis property, we reason as follows. For $n = 4$, it is trivial because there is only one generator. For $n = 5, 6, 7$, this is a direct computation, e.g. using Macaulay2. One checks that the S-polynomial of any two quadrics in (5.4) reduces to zero. Suppose that $n \geq 8$ and consider two Plücker quadrics. These involve at most 8 distinct indices. If the number of distinct indices is 7 or less then we are done by the aforementioned computation, which verified the claim for $n \leq 7$. Hence we may assume that all eight indices occurring in the two Plücker quadrics are distinct. In that case, the two underlined leading monomials are relatively prime. Here, Buchberger’s Second Criterion applies, and we can conclude that the S-polynomial automatically reduces to zero. In conclusion, all S-polynomials formed by pairs from (5.4) reduce to zero. This completes the proof. \qed
The Plücker relations for arbitrary Grassmannians are hard-wired in the computer algebra system Macaulay2. One finds generators for the ideal \( I(G(k,n)) \) with the convenient command \( \text{Grassmannian}(k-1,n-1) \). Here, the parameters \( k \) and \( n \) are decreased by one because Macaulay2 refers to the projective geometry interpretation: points in \( G(k,n) \) correspond to projective spaces of dimension \( k - 1 \) in an ambient projective space of dimension \( n - 1 \). Another thing that is tricky about the command \( \text{Grassmannian} \) is the ordering of the Plücker coordinates in Macaulay2. Here is how it works.

**Example 5.9** \((k = 3, n = 6)\). The following two command lines yield equations defining the Grassmannian of 3-dimensional vector subspaces in \( K^6 \).

```plaintext
R = QQ[p123,p124,p134,p234,p125,p135,p235,p145,p245,
p345,p126,p136,p236,p146,p246,p346,p156,p256,p356,p456];
I = Grassmannian(2,5,R)
```

This produces 35 quadratic relations. Note that \( G(3,6) \) is a smooth projective variety of dimension 9 and degree 42 in \( \mathbb{P}^{19} \), as is seen by also typing \( \text{dim I}, \text{degree I}, \text{betti mingens I} \).

Among the 35 minimal generators of \( I(G(3,6)) \), there are 30 trinomials, like \( p_{134}p_{125} - p_{124}p_{135} + p_{123}p_{145} \), plus five additional relations that involve all six indices, like \( p_{345}p_{126} - p_{125}p_{346} + p_{124}p_{356} - p_{123}p_{456} \). What is the combinatorial pattern? Can you generalize it to larger values of \( k \) and \( n \)? You will find answers and much more information in the books [42, 44, 56].

### 5.3. Schubert Calculus

We now show how Grassmannians can help us to answer enumerative questions. We would like to know how many lines or planes in space satisfy some properties. This general subject area is known as enumerative geometry, and the specific answers we provide are based on Schubert calculus. Schubert calculus furnishes an intersection theory for subvarieties of a Grassmannian.

We illustrate the concepts for the special case of \( G(2,4) \). Here is the simplest question of Schubert calculus: How many lines \( L \) intersect four general lines \( L_1, L_2, L_3, L_4 \) in \( \mathbb{P}^3 \)? The answer to this question is: two. To see this, we represent the line \( L \) by its corresponding point \( p = (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) \) in \( G(2,4) \subset \mathbb{P}^5 \). As we will see, the condition that \( L \) intersects a fixed \( L_i \) is a linear condition in \( p \). We must solve four such linear equations, and these have general coefficients since \( L_1, L_2, L_3, L_4 \) are general lines. In addition, there is the quadratic equation \( p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0 \). Thus, we are solving five equations in \( \mathbb{P}^5 \) of degrees 1, 1, 1, 1, 2. By Bézout’s Theorem, this system has two solutions \( p \), representing the two lines \( L \).
To study such intersection problems more systematically, one introduces some special subvarieties of Grassmannians. We continue with the example of $G(2, 4)$, the manifold of all lines in $\mathbb{P}^3$. We fix a complete flag in $\mathbb{P}^3$, consisting of a point in a line in a plane: $f_0 = \mathbb{P}^0 \subset f_1 = \mathbb{P}^1 \subset f_2 = \mathbb{P}^2 \subset \mathbb{P}^3$. Our aim is to group families of projective lines according to how they intersect that flag. These will be subvarieties $X_i$ of dimension $i$ in $G(2, 4)$.

First, the flag distinguishes a point in $G(2, 4)$, namely $X_0 := \{ f_1 \} \subset G(2, 4)$. It also distinguishes a curve $X_1$ in the Grassmannian $G(2, 4)$. The points of $X_1$ are the lines $l$ such that $f_0 \in l \subset f_2$. The most interesting is the case of surfaces in our family. There are two types of those in $G(2, 4)$:

1. the surface $X_2$ consisting of all lines $l$ such that $f_0 \in l$ and
2. the surface $X_2'$ consisting of all lines $l$ such that $l \subset f_2$.

Finally, there is also one three-dimensional variety $X_3$, consisting of all lines that intersect the given line $f_1$. The varieties $X_1$, $X_2$, $X_2'$ and $X_3$ we described are called the Schubert varieties in $G(2, 4)$. In Exercise [1] you will generalize the construction of Schubert varieties to larger Grassmannians.

Let us now fix a basis $v_1, v_2, v_3, v_4$ for $K^4$ and take $f_1$ to be the linear subspace in $\mathbb{P}^3$ spanned by $v_1, \ldots, v_{i+1}$. The point $f_1 \in G(2, 4)$ is given in $\mathbb{P}^5$ by the vanishing of all coordinates $p_{ij}$ apart from $p_{12}$. We have $f_0 \in l \subset f_2$ if and only if the line $l$ is spanned by the rows of a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & f & g & 0 \end{bmatrix}.$$ 

Hence, the curve $X_1$ is a line $\mathbb{P}^1$ in $G(2, 4) \subset \mathbb{P}^5$, defined by the vanishing of all $p_{ij}$ apart from $p_{12}$ and $p_{13}$. Similarly, $X_2$ is the $\mathbb{P}^2$ with coordinates $p_{12}, p_{13}, p_{14}$, and $X_2'$ is a different $\mathbb{P}^2$ with coordinates $p_{12}, p_{13}, p_{23}$. The common span of these two planes is the 3-dimensional variety $X_3 = \mathcal{V}(p_{34})$.

The relationship between $X_2$, $X_2'$ and $X_1$ inside $G(2, 4)$ can also be understood as follows. For any integer $k \leq 1$, let $Q$ be a nonsingular quadratic hypersurface in $\mathbb{P}^{2k+1}$ and consider a linear subspace $L = \mathbb{P}^{k-1}$ that is contained in $Q$. Then there exist precisely two $k$-dimensional subspaces that contain $L$ and are contained in $Q$. For $k = 2$ and $Q = G(2, 4)$ containing the line $L = X_1$ in $\mathbb{P}^5$, the two subspaces are the planes $X_2$ and $X_2'$ in $\mathbb{P}^5$.

For $k = 1$, our statement is a classical fact of projective geometry. A quadratic surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. For point $p = L$ in the quadric, there are precisely two lines that contain $p$ and lie on the quadric. In Figure [1] the blue quadric contains the red point $p$, and the two lines are green. This 3-dimensional picture arises by intersecting $G(2, 4)$ with the subspace $H = \mathbb{P}^3$ that is defined by $p_{12} = p_{34} = 0$ inside $\mathbb{P}^5$. The red point
Figure 1. Every (red) point on a smooth quadratic surface lies on two (green) lines that are contained in the surface. This picture illustrates the relationship among the Schubert subvarieties \(X_1, X_2, X_2'\) of \(G(2, 4)\).

\(p\) equals \(H \cap X_1\). The quadric is simply \(H \cap G(2, 4) = \mathcal{V}(p_{23}p_{14} - p_{13}p_{24})\), and the two green lines are the intersections \(X_2 \cap H\) and \(X_2' \cap H\).

The Schubert subvarieties of a Grassmannian represent a basis for the cohomology ring of the Grassmannian when the underlying field is \(K = \mathbb{C}\). One learns in algebraic topology that the multiplication in a cohomology ring corresponds to intersections of submanifolds. This can then be used to answer enumerative questions, namely for counting the number of points in an intersection that turns out to be 0-dimensional. Schubert calculus is the art of making this precise when the ambient manifold is a Grassmannian.

When intersecting Schubert varieties \(X_i\) as cohomology classes \([X_i]\), one should consider them coming from distinct general flags. For instance, consider the two surfaces \(X_2\) and \(X_2'\). They intersect in the line \(X_1\). However, when thinking of the classes, the former represents all lines through some arbitrary point in \(\mathbb{P}^3\) and the latter represents all lines contained in some entirely unrelated plane in \(\mathbb{P}^3\). There are no lines satisfying both condition, so the intersection of \([X_2]\) and \([X_2']\) is the class of the empty set. We write this as \([X_2] : [X_2'] = 0\). On the other hand, if we ask for lines going through two distinct points, or for lines contained in two distinct planes, then there is one solution. The selfintersections of the classes \([X_2]\) and \([X_2']\) give one point. That one point is represented by the element 1 in the cohomology ring. Our discussion is summarized by the following relations that hold in
the cohomology ring of the Grassmannian $G(2, 4)$ of lines in $\mathbb{P}^3$:

\[ [X_3][X_3] = [X_2] + [X_2'], \quad [X_2][X_2] = [X_2'][X_2'] = [X_3][X_1] = 1, \quad [X_2][X_2'] = 0. \]

Recall that multiplication represents intersection and sum represents union. The fact that the Schubert classes $[X_i]$ form a basis for the cohomology ring means that the class $[Z]$ of any subvariety $Z$ is a linear combination of the $[X_i]$. Finding that linear combination for a given $Z$ is similar to computing the degree of a subvariety in $\mathbb{P}^n$, as discussed at the end of Chapter 1.

We now have a conceptual framework for studying enumerative questions like: How many lines pass through four general lines in $\mathbb{P}^3$? The set of such lines is a finite subset in $G(2, 4)$. It is the intersection of four hypersurfaces, all of the form $X_3$. Since intersections are represented by multiplication in the cohomology ring, the following formal computation reveals the answer:

\[
\begin{align*}
[X_3]^4 &= ([X_3][X_3])^2 = ([X_2] + [X_2'])^2 = \\
[X_2]^2 + 2[X_2][X_2'] + [X_2']^2 &= 1 + 2 \cdot 0 + 1 = 2.
\end{align*}
\]

Here is another question to be answered by Schubert calculus: How many lines are simultaneously tangent to four general quadratic surfaces in $\mathbb{P}^3$? Consider a quadric $Q$ in $\mathbb{P}^3$, represented as a symmetric $4 \times 4$-matrix. Let $\wedge^2 Q$ be the symmetric $6 \times 6$ matrix with entries given by the $2 \times 2$ minors of the matrix representing $Q$. This is known as the second compound matrix of $Q$. The condition for a line to be tangent to $Q$ is expressed by the vanishing of the quadratic form $P(\wedge^2 Q)P^T$ in the Plücker coordinates $P = (p_{12}, p_{13}, \ldots, p_{34})$ of that line. This defines a threefold in $G(2, 4)$. The cohomology class of that threefold is $2[X_3]$, since its equation is quadratic.

We conclude that the number of lines that are tangent to four given general quadratic surfaces in $\mathbb{P}^3$ is equal to

\[ (2[X_3])^4 = 16[X_3]^4 = 16 \cdot 2 = 32. \]

In order to actually compute the 32 lines over $\mathbb{C}$, given four concrete quadrics in $\mathbb{P}^3$, we would need to carry out some serious Gröbner basis computations.

Exercise 1 suggests that Schubert varieties in $G(k, n)$ correspond to Young diagrams contained in a $k \times (n - k)$ rectangle. Young diagrams play a fundamental role in representation theory. A formal definition appears in Definition 10.19. Classes of Schubert varieties form a $\mathbb{Z}$-basis of the cohomology ring of a Grassmannian. There is a purely combinatorial rule, named after Littlewood and Richardson, for multiplying (i.e. intersecting) Schubert varieties. This is expressed via Young diagrams; see [23] Appendix A.1.

The number of boxes of the Young diagram corresponds to the codimension of the Schubert variety. In particular, as there is only one Young diagram with one box, there is only one codimension one class: the class
of a hyperplane section under the Plücker embedding. The full $k \times (n-k)$ rectangle corresponds to the class of a point. We will not present the full Littlewood-Richardson rule here. Instead, we focus on the simpler Pieri rule.

Let $\mu$ be any Young diagram and let $\lambda$ be the Young diagram with $s$ boxes and contained in one column (resp. row). The product $\mu \cdot \lambda$ is the sum of all Young diagrams obtained by adding $s$ boxes to $\mu$, no more than one in a row (resp. column). Of course if we work with the Grassmannian $G(k,n)$ we disregard all diagrams that do not fit in a $k \times (n-k)$ rectangle.

**Example 5.10.** For $k=3, n=6$, Pieri’s rule gives the following identity:

\[
\begin{array}{c}
\ \ \ \ | \ \ \\
\ \ | \ \ \\
\ | \ \\
\end{array}
\cdot
\begin{array}{c}
\ \ \ | \ \ \\
\ \ | \ \\
\ | \\
\end{array}
= \begin{array}{c}
\ \ | \ \\
\ | \ \\
\end{array} + \begin{array}{l}
\ \ \\
\ \\
\end{array} + \begin{array}{l}
\ \\
\ \\
\end{array}.
\]

We close this section by deriving the degree of the Grassmannian $G(k,n)$ as a projective variety in $\mathbb{P}^{(n)}-1$. There are two different methods to do this. The first method is geometric, namely, we intersect the hyperplane class with itself $k(n-k)$ times. Here, in each step, we apply Pieri’s rule.

**Definition 5.11.** A Young diagram with $s$ boxes, filled with the numbers $1, 2, \ldots, s$, is called a standard Young tableau if the entries are increasing in every row and column. These are ubiquitous in algebraic combinatorics.

**Proposition 5.12.** The degree of the Grassmannian $G(k,n) \subset \mathbb{P}^{(n)}-1$ equals the number of standard Young tableaux of rectangle shape $k \times (n-k)$.

**Proof.** The entries in such a standard Young tableaux indicate where the box in each step is added when we apply Pieri’s rule $k(n-k)$ times. □

The second method is more algebraic, utilizing a Gröbner basis for the ideal $I(G(k,n))$. We explain this for $k=2$. By Theorem 5.8 the initial monomial ideal equals $M = \langle p_{il}p_{jk} : 1 \leq i < j < k < l \leq n \rangle$. The variety $V(M)$ is a union of coordinate subspaces of dimension $2n-3$; see e.g. Exercise 12 in Chapter 2. The number of these subspaces is the degree of $M$ and hence the degree of $G(2,n)$. We find that this is the Catalan number:

\[
\text{degree}(G(2,n)) = \frac{1}{n-1} \binom{2n-4}{n-2}.
\]

By Proposition 5.12 this equals the number of standard Young tableaux of shape $2 \times (n-2)$. For more on Catalan numbers we refer to the entry A000108 in the Online Encyclopedia of Integer Sequences. In general, a closed formula for the degree of the Grassmannian may be derived from the hook-length formula for the number of standard Young tableaux [23, 4.12].
Theorem 5.13. The degree of the Grassmannian $G(k, n) \subset \mathbb{P}^{\binom{n}{k}-1}$ equals
\[
\frac{(k(n-k))!}{\prod_{j=1}^{k} j \cdot (j+1) \cdots (j+n-3)}.
\]

Remark 5.14. Grassmannians are named after Hermann Grassmann. However, it was Julius Plücker who first noted that lines in 3-space may be studied as a four-dimensional object \[47\]. Yet, Grassmann’s earlier discoveries were fundamental: he was the one to realize that the algebraic setting of geometry allows to consider objects not only in 3-dimensional space, but in any dimension. Can you imagine data science without Grassmann’s insight?

Exercises

1. Fix a complete flag in $\mathbb{P}^n$. Construct a bijection between:
   - subvarieties of $G(k, n)$ that consist of $l \in G(k, n)$ that intersect each element of the flag in at most the given dimension, and
   - Young diagrams contained in a $k \times (n-k)$ rectangle.

   Either the codimension or the dimension of the subvariety in $G(k, n)$ should equal the number of boxes in the corresponding Young diagram.

2. Let $G$ be a subgroup of the general linear group $GL(V)$, and let $X \subset V$ be a variety such that the action of $G$ on $V$ restricts to $X$. Prove that if $x$ is a smooth point of $X$ and $g \in G$, then $gx$ is also a smooth point.

3. Consider the inclusion $G(2, 4) \times G(2, 4) \subset \mathbb{P}^5 \times \mathbb{P}^5$. Using Plücker coordinates, describe the locus of pairs of lines $(l_1, l_2) \in G(2, 4) \times G(2, 4)$ such that $l_1$ intersects $l_2$ in $\mathbb{P}^3$. Hint: Present both lines as $2 \times 4$ matrices. Note that two lines in $\mathbb{P}^3$ intersect if and only if they do not span the whole ambient space. Apply Laplace expansion of the determinant.

4. For a variety $X \subset \mathbb{P}^n$, one considers a subset of $G(k+1, n+1)$ of $\mathbb{P}^k \subset X$. This is known as the Fano variety of $k$-dimensional subspaces of $X$. Fix a nondegenerate quadric $Q \subset \mathbb{P}^3$. Describe the Fano variety of lines in it. Hint: One may solve this exercise either theoretically or using algebra software. Also Figure 1 gives a hint about the answer.

5. How many real lines in 3-space can be simultaneously tangent to four given spheres? It is best to start by computing some explicit examples.

6. The two lines incident to four given real lines in $\mathbb{P}^3$ can be either real or complex. In the latter case they form a complex conjugate pair. Write down a polynomial in the $24 = 4 \cdot 6$ Plücker coordinates of four given lines whose sign distinguishes the two cases.
(7) How many lines in \( \mathbb{P}^3 \) are simultaneously incident to two given lines and tangent to two given quadratic surfaces?

(8) Consider the set of all lines in \( \mathbb{P}^3 \) that are tangent to the cubic Fermat surface \( \{ x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \} \). This set is an irreducible hypersurface in the Grassmannian \( G(2, 4) \). Compute a polynomial in \( p_{12}, p_{13}, \ldots, p_{34} \) that defines this hypersurface.

(9) Find a minimal generating set for the ideal of the Grassmannian \( G(3, 7) \).

(10) Prove that the determinant of a skew-symmetric \( n \times n \)-matrix is zero if \( n \) is odd, and it is the square of a polynomial when \( n \) is even.

(11) Examine the monomial ideal that is generated by the underlined initial monomials in (5.4). Express this ideal as the intersection of prime ideals. How many primes occur? Use this to derive the identity in (5.5).

(12) Fix six general planes \( \mathbb{P}^2 \) in \( \mathbb{P}^4 \). How many lines in \( \mathbb{P}^4 \) intersect all six planes? Describe Schubert calculus for the Grassmannian \( G(2, 5) \).

(13) Let \( n = 2k \) and suppose that the \( n \times n \)-matrix \( A = (a_{ij}) \) in (5.2) is symmetric, i.e. its entries satisfy the equations \( a_{ij} = a_{ji} \). Express these equations in terms of the \( \binom{2n}{n} \) Plücker coordinates. The resulting subvariety of \( G(n, 2n) \) is known as the Lagrangian Grassmannian.

(14) Can you find four smooth quadratic surfaces in real 3-space such that 32 real lines are tangent to all four of surfaces? Are they ellipsoids?

(15) The identity in Example 5.10 expresses a geometric statement about intersections inside the Grassmannian \( G(3, 6) \subset \mathbb{P}^{19} \). Explain this statement in your own words. Illustrate it by computing an explicit example.
Nullstellensätze

“This is not mathematics, it is theology!”, Paul Gordan

The German noun Nullstellensatz refers to a theorem that characterizes the existence of a zero (= Nullstelle) for a system of polynomials. The classical version, due to Hilbert, works over algebraically closed fields. It says that the nonexistence of zeros is equivalent to the existence of a partition of unity for the given polynomials. A more general version furnishes a bijection between varieties and radical ideals. In this chapter we also discuss the analogous theme over the field of real numbers. Here the main results are the real Nullstellensatz and the Positivstellensatz. These theorems furnish criteria for polynomial equations and inequalities to have no real solutions. This leads us to real radical ideals and their characterization via sums of squares.

6.1. Certificates for Infeasibility

In Chapter 3 we discussed how to find and represent solutions to a system of polynomial equations. But what if such a solution does not exist? In this chapter we present methods to prove that a given system has no solution.

Throughout this section, we fix an algebraically closed field $K$, such as the complex numbers $K = \mathbb{C}$. We write $K[x] = K[x_1, \ldots, x_n]$ for its polynomial ring in $n$ variables. For an ideal $I \subseteq K[x]$ we denote the associated variety in $K^n$ by $\mathcal{V}(I)$, as in Chapter 2. We begin with the following version of the Nullstellensatz. This result appears as Theorem 1 in [11, §4.1].

**Theorem 6.1.** If $I$ is an ideal in $K[x]$ that is proper, i.e. $1 \notin I$, then its variety $\mathcal{V}(I)$ in $K^n$ is non-empty. Equivalently, if $\mathcal{V}(I) = \emptyset$ then $1 \in I$. 
Proof. Our proof follows that in [11] §4.1. We use induction on \( n \). For \( n = 1 \), the statement holds because \( K[x_1] \) is a PID and every non-constant polynomial in one variable has a zero in the algebraically closed field \( K \).

Let now \( n \geq 2 \). For \( a \in K \), we write \( I_{x_n=a} \) for the ideal in \( K[x_1, \ldots, x_{n-1}] \) that is obtained by setting \( x_n = a \) in each element of \( I \). One easily checks that this is indeed an ideal. We claim that there exists a scalar \( a \in K \) such that \( 1 \notin I_{x_n=a} \). In such a case, by induction, there is a point \( (a_1, \ldots, a_{n-1}) \) in \( V(I_{x_n=a}) \). This implies that \((a_1, \ldots, a_{n-1}, a)\) is a point in the variety \( V(I) \).

Consider the elimination ideal \( I \cap K[x_n] \). To prove the claim, we distinguish two cases. First suppose that this ideal is not the zero ideal. Since \( 1 \in K \), we have \( I \subseteq I \cap K[x_n] \). Hence there exist polynomial multipliers \( g \) for \( I \), such that the product \( \prod_{i=1}^{r} g_i(x_{n-1}) \) is a nonconstant polynomial in \( K[x_{n-1}] \) which is congruent to 1 modulo \( I \).

Suppose that \( 1 \in I_{x_n=b_i} \) for \( i = 1, 2, \ldots, r \). If this is not the case then we are done. Hence there exist \( B_1, \ldots, B_r \in I \) such that \( B_i(x_{n-1}, b_i) = 1 \) for all \( i \). Note that \( B_i \) is congruent to 1 modulo \( x_n - b_i \) in \( K[x] \). This implies that the product \( \prod_{i=1}^{r} (B_i - 1)^{m_i} \) belongs to the ideal \( \langle f \rangle \). Since \( f \in I \) and \( B_i \in I \), the following identity holds modulo the ideal \( I \):

\[
0 = \prod_{i=1}^{r} (B_i - 1)^{m_i} = \prod_{i=1}^{r} (-1)^{m_i} = \pm 1, \quad \text{i.e. } 1 \in I.
\]

Next suppose \( I \cap K[x_n] = \{0\} \). Let \( \{g_1, \ldots, g_r\} \) be a Gröbner basis for \( I \) with respect to the lexicographic order with \( x_1 > \cdots > x_n \). Write \( g_i = c_i(x_n) x^{\alpha_i} + \) lower order terms, where \( x^{\alpha_i} \) is a monomial in \( x_1, \ldots, x_{n-1} \). Since the field \( K \) is infinite, we can choose \( a \in K \) such that \( c_i(a) \neq 0 \) for all \( i \). The polynomials \( g_i(x_1, \ldots, x_{n-1}, a) \) form a Gröbner basis for \( I_{x_n=a} \), for the lexicographic monomial order, with leading monomials \( x^{\alpha_i} \) for \( i = 1, \ldots, r \). None of these monomials is 1, since \( I \cap K[x_n] = \{0\} \). This implies that 1 is not in the ideal \( I_{x_n=a} \).

Theorem 6.1 gives a certificate for the non-existence of solutions to a system of polynomial equations, namely the partition of unity we had promised.

Corollary 6.2. A collection of polynomials \( f_1, \ldots, f_r \in K[x] \) either has a common zero in \( K^n \) or there exists an identity \( g_1 f_1 + \cdots + g_r f_r = 1 \) with polynomial multipliers \( g_1, \ldots, g_r \in K[x] \). This is the desired certificate.

Proof. Let \( I = \langle f_1, \ldots, f_r \rangle \). The either \( V(I) \neq \emptyset \) or \( V(I) = \emptyset \). In the latter case, \( 1 \in I \), meaning that 1 is a polynomial linear combination of the \( f_i \).

Example 6.3. Let \( n = 2 \) and consider the following three polynomials

\[
f_1 = (x+y-1)(x+y-2), \quad f_2 = (x-y+3)(x+2y-5), \quad f_3 = (2x-y)(3x+y-4).
\]
These do not have a common zero. This is proved by a Nullstellensatz certificate \( g_1 f_1 + g_2 f_2 + g_3 f_3 = 1 \). One choice of multipliers is given by

\[
\begin{align*}
g_1 &= \frac{895}{756} x^2 - \frac{6263}{2160} x - \frac{2617}{2520} y + \frac{4327}{1008}, \\
g_2 &= \frac{5191}{3780} x^2 + \frac{358}{945} xy - \frac{6907}{3024} x - \frac{2123}{15120} y + \frac{3823}{7560}, \\
g_3 &= -\frac{179}{420} x^2 - \frac{716}{945} xy + \frac{1453}{1080} x - \frac{716}{945} y + \frac{13771}{7560}.
\end{align*}
\]

There are two possible methods for computing the multipliers \((g_1, \ldots, g_r)\) for the Nullstellensatz certification, as in Corollary 6.2. The first method is to use the Extended Buchberger Algorithm. This is analogous to the Extended Euclidean Algorithm for the ring of integers or the ring of polynomials in one variable. For instance, given any collection of relatively prime integers, this method writes 1 as a \( \mathbb{Z} \)-linear combination of these integers.

In the Extended Buchberger Algorithm one keeps track of the polynomial multipliers that are used to generate new \( S \)-polynomials from current basis polynomials. In the end, each element in the final Gröbner basis is written explicitly as a polynomial linear combination of the input polynomials. If \( \mathcal{V}(I) = \emptyset \) then that final reduced Gröbner basis is the singleton \( \{1\} \).

The second method for computing Nullstellensatz certificates is to use degree bounds plus linear algebra. Let \( d \) be any integer that exceeds the degree of each \( f_i \). Let \( g_i \) be a polynomial of degree \( d - \deg(f_i) \) with coefficients that are unknowns, for \( i = 1, 2, \ldots, r \). The desired identity \( \sum_{i=1}^r g_i f_i = 1 \) translates into a system of linear equations in all of these unknowns. We solve this system. If a solution is found then this gives a certificate. If not then there is no certificate in degree \( d \), and we try a higher degree.

The two methods, in complete generality, can be very complicated to carry out in practice. The computation of Gröbner bases does not run in polynomial time. Worst-case complexity bounds for Gröbner bases are doubly exponential in the number of variables. Furthermore, the degree of the multipliers \( g_i \) above are not polynomial in the input degrees either. Many mathematicians and computer scientists believe that there is no polynomial-time algorithm to decide if a given polynomial system has a complex solution. This is an active area of research. The situation is even worse if we want solutions with coordinates in \( \mathbb{R} \) or \( \mathbb{Q} \) or \( \mathbb{Z} \). In the last case, it is known there exists no algorithm at all — irrespective of complexity — to decide if a system has an integral solution. This was Hilbert’s 10\(^{th}\) problem.

### 6.2. Hilbert’s Nullstellensatz

Hilbert’s Nullstellensatz offers a characterization of the set of all polynomials that vanish on a given variety. This classical result from 1890 works over
any algebraically closed field $K$, such as the complex numbers $K = \mathbb{C}$. In this section we present this theorem and we discuss some of its ramifications.

Recall that the radical of an ideal $I$ in $K[x]$ is the (possibly larger) ideal
$$
\sqrt{I} = \{ f \in K[x] : f^m \in I \text{ for some } m \in \mathbb{N} \}.
$$
This is a radical ideal, hence it is an intersection of prime ideals.

**Example 6.4.** Consider the ideal $I = \langle x_1x_3, x_1x_4 + x_2x_3, x_2x_4 \rangle$ in the polynomial ring in four variables. This is not radical. To see this, note that the monomial $f = x_1x_4$ is not in $I$ but $f^2$ is in $I$. The radical of $I$ equals
$$
\sqrt{I} = \langle x_1x_3, x_1x_4, x_2x_3, x_2x_4 \rangle = \langle x_1, x_2 \rangle \cap \langle x_3, x_4 \rangle.
$$

How many associated primes does the ideal $I$ have? Do Gröbner bases of $I$ give any hints? We refer to Example 3.28 for the answer.

We now show that $\sqrt{I}$ comprises all polynomials that vanish on $\mathcal{V}(I)$.

**Theorem 6.5** (Hilbert’s Nullstellensatz). For any ideal $I$ in the polynomial ring $K[x]$ in $n$ variables over an algebraically closed field $K$, we have

$$
I(\mathcal{V}(I)) = \sqrt{I}.
$$

**Proof.** The radical $\sqrt{I}$ is contained in the vanishing ideal $I(\mathcal{V}(I))$, because $f^m(a) = 0$ implies $f(a) = 0$ for all $a \in K^n$. We must show the left hand side is a subset of the right hand side in (6.1). Let $I = \langle f_1, \ldots, f_r \rangle$ and suppose that $f$ is a polynomial which vanishes on $\mathcal{V}(I)$. Let $y$ be a new variable and consider the ideal $J = \langle f_1, \ldots, f_r, yf − 1 \rangle$ in the enlarged polynomial ring $K[x,y] = K[x_1, \ldots, x_n,y]$. The variety $\mathcal{V}(J)$ in $K^{n+1}$ is the empty set because $f = 0$ on every zero of $f_1, \ldots, f_r$ and $f \neq 0$ on every zero of $yf − 1$.

By Theorem 6.1 there exist multipliers $g_1, \ldots, g_r, h$ in $K[x, y]$ such that

$$
\sum_{i=1}^r g_i(x, y) \cdot f_i(x) + h(x, y) \cdot (yf(x) − 1) = 1.
$$

We now substitute $y = 1/f(x)$ into this identity. This yields the following identity of rational functions in $n$ variables:

$$
\sum_{i=1}^r g_i(x, 1/f(x)) \cdot f_i(x) = 1.
$$

The common denominator is $f(x)^m$ for some $m \in \mathbb{N}$. Multiplying both sides with this common denominator, we obtain a polynomial identity of the form

$$
\sum_{i=1}^r p_i(x) \cdot f_i(x) = f(x)^m.
$$

This shows that $f^m$ lies in $I$, and hence $f$ lies in the radical $\sqrt{I}$. $\square$
Example 6.6. Which polynomial functions vanish on all nilpotent $3 \times 3$-matrices? We set $n = 9$ and take $I$ to be ideal generated by the entries of $X^3$, where $X = (x_{ij})$ is a $3 \times 3$-matrix with variables as entries. These are nine homogeneous cubic polynomials in nine unknowns $x_{ij}$. One of them is
\[ x_{11}^3 + 2x_{11}x_{12}x_{21} + x_{12}x_{21}x_{22} + 2x_{11}x_{13}x_{31} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} + x_{13}x_{31}x_{33}. \]

But what are all polynomials that vanish on nilpotent matrices? Can they be written as polynomial linear combinations of these nine cubics? The answer is no. The ideal $I(\mathcal{V}(I))$ is larger than $I$. The radical of $I$ is
\[ \langle x_{11} + x_{22} + x_{33}, x_{11}x_{22} + x_{11}x_{33} - x_{12}x_{21} - x_{13}x_{31} + x_{22}x_{33} - x_{23}x_{32}, \det(X) \rangle. \]

In words, the ideal $\sqrt{I}$ is generated by the three trailing coefficients of the characteristic polynomial of $X$. This reflects the familiar fact that a square matrix is nilpotent if and only if it has no eigenvalues other than zero. Theorem 6.5 implies that every polynomial that vanishes on nilpotent $3 \times 3$-matrices is a polynomial linear combination of the three generators above.

The Nullstellensatz implies a one-to-one correspondence between varieties in affine $n$-space and radical ideals in the polynomial ring in $n$ variables.

Corollary 6.7. The map $V \mapsto \mathcal{I}(V)$ defines a bijection between varieties in $K^n$ and radical ideals in $K[x]$. The inverse map that takes radical ideals to varieties is given by $I \mapsto \mathcal{V}(I)$.

Proof. A variety $V$ is Zariski closed and hence satisfies $V = \mathcal{V}(\mathcal{I}(V))$. The Nullstellensatz yields $I = \mathcal{I}(\mathcal{V}(I))$. These identities imply that both maps are one-to-one and onto, and that they are the inverses of each other. \hfill \Box

Corollary 6.8. The map $V \mapsto \mathcal{I}(V)$ defines a bijection between irreducible varieties in the affine space $K^n$ and prime ideals in the polynomial ring $K[x]$. As before, the inverse map is given by $I \mapsto \mathcal{V}(I)$.

Proof. By Proposition 2.3 a variety $V$ is irreducible if and only if its associated radical ideal $\mathcal{I}(V)$ is prime. \hfill \Box

Example 6.9 ($n = 2$). There are only two kinds of proper irreducible subvarieties in the affine plane $K^2$. First, we have the points $(a, b)$, corresponding to maximal ideals $I = (x - a, y - b)$. Second, there are irreducible curves, one for each principal ideal $I = (f)$ where $f$ is an irreducible polynomial in $K[x, y]$. Arbitrary varieties are unions of these. For instance, consider
\[ J = \langle x^4 + 2x^2 + y^2 + 1 \rangle \cap \langle y^3 - 4, 2x - y^2 \rangle \quad \text{in} \quad \mathbb{C}[x, y]. \]

This ideal is radical. Its variety $\mathcal{V}(J)$ in $\mathbb{C}^2$ has five irreducible components, namely two quadratic curves and three points. Check that $\mathcal{I}(\mathcal{V}(J)) = J$. 
In many applications one is interested in solving polynomial equations over the real numbers, and one cares less about non-real complex solutions. This raises the following important question: Does there exist an analog of Hilbert’s Nullstellensatz over an ordered field, such as the real numbers $K = \mathbb{R}$? We shall see that the answer is affirmative. In the next section we discuss the real Nullstellensatz and the Positivstellensatz. These concern systems of polynomial equations and inequalities over the real numbers. They generalize Linear Programming Duality, for systems of linear equations and linear inequalities over $\mathbb{R}$. Moreover, as we shall see in Chapter 12 the Positivstellensatz plays an important role in Nonlinear Optimization.

The theorems above are false when $K = \mathbb{R}$ is the field of real numbers. To see this, let $n = 2$ and consider varieties in the plane $\mathbb{R}^2$. Theorem 6.1 fails for $I = \langle x^2 + y^2 + 1 \rangle$. This is a proper ideal in $\mathbb{R}[x,y]$ but $\mathcal{V}_{\mathbb{R}}(I) = \emptyset$. Theorem 6.5 is also false for $I = \langle x^2 + y^2 \rangle$. This is a radical ideal, but $\mathcal{I}(\mathcal{V}_{\mathbb{R}}(I)) = \langle x,y \rangle$ strictly contains $\sqrt{I} = I$.

We ask these two questions about ideals $I$ in $\mathbb{R}[x]$ and their varieties in $\mathbb{R}^n$:

- How to best certify that the real variety $\mathcal{V}_{\mathbb{R}}(I)$ is empty?
- How to compute the ideal $\mathcal{I}(\mathcal{V}_{\mathbb{R}}(I))$ from generators of $I$?

The goal of the next section is to give answers to these questions.

### 6.3. Let’s Get Real

We here present the real Nullstellensatz. Our point of departure is the fact that a polynomial $f$ in $\mathbb{R}[x]$ which is a sum of squares must be nonnegative, i.e. the inequality $f(u) \geq 0$ holds for all $u \in \mathbb{R}^n$. A natural question is whether the converse holds: can every nonnegative polynomial be written as a sum of squares? The answer depends on what is being squared.

Hilbert showed in 1893 that the answer is no if one asks for squares of polynomials. However, it is yes if one allows squares of rational functions. This was the 17th problem in Hilbert’s famous list from the International Congress of Mathematicians in 1900. It was solved by Emil Artin in 1927.

**Theorem 6.10 (Artin’s Theorem).** If $f \in \mathbb{R}[x]$ is nonnegative on $\mathbb{R}^n$ then there exist polynomials $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_r \in \mathbb{R}[x]$ such that

$$f = \left(\frac{p_1}{q_1}\right)^2 + \left(\frac{p_2}{q_2}\right)^2 + \cdots + \left(\frac{p_r}{q_r}\right)^2.$$  

**Example 6.11** $(n = 2)$. The Motzkin polynomial $M(x,y)$ equals

$$x^4y^2 + x^2y^4 + 1 - 3x^2y^2 = \frac{(x^2+y^2+1) \cdot x^2y^2(x^2+y^2-2)^2 + (x^2-y^2)^2}{(x^2+y^2)^2}.$$
Distributing the three terms of the factor \((x^2 + y^2 + 1)\), we see that the right hand side is a sum of four squares of rational functions. This shows that \(M(x, y)\) is nonnegative. However, it is not a sum of squares in \(\mathbb{R}[x, y]\). Suppose it were equal to \(\sum_i f_i^2\) where \(f_i\) are polynomials. None of the \(f_i\)'s may contain a monomial \(x^d\) or \(y^d\) for \(d > 0\) — otherwise the largest such \(d\) contributes positively to the coefficient of \(x^{2d}\) in \(M(x, y)\). We have \(f_i = \alpha_i + \beta_i xy + \tilde{f}_i\), where \(\tilde{f}_i\) have all terms of degree \(\geq 3\) and \(\alpha_i, \beta_i \in \mathbb{R}\). Further, \(\tilde{f}_i\) cannot have terms of degree strictly greater than three. Indeed, let us fix any degree compatible monomial order. The leading term of \(\sum f_i^2\) is the sum of squares of the top leading term in any of the \(f_i\)'s that may contribute to it. In particular, the terms of top degree do not cancel. Hence, the coefficient \(-3\) of \(x^2 y^2\) in \(M(x, y)\) would then be equal to \(\sum_i \beta_i^2 \geq 0\).

We shall derive Artin’s Theorem 6.10 as a special case from the following more general statement. Theorem 6.12 is the real number analogue to the weak form of the Nullstellensatz which was established in Theorem 6.1.

**Theorem 6.12.** Let \(I\) be an ideal in \(\mathbb{R}[x]\) whose variety \(V_\mathbb{R}(I)\) is empty. Then \(-1\) is a sum of squares of polynomials modulo \(I\), i.e. we have

\[
1 + p_1^2 + p_2^2 + \cdots + p_r^2 \in I \quad \text{for some } p_1, p_2, \ldots, p_r \in \mathbb{R}[x].
\]

For the proof of Theorem 6.12 see Murray Marshall’s book \[43\] §2.3.

**Proof of Theorem 6.10.** Let \(y\) be a new variable. Consider the polynomial \(g = f(x)y^2 + 1\) in \(\mathbb{R}[x, y]\). Since \(f\) is nonnegative, the real variety \(V_\mathbb{R}(g)\) is empty in \(\mathbb{R}^{n+1}\). By Theorem 6.12 there exists a polynomial identity

\[
1 + p_1(x, y)^2 + p_2(x, y)^2 + \cdots + p_r(x, y)^2 + h(x, y)g(x, y) = 0.
\]

We substitute \(y = \pm \frac{1}{\sqrt{-f(x)}}\) into (6.3). This makes the last term cancel in both substitutions. Note that \(p_i(x, \frac{1}{\sqrt{-f(x)}}) = a_i(x) + \frac{1}{\sqrt{-f(x)}} b_i(x)\) and \(p_i(x, -\frac{1}{\sqrt{-f(x)}}) = a_i(x) - \frac{1}{\sqrt{-f(x)}} b_i(x)\) where \(a_i\) and \(b_i\) are rational functions.

We see that the two substitutions in (6.3) lead to the identities

\[
1 + \sum_{i=1}^r a_i(x)^2 - \frac{1}{f(x)} \sum_{i=1}^r b_i(x)^2 \pm \frac{2}{\sqrt{-f(x)}} \sum_{i=1}^r a_i(x)b_i(x) = 0.
\]

After adding these two expressions and dividing by two, we obtain:

\[
1 + \sum_{i=1}^r a_i(x)^2 - \frac{1}{f(x)} \sum_{i=1}^r b_i(x)^2 = 0.
\]

Clearing the denominator, we obtain

\[
f = \frac{\sum_{i=1}^r b_i^2}{1 + \sum_{i=1}^r a_i^2} = \frac{(\sum_{i=1}^r b_i^2)(1 + \sum_{i=1}^r a_i^2)}{(1 + \sum_{i=1}^r a_i^2)^2}.
\]
The right hand side is a sum of squares of rational functions in \( n \) variables. This establishes Artin’s Theorem 6.10. □

For systems consisting of both equations and inequalities, there is the Positivstellensatz. To motivate this, we review the corresponding result for linear polynomials known as Farkas’ Lemma. It lies at the heart of Linear Programming Duality. Informally, Farkas’ Lemma states that a system of linear equations and inequalities either has a solution in \( \mathbb{R}^n \), or it has a dual solution which certifies that the original system has no solution. Farkas’ Lemma has many equivalent formulations. Here is one of them, selected to make the extension to higher-degree polynomials transparent.

Let \( f_1, \ldots, f_r, g_1, \ldots, g_s \) be polynomials of degree 1 in \( \mathbb{R}[x] \), and consider

\[
(6.4) \quad f_1(u) = 0, \ldots, f_r(u) = 0, \quad g_1(u) \geq 0, \ldots, g_s(u) \geq 0.
\]

In the dual problem, we seek numbers \( a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{R} \) such that

\[
(6.5) \quad a_1 \cdot f_1 + \cdots + a_r \cdot f_r + b_1^2 \cdot g_1 + \cdots + b_s^2 \cdot g_s = -1 \text{ in } \mathbb{R}[x].
\]

At most one of these two can have a solution. Indeed, since \( b_1^2, \ldots, b_s^2 \geq 0 \), the left hand side of (6.5) is nonnegative for every vector \( u \) that solves (6.4).

**Theorem 6.13 (Farkas’ Lemma).** Given any choice of polynomials \( f_1, \ldots, f_r \) and \( g_1, \ldots, g_s \) in \( \mathbb{R}[x] \), exactly one of the following two statements is true:

- **(P)** There exists a point \( u \in \mathbb{R}^n \) such that (6.4) holds.
- **(D)** There exist real numbers \( a_1, \ldots, a_r, b_1, \ldots, b_s \) such that (6.5) holds.

Consider the system (6.4) where the \( f_i \) and \( g_j \) are now arbitrary polynomials. In the dual problem, we seek polynomials \( a_i \) and \( b_{j\nu} \) in \( \mathbb{R}[x] \) such that

\[
(6.6) \quad a_1 \cdot f_1 + \cdots + a_r \cdot f_r + \sum_{\nu \in \{0,1\}^s} \left( \sum_j b_{j\nu}^2 \right) \cdot g_1^{\nu_1} \cdots g_s^{\nu_s} = -1.
\]

In the double sum on the right, we see linear combinations of squarefree monomials in \( g_1, \ldots, g_s \) whose coefficients are sums of squares. The set of polynomials that admit such a representation is the quadratic module generated by \( g_1, \ldots, g_s \). Quadratic modules associated with inequality constraints are fundamental in the study of semi-algebraic sets [43, §2.1].

**Theorem 6.14 (Positivstellensatz).** Given any polynomials \( f_1, \ldots, f_r \) and \( g_1, \ldots, g_s \) in \( \mathbb{R}[x] \), exactly one of the following two statements is true:

- **(P)** There exists a point \( u \in \mathbb{R}^n \) such that (6.4) holds.
- **(D)** There exist polynomials \( a_i \) and \( b_{j\nu} \) in \( \mathbb{R}[x] \) such that (6.6) holds.

**Proof.** See [43, §2.3]. □
6.3. Let’s Get Real

The dual solution (D) in Theorem 6.14 is similar to that in Farkas’ Lemma. One extra complication is that we now need products of the $g_i$. The result can be rephrased as follows: if a system of polynomial equations and inequalities is infeasible then $-1$ lies in the sum of the ideal of equations and the quadratic module of inequalities. There is a more general version of the Positivstellensatz which also incorporates strict inequalities $h_1 > 0, \ldots, h_t > 0$. This is stated in [58, Theorem 7.5] and it is also proved in [43, §2.3].

The radical $\sqrt{I}$ of a polynomial ideal $I$ was the main player in the strong form of Hilbert’s Nullstellensatz (Theorem 6.5). It offers an algebraic representation for polynomials that vanish on a given complex variety. We now come to the analogous result for varieties over the real numbers.

Given an ideal $I$ in $\mathbb{R}[x]$, we define its real radical $\sqrt[\mathbb{R}]{I}$ to be the set

$$\{ f \in \mathbb{R}[x] : f^{2m} + g_1^2 + \cdots + g_s^2 \in I \text{ for some } m \in \mathbb{N} \text{ and } g_1, \ldots, g_s \in \mathbb{R}[x] \}.$$ 

One checks that $\sqrt[\mathbb{R}]{I}$ is an ideal in $\mathbb{R}[x]$. Here is the analogue to Theorem 6.5:

**Theorem 6.15 (Real Nullstellensatz).** For any ideal in $\mathbb{R}[x]$, we have

$$(6.7) \quad I(\mathcal{V}_\mathbb{R}(I)) = \sqrt[\mathbb{R}]{I}.$$ 

**Proof.** The argument is similar to the proof of Theorem 6.5. Clearly, $\sqrt[\mathbb{R}]{I}$ is contained in $I(\mathcal{V}_\mathbb{R}(I))$. We need to show the reverse inclusion. Suppose that $f$ vanishes on the real variety of $I = \langle f_1, \ldots, f_r \rangle \subset \mathbb{R}[x]$. We introduce a new variable $y$ and consider ideal $J = \langle f_1, \ldots, f_r, yf - 1 \rangle$ in $\mathbb{R}[x,y]$. It satisfies $\mathcal{V}_\mathbb{R}(J) = \emptyset$. By Theorem 6.12 there exists an identity of the form (6.2) for the ideal $J$. Substituting $y = 1/f(x)$ into that identity and clearing denominators, we find that some even power of $f$ plus a sum of squares lies in $I$. This means that the polynomial $f$ is in the real radical $\sqrt[\mathbb{R}]{I}$. \hfill \Box

**Example 6.16.** Fix the ideal generated by the Motzkin polynomial

$$I = \langle M(x,y) \rangle = \langle x^4y^2 + x^2y^4 + 1 - 3x^2y^2 \rangle.$$ 

Building on Example 6.11 we wish to compute the real radical $\sqrt[\mathbb{R}]{I}$. It must contain the numerators of the four summands in the rational sum of squares representation of $M$. This leads us to consider the ideal

$$J = \langle M, xy(x^2 + y^2 - 2), x^2 - y^2 \rangle.$$ 

We find that the radical of $J$ is the Jacobian ideal of the Motzkin polynomial:

$$\sqrt{J} = \langle M, \frac{\partial M}{\partial x}, \frac{\partial M}{\partial y} \rangle.$$ 

Furthermore, this radical ideal is precisely the real radical we are looking for:

$$\sqrt[\mathbb{R}]{I} = \sqrt{J} = \langle x - 1, y - 1 \rangle \cap \langle x - 1, y + 1 \rangle \cap \langle x + 1, y - 1 \rangle \cap \langle x + 1, y + 1 \rangle.$$
The real variety $V_{\mathbb{R}}(M)$ defined by the Motzkin polynomial consists of the four points $(1,1), (1,-1), (-1,1)$ and $(-1,-1)$ in $\mathbb{R}^2$. Since $M$ is non-negative, these zeros are singular points of the complex curve $V(M) \subset \mathbb{C}^2$.

## Exercises

1. Show that the real radical of an ideal $I$ in $\mathbb{R}[x]$ is a radical ideal and that it contains the radical of $I$.

2. Find univariate polynomials $g_1, g_2, g_3, g_4$ in $\mathbb{Q}[x]$ such that
   \[
   g_1(x-2)(x-3)(x-4) + g_2(x-1)(x-3)(x-4) + g_3(x-1)(x-2)(x-4) + g_4(x-1)(x-2)(x-3) = 1.
   \]

3. An ideal $I$ in $\mathbb{C}[x]$ contains a monomial if and only if each point in its variety $V(I)$ has at least one zero coordinate. Prove this fact, and describe an algorithm for testing whether an ideal contains a monomial.

4. Let $M$ be an ideal generated by monomials in $K[x]$. How to compute the radical $\sqrt{M}$? How to compute the real radical $\sqrt{M}$?
   Hint: It could help to investigate the ideal in Example 3.23.

5. Let $I$ be the ideal generated by the two cubics $x_1^3x_2-x_3^3x_4$ and $x_1x_2^3-x_4^3$. Describe the projective variety $V(I)$ in $\mathbb{P}^3$. Find the radical ideal $\sqrt{I}$. How many minimal generators does $\sqrt{I}$ have and what are their degrees?

6. Let $V \subset \mathbb{R}^7$ be the variety of orthogonal Hankel matrices of format $4 \times 4$. Describe the ideal $I(V)$. What are the irreducible components of $V$?

7. Let $I$ be the ideal generated by the two quartics $x_1^4-x_1^2x_2^2$ and $x_3^4-x_4^4$ in $\mathbb{R}[x_1,x_2,x_3]$. Determine the radical $\sqrt{I}$ and the real radical $\sqrt[\mathbb{R}]{I}$. Write each of these two radical ideals as an intersection of prime ideals.

8. Let $f_1, \ldots, f_r$ and $f$ be polynomials in $\mathbb{Q}[x]$. Explain how Gröbner bases can be used to test whether $f$ lies in the radical of $I = \langle f_1, \ldots, f_r \rangle$.

9. Let $f \in \mathbb{R}[x]$ be a non-negative polynomial. Show that the real radical of the ideal generated by partial derivatives of $f$ is contained in the real radical of $\langle f \rangle$. Find an example where the containment is strict.

10. The circle given by $f = x^2 + y^2 - 4$ does not intersect the hyperbola given by $g = xy - 10$ in the plane $\mathbb{R}^2$. Find a real Nullstellensatz certificate for this, i.e. write $-1$ as a sum of squares modulo the ideal $\langle f, g \rangle$ in $\mathbb{R}[x, y]$.

11. For $d \in \mathbb{N}$, exhibit a polynomial $f$ and an ideal $I$ in $K[x]$ with $f^d \notin I$ but $f^{d+1} \in I$. How small can the degrees of the generators of $I$ be?

12. Let $I$ be the ideal in $\mathbb{R}[x, y, z]$ generated by the *Robinson polynomial*
   \[
   x^6 + y^6 + z^6 + 3x^2y^2z^2 - x^4y^2 - x^4z^2 - x^2y^4 - x^2z^4 - y^4z^2 - y^2z^4.
   \]
Determine the real radical $\sqrt{I}$ and the real variety $V_{\mathbb{R}}(I)$ in $\mathbb{P}^2$. (13) Show that Theorem 6.15 implies Theorem 6.10. (14) What is the Effective Nullstellensatz? (15) Find the radical and the real radical of the ideal $I = \langle x^7 - y^7, x^8 - z^8 \rangle$ in $\mathbb{R}[x, y, z]$. Explain the difference between these two radical ideals. (16) Prove Farkas’ Lemma (Theorem 6.13). (17) Consider the unit circles centered at the points $(0, 0)$, $(1, \frac{3}{2})$ and $(\frac{3}{2}, 1)$ in the plane $\mathbb{R}^2$. Do they intersect? Give a formulation (6.4), and prove that the answer is no by exhibiting a Positivstellensatz certificate (6.6).
Tropical Algebra

“It is a theorem from algebraic geometry that all log-log plots look like straight lines”, Maciej Zworski

The operations of addition and multiplication are familiar from primary school. We here introduce tropical arithmetic. The new operations may at first seem unnatural to the reader, but we justify them with several applications, e.g. in the design of dynamic programming algorithms. A big part of our discussion centers around tropical linear algebra. The point is that the piecewise linear structures of tropical mathematics offer yet another transition point between linear and nonlinear algebra. On the fully nonlinear side lies tropical algebraic geometry. We briefly touch on this subject with a discussion of tropical varieties and their geometric properties. This chapter is meant as an invitation to the textbook [41] whose setup we follow closely.

7.1. Arithmetic and Valuations

The tropical semiring \((\mathbb{R} \cup \{\infty\}, \oplus, \odot)\) consists of the real numbers \(\mathbb{R}\), together with an extra element \(\infty\) that represents plus-infinity. The arithmetic operations of addition and multiplication in the tropical semiring are

\[
\begin{align*}
x \oplus y & := \min(x,y) \quad \text{and} \quad x \odot y := x + y.
\end{align*}
\]

The tropical sum of two real numbers is their minimum, and the tropical product of two real numbers is their usual sum. It takes some practice to carry out arithmetic in the tropical world. Here is an example with numbers:

\[
4 \oplus 5 = 4 \quad \text{and} \quad 4 \odot 5 = 9.
\]
Tropical addition and tropical multiplication are both commutative and associative. The distributive law holds and the times operator $\odot$ takes precedence when plus $\oplus$ and times $\odot$ occur in the same expression. For example:

$$4 \odot (5 \oplus 7) = 4 \odot 5 = 9,$$
$$4 \odot 5 \oplus 4 \odot 7 = 9 \oplus 11 = 9.$$ 

Both arithmetic operations have an identity element. Infinity is the identity element for addition and zero is the identity element for multiplication:

$$x \oplus \infty = x \quad \text{and} \quad x \odot 0 = x.$$ 

We also note the following identities involving the two identity elements:

$$x \odot \infty = \infty \quad \text{and} \quad x \oplus 0 = \begin{cases} 0 \text{ if } x \geq 0, \\ x \text{ if } x < 0. \end{cases}$$ 

There is no subtraction in tropical arithmetic. There is no real number $x$ to be called “17 minus 8” because the equation $8 \oplus x = 17$ has no solution $x$. Tropical division is defined to be classical subtraction, so $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ satisfies all ring axioms except for the existence of an additive inverse.

Such algebraic structures without division are called semirings, whence the name tropical semiring. It is essential to remember that “0” is the multiplicative identity element. If we write a term without explicit coefficient then that coefficient is zero. For instance, $x \oplus y$ means $0 \odot x \oplus 0 \odot y$.

**Example 7.1 (Binomial Theorem).** We consider the third tropical power of a tropical sum. The following identities hold for all real numbers $x, y \in \mathbb{R}$:

$$(x \oplus y)^\odot 3 = (x \oplus y) \odot (x \oplus y) \odot (x \oplus y)$$

$$= 0 \odot x^\odot 3 \oplus 0 \odot x^\odot 2 \odot y \oplus 0 \odot x \odot y^\odot 2 \oplus 0 \odot y^\odot 3.$$ 

Of course, the zero coefficients can here be dropped:

$$(x \oplus y)^\odot 3 = x^\odot 3 \oplus x^\odot 2 \odot y \oplus x \odot y^\odot 2 \oplus y^\odot 3 = x^\odot 3 \oplus y^\odot 3.$$

What is the relationship between classical arithmetic and tropical arithmetic? An informal answer is that the latter is the image of the former under taking logarithms. Indeed, if $u$ and $v$ are small positive real numbers then $\log(u \cdot v)$ equals $\log(u) \odot \log(v)$, and $\log(u + v)$ is approximately the same as $\log(u) \oplus \log(v)$. Thus tropical geometry arises naturally when one draws a log-log-plot of figures in $\mathbb{R}^2_{>0}$. This was the point of the quote of the previous page. We refer to [41, Chapter 1] for further inspirations.

Applications of logarithms to simplify mathematical objects go back to the 17th century. The original idea was really to reduce multiplication to addition. According to Laplace, the invention of logarithms “doubles the life of the astronomer, and spares him the errors and disgust inseparable
from long calculations”. For several centuries, scientists and engineers used slide rules to perform multiplications. These marvelous devices embody the logarithm. They were used until the late 1970’s, also by school children, including the second author of this book. Eventually, slide rules were replaced by calculators. However, the idea of addition as a model for multiplication remains powerful, as it simplifies both algebra and geometry.

A more formal way of understanding the relationship between classical arithmetic and tropical arithmetic is to introduce fields with valuations.

**Definition 7.2.** A valuation on a field $K$ is a function $\text{val} : K \to \mathbb{R} \cup \{\infty\}$ that satisfies the following three axioms for all $a,b \in K$:

1. $\text{val}(ab) = \text{val}(a) + \text{val}(b)$,
2. $\text{val}(a + b) \geq \min\{\text{val}(a), \text{val}(b)\}$, and
3. $\text{val}(a) = \infty$ if and only if $a = 0$.

We often identify a valuation with its restriction to $K^* = K \setminus \{0\}$. With this, the image of $\text{val}$ is an additive subgroup of $\mathbb{R}$, known as the value group.

A field $K$ with valuation is a metric space. Namely, the valuation induces a norm $|\cdot| : K \to \mathbb{R}$ by setting $|a| = \exp(-\text{val}(a))$ for $a \in K^*$ and $|0| = 0$. The field $K$ is a metric space with distance $|a - b|$ between two elements $a,b \in K$. In fact, this metric on $K$ is an ultrametric, which means that

$$|a + b| \leq \max(|a|, |b|) \leq |a| + |b|.$$

This allows the use of analytical and topological methods for studying $K$.

An important example is the field of Puiseux series in a variable $t$ with complex coefficients. This field is denoted $K = \mathbb{C}\{\{t\}\}$. It contains the field $\mathbb{C}(t)$ of rational functions and its algebraic closure $\overline{\mathbb{C}(t)}$. Indeed, every element in $\overline{\mathbb{C}(t)}$ can be expanded into a Puiseux series with integer exponents.

The valuation of a scalar $c$ in $K$ is the smallest exponent $a$ of any term $c_a t^a$ with $c_a \neq 0$ that appears in the series expansion of $c$. We write $a = \text{val}(c)$. This is an element in the value group $(\mathbb{Q}, +)$ of $K$. Here are two examples of scalars in the Puiseux series field $K$ and their valuations:

$$c = \frac{1}{t^2 + 2t^3 + t^5} = t^{-2} - 2t^{-1} + 4 - 9t + 20t^2 - 44t^3 + 97t^4 - 214t^5 + 472t^6 - \cdots$$

has $\text{val}(c) = -2$, while the following scalar has $\text{val}(c') = \frac{2}{7}$:

$$c' = t^{2/7}\sqrt{1 - t^{2/3}} = t^{2/7} - \frac{1}{2} t^{20/21} - \frac{1}{8} t^{34/21} - \frac{1}{16} t^{16/7} - \frac{5}{128} t^{62/21} - \cdots$$

It is known that the field $K$ is algebraically closed [41 Theorem 2.1.5]. So, every polynomial of degree $d$ in $K[x]$ has $d$ roots, counting multiplicities.
Example 7.3 (Puiseux series). Every cubic polynomial in $K[x]$ has three roots. For instance, the three roots of \( f(x) = tx^3 - x^2 + 3tx - 2t^5 \) are
\[
\begin{align*}
t^{-1} - 3t - 9t^3 - 54t^5 + 2t^6 - 405t^7 + 18t^8 - 3402t^9 + 180t^{10} - 30618t^{11} + \cdots \\
3t + 9t^3 - \frac{2}{3}t^4 + 54t^5 - 2t^6 + \frac{10931}{288}t^7 - 18t^8 + 3402t^9 - \frac{43756}{2187}t^{10} + 30618t^{11} + \cdots \\
\frac{2}{3}t^4 + \frac{4}{27}t^7 + \frac{16}{243}t^{10} - \frac{8}{81}t^{12} + \frac{80}{2187}t^{13} - \frac{80}{729}t^{15} + \frac{448}{19683}t^{16} - \frac{224}{2187}t^{18} + \cdots
\end{align*}
\]
Such Puiseux series can be computed in a computer algebra system. The valuations of the three roots are $-1$, $1$ and $4$. These characterize the asymptotic behavior of the roots when $t$ is a real number that is close to zero.

From now on we assume that $K$ is an algebraically closed field of characteristic zero and with valuation. Let $f \in K[x]$ be a polynomial in $n$ variables. We define its tropicalization $\text{trop}(f)$ to be the polynomial over the tropical semiring obtained by replacing each coefficient in $f$ by its valuation. In what follows we restrict to $n = 1$, but later we also take $n \geq 2$.

For instance, if $f$ is the cubic in Example 7.3 then its tropicalization is
\[
\text{trop}(f) = 1 \odot x^3 \oplus 0 \odot x^2 \oplus 1 \odot x \oplus 5.
\]
To evaluate a tropical polynomial, one takes the minimum of its tropical monomials. Tropical monomials are affine-linear functions. In (7.1) we have
\[
\text{trop}(f)(u) = \min\{1 + 3u, 0 + 2u, 1 + u, 5\} \quad \text{for all } u \in \mathbb{R}.
\]

Definition 7.4. A tropical polynomial is a function that is the minimum of finitely many affine-linear functions. A number $u \in \mathbb{R}$ is called a tropical root of a tropical polynomial if that minimum is attained at least twice for the argument $u$. For which values of $u$ does this happen in (7.2)?

If $u = \text{val}(c)$ for some $c \in K$ and $u$ is not a tropical root of $\text{trop}(f)$ then
\[
\text{val}(f(c)) = (\text{trop}(f))(u).
\]
In the following lemma we are claiming that two functions $\mathbb{R} \to \mathbb{R}$ agree.

Lemma 7.5. Multiplication of polynomials is compatible with tropicalization. Namely, if $f, g \in K[x]$ then $\text{trop}(fg) = \text{trop}(f) \odot \text{trop}(g)$.

Proof. By enlarging the field $K$, we may assume that the value group equals $\mathbb{R}$. Suppose that $u \in \mathbb{R}$ is not a tropical root of $\text{trop}(f)$, $\text{trop}(g)$ or $\text{trop}(fg)$. Choose any $c \in K$ with $\text{val}(c) = u$. A computation shows
\[
\text{trop}(fg)(u) = \text{val}((fg)(c)) = \text{val}(f(c)) + \text{val}(g(c)) = \text{trop}(f)(u) \odot \text{trop}(g)(u).
\]
Hence the two functions agree on all but finitely many arguments $u \in \mathbb{R}$. As the functions are continuous, we conclude that they are equal.

The following facts relate classical root finding and tropical root finding.
Theorem 7.6. Fix a univariate polynomial $f \in K[x]$ and let $g = \text{trop}(f)$ be its tropicalization. If $c \in K$ satisfies $f(c) = 0$ then $u = \text{val}(c)$ is a tropical root of $g$. Conversely, every tropical root $u$ of $g$ arises from a zero $c$ of $f$.

Proof. Let $f(x) = \sum_{i=0}^{d} b_i x^i$ and suppose $f(c) = \sum_{i=0}^{d} b_i c^i$ is zero. Then $\infty = \text{val}(f(c))$ but $\text{val}(b_i c^i) < \infty$ for $i = 1, \ldots, d$. Using [41, Lemma 2.1.1], this implies $\text{val}(b_i c^i) = \text{val}(b_j c^j) \leq \text{val}(b_k c^k)$ for some $i \neq j$ and all $k \neq i, j$. This means that $u = \text{val}(c)$ is a tropical zero of the $g = \text{trop}(f)$.

Our proof of the second statement follows that of [41, Proposition 3.1.5]. We assume that $u$ is a tropical root of $g = \text{trop}(f)$. Since $K$ is algebraically closed, we can factor $f(x) = \prod_{j=1}^{d} (a_j x - b_j)$. By Lemma 7.5, the tropicalization $g = \text{trop}(f)$ is the classical sum of the $d$ simple tropical functions

$$\text{trop}(a_j x - b_j) = \text{val}(a_j) \odot x \oplus \text{val}(b_j) = \min\{\text{val}(a_j) + x, \text{val}(b_j)\}.$$ 

Each of these functions has precisely one tropical root, namely $\text{val}(b_j) - \text{val}(a_j)$, and each tropical root of their classical sum $g$ must be one of these.

Let $u$ be a tropical root of $g$. Then there is an index $j$ such that $u = \text{val}(b_j) - \text{val}(a_j)$. If we set $c = b_j / a_j$ in $K$ then $f(c) = 0$ and $u = \text{val}(c)$. □

Example 7.7 ($d = 3$). Let $f$ be as in Example 7.3 and $g = \text{trop}(f)$ as in (7.1). The tropical roots of $g$ are the numbers $u$ such that minimum of $1 + 3u$, $0 + 2u$, $1 + u$ and $5$ is attained twice. The solutions to these constraints are $u = -1$, $u = 1$, $u = 4$. These are the valuations of the three roots of $f$.

Fields with valuations provide a systematic way of speaking algebraically about logarithms. This explains the connection between classical arithmetic and tropical arithmetic. Here is a scenario that appears in number theory.

Example 7.8 (The $p$-adic valuation). For every prime number $p$, the field $K = \mathbb{Q}$ of rational numbers has a valuation $\text{val}_p$ with value group $\mathbb{Z}$. Indeed, every rational number $c$ can be written uniquely as $c = p^u \cdot \frac{q}{r}$ where $q$ and $r$ are relatively prime integers not divisible by $p$ and $u \in \mathbb{Z}$. We have $\text{val}_p(c) = u$. The completion of $\mathbb{Q}$ with respect to the norm induced by $\text{val}_p$ is the field of $p$-adic numbers. This field is important in number theory.

We shall return to fields and their valuations in our exploration of varieties in Subsection 7.3. First, however, let us develop some more purely tropical machinery, in the familiar setting of matrices and linear algebra.
7.2. Linear Algebra

Vectors and matrices make sense over the tropical semiring. For instance, the tropical scalar product in \( \mathbb{R}^3 \) of a row vector with a column vector equals

\[
(u_1, u_2, u_3) \odot (v_1, v_2, v_3)^\top = u_1 \odot v_1 \oplus u_2 \odot v_2 \oplus u_3 \odot v_3 = \min\{u_1 + v_1, u_2 + v_2, u_3 + v_3\}.
\]

Here is the product of a column vector and a row vector of length three:

\[
(u_1, u_2, u_3)^\top \odot (v_1, v_2, v_3) = 
\begin{pmatrix}
  u_1 \odot v_1 & u_1 \odot v_2 & u_1 \odot v_3 \\
  u_2 \odot v_1 & u_2 \odot v_2 & u_2 \odot v_3 \\
  u_3 \odot v_1 & u_3 \odot v_2 & u_3 \odot v_3
\end{pmatrix}
= 
\begin{pmatrix}
  u_1 + v_1 & u_1 + v_2 & u_1 + v_3 \\
  u_2 + v_1 & u_2 + v_2 & u_2 + v_3 \\
  u_3 + v_1 & u_3 + v_2 & u_3 + v_3
\end{pmatrix}.
\]

Any matrix which can be expressed as such a product has tropical rank one.

Fix a \( d \times n \)-matrix \( A \). We may wish to find its image \( \{A \odot x : x \in \mathbb{R}^n\} \) and to solve linear systems \( A \odot x = b \) for various right hand sides \( b \). For an introduction to tropical linear systems see the book on Max-linear Systems by Butkovič [7] and Essentials of Tropical Combinatorics by Joswig [31].

For a first application of tropical linear algebra, consider the problem of finding shortest paths in a weighted directed graph \( G \) with \( n \) nodes. Every directed edge \((i, j)\) in \( G \) has an associated length \( d_{ij} \) which is a non-negative real number. If \((i, j)\) is not an edge of \( G \) then we set \( d_{ij} = +\infty \).

We represent \( G \) by its \( n \times n \) adjacency matrix \( D_G = (d_{ij}) \) with zeros on the diagonal. The off-diagonal entries are the edge lengths \( d_{ij} \). The matrix \( D_G \) need not be symmetric: we allow \( d_{ij} \neq d_{ji} \). If \( G \) is an undirected graph, then we view \( G \) as a directed graph with two directed edges \((i, j)\) and \((j, i)\) for each undirected edge \(\{i, j\} \). In that case, \( D_G \) is a symmetric matrix, where \( d_{ii} = 0 \) and \( d_{ij} = d_{ji} \) is the distance between node \( i \) and node \( j \).

Consider the \( n \times n \)-matrix with entries in \( \mathbb{R}_{\geq 0} \cup \{\infty\} \) that results from tropically multiplying the given adjacency matrix \( D_G \) with itself \( n-1 \) times:

\[
D_G^{\odot(n-1)} = D_G \odot D_G \odot \cdots \odot D_G.
\]

**Proposition 7.9.** Let \( G \) be a weighted directed graph on \( n \) nodes with adjacency matrix \( D_G \). The entry of the matrix \( D_G^{\odot(n-1)} \) in row \( i \) and column \( j \) equals the length of a shortest path from node \( i \) to node \( j \) in the graph \( G \).

**Proof.** Let \( d_{ij}^{(r)} \) denote the minimum length of any path from node \( i \) to node \( j \) using at most \( r \) edges in \( G \). We have \( d_{ij}^{(1)} = d_{ij} \) for any two nodes \( i \) and \( j \). Since the edge weights \( d_{ij} \) were assumed to be non-negative, for each two nodes \( i, j \), there exists a shortest path from \( i \) to \( j \) that visits each node of \( G \) at most once. Hence the length of a shortest path from \( i \) to \( j \) equals \( d_{ij}^{(n-1)} \).
For $r \geq 2$ we have a recursive formula for the length of a shortest path:

\[ d_{ij}^{(r)} = \min \{ d_{ik}^{(r-1)} + d_{kj} : k = 1, 2, \ldots, n \}. \]  

Using tropical arithmetic, this formula can be rewritten as follows:

\[ d_{ij}^{(r)} = d_{i1}^{(r-1)} \circ d_{1j} + d_{i2}^{(r-1)} \circ d_{2j} + \cdots + d_{in}^{(r-1)} \circ d_{nj}. \]

\[ = (d_{i1}^{(r-1)}, d_{i2}^{(r-1)}, \ldots, d_{in}^{(r-1)}) \circ (d_{1j}, d_{2j}, \ldots, d_{nj})^T. \]

From this it follows, by induction on $r$, that $d_{ij}^{(r)}$ equals the entry in row $i$ and column $j$ of the $n \times n$ matrix $D_G^\circ r$. Indeed, the right hand side of the recursive formula is the tropical product of row $i$ of $D_G^\circ (r-1)$ and column $j$ of $D_G$, which is the $(i,j)$ entry of $D_G^\circ r$. In particular, $d_{ij}^{(n-1)}$ is the entry in row $i$ and column $j$ of $D_G^\circ (n-1)$. This proves the claim.

The above algorithm belongs to what is known as Dynamic Programming in Computer Science. For us, it means performing the matrix multiplication

\[ D_G^\circ r = D_G^\circ (r-1) \circ D_G \quad \text{for} \quad r = 2, \ldots, n-1. \]

We next consider the notion of the tropical determinant. This is the tropicalization of the classical determinant, here regarded as a polynomial of degree $n$ in $n^2$ unknowns $x_{ij}$. Thus, we form the tropical sum over the tropical diagonal products obtained by taking all $n!$ permutations $\pi$ of $\{1, 2, \ldots, n\}$:

\[ \text{tropdet}(X) := \bigoplus_{\pi \in S_n} x_{1\pi(1)} \circ x_{2\pi(2)} \circ \cdots \circ x_{n\pi(n)}. \]

Here $S_n$ is the symmetric group of permutations of $\{1, 2, \ldots, n\}$. Evaluating the tropical determinant means solving the classical assignment problem of combinatorial optimization. Imagine a company that has $n$ jobs and $n$ workers, and each job needs to be assigned to exactly one of the workers. Let $x_{ij}$ be the cost of assigning job $i$ to worker $j$. The company wishes to find the cheapest assignment $\pi \in S_n$. The optimal total cost equals

\[ \min \{ x_{1\pi(1)} + x_{2\pi(2)} + \cdots + x_{n\pi(n)} : \pi \in S_n \}. \]

That minimum is the tropical determinant (7.6) of the matrix $X = (x_{ij})$.

**Proposition 7.10.** The tropical determinant solves the assignment problem.

In the assignment problem we seek the minimum over $n!$ quantities. This appears to require exponentially many operations. However, there is a polynomial-time method for this task, namely the Hungarian Algorithm.

The square matrix $X$ is called tropically singular if the minimum in (7.7) is attained at least twice. Following Definition 7.4, this is equivalent to saying that $X$ is a tropical root of the tropical determinant (7.6). If $Y$
7. Tropical Algebra

is any rectangular matrix then we define the tropical rank of $Y$ to be the size of the largest square submatrix that is not tropically singular. For rank one, this consistent with our earlier definition, given after (7.3). Namely, the matrix $Y$ has tropical rank one if and only if it the tropical product of a column vector and a row vector. For a detailed discussion of the tropical rank, and other related notions of matrix ranks, we refer to [41, Section 5.3].

Eigenvectors and eigenvalues of square matrices are central to linear algebra. The same is true in tropical linear algebra. We fix an $n \times n$-matrix $A = (a_{ij})$ over $\mathbb{R} = \mathbb{R} \cup \{\infty\}$. An eigenvalue of $A$ is a real number $\lambda$ such that

$$A \circ \mathbf{v} = \lambda \circ \mathbf{v} \quad \text{for some } \mathbf{v} \in \mathbb{R}^n.$$  

We say that $\mathbf{v}$ is an eigenvector of the matrix $A$. The arithmetic operations in (7.8) are tropical. For instance, for $n = 2$, the left hand side of (7.8) is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \circ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_{11} \circ v_1 \oplus a_{12} \circ v_2 \\ a_{21} \circ v_1 \oplus a_{22} \circ v_2 \end{pmatrix} = \begin{pmatrix} \min\{a_{11} + v_1, a_{12} + v_2\} \\ \min\{a_{21} + v_1, a_{22} + v_2\} \end{pmatrix}.$$  

The right hand side of (7.8) equals

$$\begin{pmatrix} \lambda \circ v_1 \\ \lambda \circ v_2 \end{pmatrix} = \begin{pmatrix} \lambda + v_1 \\ \lambda + v_2 \end{pmatrix}.$$  

Let $G(A)$ denote the directed graph with adjacency matrix $A$. Its nodes are labeled by $[n] = \{1, 2, \ldots, n\}$. There is an edge from node $i$ to node $j$ if and only if $a_{ij} < \infty$. The edge has length $a_{ij}$. In particular, $a_{ii} \neq \infty$ if and only if there is a loop at vertex $i$. The normalized length of a directed path $i_0, i_1, \ldots, i_k$ in $G(A)$ is $(a_{i_0i_1} + a_{i_1i_2} + \cdots + a_{i_{k-1}i_k})/k$, computed in classical arithmetic. If $i_k = i_0$ then the path is a directed cycle, and this quantity is the normalized length of the cycle. Recall that a directed graph is strongly connected if there is a directed path from any node to any other node.

**Theorem 7.11.** Let $A$ be an $n \times n$-matrix such that $G(A)$ is strongly connected. Then $A$ has precisely one eigenvalue $\lambda(A)$. It equals the minimum normalized length of a directed cycle.

**Proof.** Let $\lambda = \lambda(A)$ be the minimum of the normalized lengths over all directed cycles in $G(A)$. We first prove that $\lambda(A)$ is the only possibility for an eigenvalue. Suppose that $z \in \mathbb{R}^n$ is any eigenvector of $A$, and let $\gamma$ be the corresponding eigenvalue. For any cycle $(i_1, i_2, \ldots, i_k, i_1)$ in $G(A)$ we have

$$a_{i_1i_2} + z_{i_2} \geq \gamma + z_{i_1}, \quad a_{i_2i_3} + z_{i_3} \geq \gamma + z_{i_2},$$

$$a_{i_3i_4} + z_{i_4} \geq \gamma + z_{i_3}, \ldots, a_{i_{k-1}i_k} + z_{i_k} \geq \gamma + z_{i_{k-1}}.$$  

Adding the left-hand sides and the right-hand sides, we find that the normalized length of the cycle is greater than or equal to $\gamma$. In particular, we have $\lambda(A) \geq \gamma$. For the reverse inequality, start with any index $i_1$. Since $z$ is an
eigenvector with eigenvalue $\gamma$, there exists $i_2$ such that $a_{i_1i_2} + z_{i_2} = \gamma + z_{i_1}$. Likewise, there exists $i_3$ such that $a_{i_2i_3} + z_{i_3} = \gamma + z_{i_2}$. We continue in this manner until we reach an index $i_l$ which was already in the sequence, say, $i_k = i_l$ for $k < l$. By adding the equations along this cycle, we find that

\[
(a_{i_ki_{k+1}} + z_{i_{k+1}}) + (a_{i_{k+1}i_{k+2}} + z_{i_{k+2}}) + \cdots + (a_{i_{l-1}i_l} + z_{i_l})
= (\gamma + z_{i_k}) + (\gamma + z_{i_{k+1}}) + \cdots + (\gamma + z_{i_{l-1}}).
\]

We conclude that the normalized length of the cycle $(i_k, i_{k+1}, \ldots, i_l = i_k)$ in $G(A)$ is equal to $\gamma$. In particular, $\gamma \geq \lambda(A)$. This proves that $\gamma = \lambda(A)$.

It remains to prove the existence of an eigenvector. Let $B$ be the matrix obtained from $A$ by (classically) subtracting $\lambda(A)$ from every entry in $A$. All cycles in $G(B)$ have non-negative length, and there exists a cycle of length zero. Using tropical matrix operations we define

\[
B^+ = B \oplus B^2 \oplus B^3 \oplus \cdots \oplus B^n.
\]

This matrix is known as the Kleene plus of the matrix $B$. The entry $B^+_{ij}$ in row $i$ and column $j$ of $B^+$ is the length of a shortest path from node $i$ to node $j$ in the weighted directed graph $G(B)$. Here, we assume that a path contains some edges, thus the shortest path from $i$ to $i$ may be strictly positive. Since $G(B)$ is strongly connected, we have $B^+_{ij} < \infty$ for all $i$ and $j$.

Fix any node $j$ that lies on a zero length cycle of $G(B)$. Let $x = B^+_{ij}$ denote the $j$th column vector of the matrix $B^+$. We have $x_j = B^+_{jj} = 0$, as there is a path from $j$ to itself of length zero, and there are no negative weight cycles. This implies $B^+ \odot x \leq B^+_{ij} = x$. Next note that $(B \odot x)_i = \min_l (B_{il} + x_l) = \min_l (B_{il} + B^+_{ij}) \geq B^+_{ij} = x_i$, since lengths of shortest paths obey the triangle inequality. In vector notation this states $B \odot x \geq x$. Since tropical linear maps preserve coordinate-wise inequalities among vectors, we have $B^2 \odot x \geq B \odot x$, and $B^3 \odot x \geq B^2 \odot x$, etc. Therefore, $B^+ \odot x = B \odot x \oplus B^2 \odot x \oplus \cdots \oplus B^n \odot x = B \odot x$. This yields $x \leq B \odot x = B^+ \odot x \leq x$. This means that $B \odot x = x$, so $x$ is an eigenvector of $B$ with eigenvalue 0. We conclude that $x$ is an eigenvector with eigenvalue $\lambda$ of our matrix $A$:

\[
A \odot x = (\lambda \odot B) \odot x = \lambda \odot (B \odot x) = \lambda \odot x.
\]

This completes the proof of Theorem 7.11.

The eigenvalue $\lambda$ of a tropical $n \times n$-matrix $A = (a_{ij})$ can be computed efficiently, using a linear program with $n + 1$ decision variables $v_1, \ldots, v_n, \lambda$:

(7.9) Maximize $\gamma$ subject to \( a_{ij} + v_j \geq \gamma + v_i \) for all $1 \leq i, j \leq n$.

**Proposition 7.12.** The unique eigenvalue $\lambda(A)$ of the given $n \times n$-matrix $A = (a_{ij})$ coincides with the optimal value $\gamma^*$ of the linear program (7.9).

**Proof.** See [41, Proposition 5.1.2].
We next determine the *eigenspace* of the matrix $A$, which is the set
\[
\text{Eig}(A) = \left\{ x \in \mathbb{R}^n : A \odot x = \lambda(A) \odot x \right\}.
\]
The set $\text{Eig}(A)$ is closed under tropical scalar multiplication: if $x \in \text{Eig}(A)$ and $c \in \mathbb{R}$ then $c \odot x$ is also in $\text{Eig}(A)$. We can thus identify $\text{Eig}(A)$ with its image in the quotient space $\mathbb{R}^n / \mathbb{R}^1 \simeq \mathbb{R}^{n-1}$. Here $1 = (1, 1, \ldots, 1)$. This space is called the *tropical projective torus*; cf. [31, Section 1.4]. We saw that every eigenvector of the matrix $A$ is also an eigenvector of the matrix $B = (\lambda(A)) \odot A$ and vice versa. Hence the eigenspace $\text{Eig}(A)$ is equal to $\text{Eig}(B) = \left\{ x \in \mathbb{R}^n : B \odot x = x \right\}$.

**Theorem 7.13.** Let $B_0^+$ be the submatrix of the Kleene plus $B^+$ given by the columns whose diagonal entry $B^+_j$ is zero. The image of this matrix, in tropical arithmetic, equals the eigenspace: $\text{Eig}(A) = \text{Eig}(B) = \text{Image}(B_0^+)$. 

**Proof.** See [41, Theorem 5.1.3].

**Example 7.14.** We demonstrate the computation of eigenvectors for $n = 3$.

In our first example, the minimal cycle lengths are attained by the loops:
\[
A = \begin{pmatrix} 3 & 4 & 4 \\ 4 & 3 & 4 \\ 4 & 4 & 3 \end{pmatrix} \Rightarrow \lambda(A) = 3 \Rightarrow B = B^+ = B_0^+ = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.
\]

The eigenspace is the image of the column vectors of $B$. Its image in $\mathbb{R}^3 / \mathbb{R}^1$ is the hexagon with vertices $(0, 1, 1)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 0, 0)$, $(1, 1, 0)$ and $(0, 1, 0)$. In our second example, the winner is the cycle $1 \rightarrow 2 \rightarrow 1$:
\[
A = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 4 & 3 \end{pmatrix} \Rightarrow \lambda(A) = 1 \Rightarrow B = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 3 & 3 & 2 \end{pmatrix} \Rightarrow B^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}.
\]

The eigenspace of $A$ is the tropical linear span of the first column of $B_+$:
\[
\text{Eig}(A) = \text{Eig}(B) = \left\{ c \odot (0, 0, 3)^T : c \in \mathbb{R} \right\} = \left\{ (c, c, c + 3)^T : c \in \mathbb{R} \right\}
\]
So, here $\text{Eig}(A)$ is just a single point in the tropical projective 2-torus $\mathbb{R}^3 / \mathbb{R}^1$.

We computed the eigenspace of a square matrix as the image of another matrix. This motivates the study of images of tropical linear maps $\mathbb{R}^m \to \mathbb{R}^n$. Such images are not tropical linear spaces. They are known as *tropical polytopes*. Indeed, one defines tropical convexity in $\mathbb{R}^n / \mathbb{R}^1$ by taking tropical linear combinations. Tropical convexity is a rich and beautiful theory with many applications. For introductions see [31, Chapter 5] and [41, §5.2].

We give a brief illustration for $m=n=3$. The image of a $3 \times 3$-matrix $A$ is the set of all tropical linear combinations of three vectors in $\mathbb{R}^3$. We represent this by its image in the plane $\mathbb{R}^3 / \mathbb{R}^1$. That image is a *tropical*
7.3. Tropical Varieties

triangle, because it is the tropical convex hull of three points in the plane. This triangle is degenerate if the three points are tropically collinear in \( R^3/R_1 \). This happens when the minimum in the tropical determinant (7.6) is attained twice. In that case, the matrix \( A \) is called tropically singular.

**Example 7.15.** Consider the tropical triangle in \( R^3/R_1 \) given by the matrix

\[
A = \begin{pmatrix}
0 & 0 & 2 \\
0 & 3 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

or

\[
A' = \begin{pmatrix}
-1 & 0 & 2 \\
-1 & 3 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Each point in \( R^3/R_1 \) is represented uniquely by a column vector \((u, v, 0)\). This tropical triangle consists of the segment between \((-1, -1, 0)\) and \((0, 0, 0)\), the segment between \((0, 3, 0)\) and \((0, 1, 0)\), the segment between \((2, 1, 0)\) and \((1, 1, 0)\), and the classical triangle with vertices \((0, 0, 0)\), \((0, 1, 0)\) and \((1, 1, 0)\).

There are five combinatorial types of tropical triangles. Similarly, there are 35 types of tropical quadrilaterals. They are shown in Figure 5.2.4.

7.3. Tropical Varieties

The previous section explored tropical counterparts of concepts from linear algebra. In what follows we move on to nonlinear algebra. Our aim is to introduce the tropical counterparts of algebraic varieties. Our point of departure is discussion of fields with valuation at the end of Section 7.1.

In what follows we assume that \( K \) is an algebraically closed field with a valuation whose value group is \( Q \). The Puiseux series field \( K = C\left\{\{t\}\right\} \) is our primary example. Another example, derived from Example 7.8 and of great interest in number theory, is the algebraic closure of the \( p \)-adic numbers.

Consider any polynomial in \( n \) variables \( x = (x_1, \ldots, x_n) \) over the field \( K \):

\[
f = c_1x^{a_1} + c_2x^{a_2} + \cdots + c_sx^{a_s}.
\]

The tropicalization of \( f \) is the following expression in tropical arithmetic:

\[
trop(f) = \text{val}(c_1) \odot x^{\odot a_1} + \text{val}(c_2) \odot x^{\odot a_2} + \cdots + \text{val}(c_s) \odot x^{\odot a_s}.
\]

To evaluate the tropical polynomial \( \text{trop}(f) \) at a point \( u = (u_1, \ldots, u_n) \), we take the minimum of the \( s \) expressions seen in the tropical sum above:

\[
\text{val}(c_i) \odot u^{\odot a_i} = \text{val}(c_i) \odot u_1^{\odot a_{i1}} \odot \cdots \odot u_n^{\odot a_{in}} = \text{val}(c_i) + a_{i1}u_1 + \cdots + a_{in}u_n.
\]

Here the index \( i \) runs over \( \{1, \ldots, s\} \). If this minimum is attained at least twice then \( u \) is a tropical zero of \( \text{trop}(f) \). The special case \( n = 1 \) appeared in Definition 7.4. The following result generalizes the first part of Theorem 7.6.

**Proposition 7.16.** If \( z = (z_1, \ldots, z_n) \in K^n \) is a zero of a polynomial \( f \) in \( K[x] \) then its coordinatewise valuation \( \text{val}(z) = (\text{val}(z_1), \ldots, \text{val}(z_n)) \in Q^n \) is a tropical zero of the associated tropical polynomial \( \text{trop}(f) \).
Proof. Let $u_i = \text{val}(z_i)$ and $u = (u_1, \ldots, u_n)$. The valuation of the term $c_iz_i^n$ in (7.10) equals $\text{val}(c_i) \odot u \odot a_i$. The sum of the $s$ scalars $c_iz_i^n$ is zero in $K$, so the summands of lowest valuation must have a cancellation. In other words, the minimum valuation in the tropical sum is attained by two or more of the expressions $\text{val}(c_i) \odot u \odot a_i$. By definition, this means that the vector $u \in \mathbb{Q}^n$ is a tropical zero of trop$(f)$. □

A celebrated result due to Kapranov states that the converse holds too. Namely, if $f \in K[x]$ and $u \in \mathbb{Q}^n$ is a tropical zero of trop$(f)$ then there is a point $z \in K^n$ such that $f(z) = 0$ and val$(z) = u$. We refer to [41, Theorem 3.1.3] for the proof and further details. For the $n = 1$ case see Theorem 7.6.

The element $\infty$ in the tropical semiring arises naturally from the arithmetic in the field $K$ because val$(0) = \infty$. Sometimes it is preferable to restrict tropical algebra to $\mathbb{R}$, or to $\mathbb{Q}$, thus excluding $\infty$. This is done by disallowing zero coordinates among the solutions of a polynomial system. For this, we set $K^* = K\setminus\{0\}$ and we introduce the \textit{algebraic torus} $(K^*)^n$.

The ring of polynomial functions on $(K^*)^n$ is the \textit{Laurent polynomial ring}

$$K[x^\pm] := K[x_1^{\pm1}, x_2^{\pm1}, \ldots, x_n^{\pm1}].$$

The elements of $K[x^\pm]$ are polynomials as in (7.10) but we now allow negative integers among the coordinates of the exponent vectors $a_i$.

In what follows we fix the field of Puiseux series $K = \mathbb{C}\{\{t\}\}$. For the extension to other fields see [41, §2.4]. Given any vector $u \in \mathbb{R}^n$, the \textit{initial form} $\text{in}_u(f)$ is the subsum of terms $\overline{c}_i x^{a_i}$ in (7.10) for which val$(c_i) \odot u \odot a_i$ is minimal. Here $\overline{c}_i$ is the coefficient of the lowest order term in the Puiseux series $c_i$. For instance, if $c = 3t + 9t^3 - \frac{3}{2}t^4 + \ldots \in K$ is the second scalar displayed in Example 7.3 then $\overline{c} = 3$. A \textit{monomial} in the Laurent polynomial ring is a scalar times a product of variables raised to integer exponents.

Lemma 7.17. For any Laurent polynomial $f \in K[x^\pm]$ and any point $u \in \mathbb{R}^n$, the following three conditions are equivalent:

- The initial form $\text{in}_u(f)$ is not a unit in $K[x^\pm]$.
- The initial form $\text{in}_u(f)$ is not a monomial.
- The point $u$ is a tropical zero of trop$(f)$.

Proof. Every monomial is invertible in the Laurent polynomial ring. To show the converse, we fix a lexicographic order on monomials. If $hg = 1$ then the product of the smallest monomials in the support of $h$ and $g$ must equal the product of the two largest. In particular, the smallest and the largest monomial appearing in $h$ are the same, i.e. $h$ is a monomial. We conclude that the first two items in Lemma 7.17 are equivalent. The equivalence of the last two items comes from the definition of tropical zeros. □
7.3. Tropical Varieties

Fix any ideal \( I \) in \( K[x^\pm] \) and let \( \mathcal{V}(I) \) be its variety in the algebraic torus \( (K^*)^n \). We define the tropical variety of \( I \) to be the following subset of \( \mathbb{R}^n \):
\[
\text{trop}(\mathcal{V}(I)) = \{ \mathbf{u} \in \mathbb{R}^n : \mathbf{u} \text{ is a tropical zero of } \text{trop}(f) \text{ for all } f \in I \}.
\]
We also refer to this set as the tropicalization of the variety \( \mathcal{V}(I) \).

The study of tropical varieties is the subject of tropical algebraic geometry. Two important results are the Fundamental Theorem \([11] \text{ Theorem 3.2.3}) and the Structure Theorem \([41] \text{ Theorem 3.3.5}] \). The former extends Kapranov’s Theorem. It states that the set of rational points in \( \text{trop}(\mathcal{V}(I)) \) is the image of the classical variety \( \mathcal{V}(I) \subseteq (K^*)^n \) under the coordinatewise valuation map. The latter states that the tropical variety \( \text{trop}(\mathcal{V}(I)) \) is a balanced polyhedral complex. Furthermore, its dimension agrees with the dimension of \( \mathcal{V}(I) \). Numerous concrete examples of such balanced polyhedral complexes are found in the textbooks \([31] \text{ and } [41] \).

**Example 7.18.** Fix \( n = 9 \) and let \( \mathbf{x} = (x_{ij}) \) be a \( 3 \times 3 \)-matrix whose entries are unknowns. Let \( I \) be the ideal in \( K[x^\pm] \) that is generated by the nine \( 2 \times 2 \)-minors of \( \mathbf{x} \). Then \( \mathcal{V}(I) \) is the 5-dimensional variety of \( 3 \times 3 \)-matrices of rank 1 in \( (K^*)^{3\times3} \). The tropical variety \( \text{trop}(\mathcal{V}(I)) \) is the set of \( 3 \times 3 \)-matrices in \( \{7,3\} \), that is, matrices \( \mathbf{u} \) of tropical rank one. This is the linear subspace of dimension 5 in \( \mathbb{R}^{3\times3} \) defined by the tropical \( 2 \times 2 \)-determinants \( u_{ij} \odot u_{kl} \oplus u_{il} \odot u_{kj} \). This minimum is attained twice if and only if \( u_{ij} + u_{kl} = u_{il} + u_{kj} = 0 \). Every matrix \( \mathbf{u} = (u_{ij}) \) that satisfies these linear equations, and has its entries in \( \mathbb{Q} \), arises as the valuation \( \mathbf{u} = \text{val}(\mathbf{z}) \) of a rank one matrix \( \mathbf{z} = (z_{ij}) \) with entries in \( K^* \). For instance, \( \mathbf{z} = (t^{u_{ij}}) \).

Consider the assignment problem in Proposition \([7,10]\). The tropical variety \( \text{trop}(\mathcal{V}(I)) \) represents scenarios where all six assignments for \( n = 3 \) have the same cost. The situation becomes more interesting when we pass from rank 1 to rank 2. Now only the two best assignments have the same cost.

To model this, let \( J \subseteq K[x^\pm] \) be the principal ideal generated by the determinant of \( \mathbf{x} \). Then \( \mathcal{V}(J) \) is a hypersurface of degree three in \( (K^*)^{3\times3} \). The tropical hypersurface \( \text{trop}(\mathcal{V}(J)) \) is defined by the tropical determinant
\[
\text{tropdet}(\mathbf{x}) = x_{11} \odot x_{22} \odot x_{33} \oplus x_{11} \odot x_{23} \odot x_{32} \oplus x_{12} \odot x_{21} \odot x_{33} \oplus x_{12} \odot x_{23} \odot x_{31} \oplus x_{13} \odot x_{21} \odot x_{32} \oplus x_{13} \odot x_{22} \odot x_{31}.
\]
Thus \( \text{trop}(\mathcal{V}(J)) \) is set of all \( 3 \times 3 \)-matrices \( \mathbf{u} = (u_{ij}) \) such that the minimum among the six terms is attained twice. For such a matrix, there is more than one optimal assignment of the three workers to the three jobs in \([7,6]\).

The set \( \text{trop}(\mathcal{V}(J)) \) is a polyhedral fan of dimension 8 in the 9-dimensional ambient space \( \mathbb{R}^{3\times3} \). It is a cone with apex \( \text{trop}(\mathcal{V}(I)) \simeq \mathbb{R}^5 \) over a 2-dimensional polyhedral complex. That complex consists of nine squares and six triangles, which are glued together as shown on the left in Figure [4].
Example 3.41 onto the positive points in the image; in symbols:

\[ \begin{pmatrix} 7 \end{pmatrix} \]

polynomial image(\(Q\) surjectively positive. Both maps have the same image, namely

\[ \begin{pmatrix} 11 \ 22 \ 33 \end{pmatrix} \begin{pmatrix} 22 \ 31 \ 23 \end{pmatrix} \begin{pmatrix} 32 \ 13 \ 21 \end{pmatrix} \begin{pmatrix} 11 \ 22 \ 33 \end{pmatrix} \]

Example 3.40 is defined by the tropical 3 \(\times\) 3-minors of a 4 \(\times\) 4-matrix of unknowns form a tropical basis from given generators of an ideal

\(I\). The supports of the matrices \(u\) and \(u'\) under tropicalization:

\[ u = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad u' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \]

The supports of the matrices \(u = \text{trop}(z)\) and \(u' = \text{trop}(z')\) match the labels of the corresponding 2-cells in Figure 1. The matrix \(u\) has support 13, 21, 32, which labels the bottom triangle on the left. The matrix \(u\) has support 13, 23, 31, 32, which labels the middle left square on the right.

We close with a remark on lifting Proposition 7.9 from tropical algebra to algebra over the field \(K = \mathbb{C}\{\{t\}\}\). Given a directed graph \(G\) with rational edge weights \(d_{ij}\), we now define a new adjacency matrix \(A_G\). The entry of \(A_G\) in row \(i\) and column \(j\) equals \(t^{d_{ij}}\) if \((i, j)\) is an edge of \(G\), and 0 otherwise.

\[ z = \begin{pmatrix} t + 1 & -1 + t & 2t \\ t & 1 & 1 + t \\ 1 & t & 1 + t \end{pmatrix} \quad \text{and} \quad z' = \begin{pmatrix} 1 & 2 & t \\ 2 & 4 & 5t \\ 3t & 6t & 7 \end{pmatrix} \]

Figure 1. The combinatorial structure of the tropical hypersurface that is defined by the tropical 3 \(\times\) 3-determinant.

The six triangles represent matrices \(u\) where the minimum of the six terms in \(\text{tropdet}(u)\) is attained by two permutations in \(S_3\) with the same sign. The nine squares on the right in Figure 1 are glued to form a torus. These represent matrices \(u'\) where the minimum is attained by two permutations in \(S_3\) with opposite signs. Concrete examples for the two cases are

\[ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \]
By construction, the valuation of the matrix $A_G$ is the adjacency matrix $D_G$ seen earlier in Section 7.1. Moreover, the tropical matrix power in (7.4) is the valuation of the corresponding power of the classical matrix $A_G$:

\[(7.11)\]

\[D_G^{(n-1)} = (\text{val}(A_G))^{(n-1)} = \text{val}(A_G^{n-1})\]

Indeed, the $(i, j)$ entry of $A_G^{n-1}$ is the generating function for all paths. To be precise, this entry is the Puiseux polynomial $\sum_c c \ell^\ell$, where $c$ is the number of paths from node $i$ to node $j$ in the graph $G$ that have length $\ell$.

**Exercises**

1. Let $u, v, w$ be real numbers and let $x, y, z$ be variables. What are the coefficients in the expansion of the expression \((u \odot x \oplus v \odot y \oplus w \odot z)^\odot n\) in tropical arithmetic?
2. Prove that the tropical matrix multiplication is an associative operation.
3. Draw the graph of the following function on the real plane: $\mathbb{R} \to \mathbb{R}, x \mapsto 1 \oplus 2 \odot x \oplus 3 \odot y \odot 6 \odot x \odot 10 \odot x^2$. What are the tropical zeros of this tropical polynomial?
5. Draw the graph of the function

\[\mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto 1 \oplus 2 \odot x \oplus 3 \odot y \oplus 6 \odot xy \oplus 10 \odot xy^2\]

What are the tropical zeros of this tropical polynomial?
6. Let $G$ be the directed graph on $n$ nodes with edge weights $d_{ij} = i \cdot j$ for $i, j \in \{1, 2, \ldots, n\}$. Compute the tropical powers $D_G^{\odot i}$ of the matrix $D_G$ for $i = 1, 2, \ldots, n - 1$. What are their tropical ranks? Interpret the entries of these matrices in terms of paths.
7. Take the graph $G$ from above with $n = 5$. Compute the powers $A_G^i$ of the matrix $A_G$ for $i < n$. What are their ranks? Interpret the entries in terms of paths. Verify equation (7.11).
8. Take the graph $G$ from above with $n = 3$. Find the eigenvalues and eigenspaces of the classical matrix $A_G$. Find the tropical eigenvalue and the tropical eigenspace of the matrix $D_G$. Do you see a relationship?
9. Take the graph $G$ from above with $n = 10$. Compute the determinant of $A_G$ and the tropical determinant of $D_G$. Do you see a relationship?
10. Take the graph $G$ from above. The matrix $D_G$ defines a tropical linear map from $\mathbb{R}^n$ to itself. Determine the image of this map for $n = 2, 3, 4$. Draw pictures in $\mathbb{R}^n/\mathbb{R}1 \simeq \mathbb{R}^{n-1}$. These are tropical polytopes.
(11) Consider the quartic polynomial \( f(x) = t + t^2x + t^3x^2 + t^6x^3 + t^{10}x^4 \) in \( K[x] \), where \( K = \mathbb{C}[[t]] \). Identify its four roots. Write the first 10 terms of these Puiseux series. What are their valuations?

(12) Let \( J \) be the ideal generated by the determinant of a symmetric \( 3 \times 3 \)-matrix. This lives in a Laurent polynomial ring with six variables. Determine the tropical hypersurface \( \text{trop}(\mathcal{V}(J)) \). Write a discussion similar to Example 7.18. Draw the analog to Figure 1 for symmetric matrices.

(13) Analyze the complexity of the algorithm described in Proposition 7.9. Can you improve the computation of \( D_{G}^{\otimes(n-1)} \)? What happens if some edge weights of \( G \) are negative? What happens if \( G \) contains cycles of negative total weight? How can you detect if such a cycle exists?

(14) The Wikipedia site for Tropical Geometry shows a tropical cubic curve. Find a tropical polynomial in two unknowns that defines this curve.
Toric Varieties

“The art of doing mathematics consists in finding that special case which contains all the germs of generality”, David Hilbert

Toric varieties are the simplest and most accessible varieties. They often appear in applications, both within mathematics and across the sciences. A toric variety is an irreducible variety that is parametrized by a vector of monomials. The relations among these monomials are binomials, i.e. polynomials with only two terms. Thus, an irreducible variety is toric if and only if its prime ideal is generated by binomials. Monomials and binomials correspond to points in an integer lattice, and we think of these as the vertices of a lattice polytope. Toric varieties appear prominently in optimization and statistics, thanks to the purely combinatorial description given above. This description also makes them a perfect “model organism” for algebraic geometers. They use toric varieties to test conjectures, teach geometric concepts, and compute invariants. For instance, the dimension and degree of a toric variety are the dimension and volume of the associated lattice polytope.

8.1. The Affine Story

The adjective toric derives from the noun torus. We begin by introducing tori from an algebraic perspective. We fix an algebraically closed field $K$ and the Laurent polynomial ring $K[x^{\pm}] = K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$. The associated variety $(K^*)^n = \text{Spec } K[x^{\pm}]$ is the algebraic torus of dimension $n$ over $K$.

The algebraic torus $(K^*)^n$ is a group under coordinatewise multiplication. The name torus comes from the special case when $n = 2$ and $K = \mathbb{C}$ is the complex numbers. Here, we have $(\mathbb{C}^*)^2 \simeq (\mathbb{R}_+ \times S^1)^2$, where $S^1$ is the
unit circle. Thus, the topological torus $S^1 \times S^1$ is equal to the 2-dimensional algebraic torus after multiplication with the contractible factor $\mathbb{R}_+ \times \mathbb{R}_+$. We recall from Section 7.3 that subvarieties of the algebraic torus $(K^*)^n$ are the objects one starts from when developing tropical algebraic geometry.

**Definition 8.1** (Character of a torus). A character of the algebraic torus $T = (K^*)^n$ is an algebraic map $\chi : T \to K^*$ that is also a group morphism.

In Exercise 1 we shall see that characters are given by Laurent monomials $x^b = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$, where $b \in \mathbb{Z}^n$. The characters of $T$ are hence the elements of $\mathbb{Z}^n$. Under this correspondence, multiplication of characters becomes addition in the group $(\mathbb{Z}^n, +)$:

$$\chi_1(x) \cdot \chi_2(x) = (\chi_1 + \chi_2)(x).$$

A group isomorphic to $\mathbb{Z}^k$, for some $k$, is called a lattice. The lattice of characters of $T$ will be denoted by $\tilde{M}_T$ or simply $\tilde{M}$. As a subgroup of a free abelian group is free, any set of characters generates a sublattice $\tilde{M} \subset M$.

Let $a_1, \ldots, a_p$ be characters in $M_T \simeq \mathbb{Z}^n$. We write $A$ for the $n \times p$ matrix whose columns are the vectors $a_i$. The lattice $\tilde{M}$ generated by the characters $a_i$ is the image of $\mathbb{Z}^p$ under the right multiplication by the matrix $A$.

**Proposition 8.2.** The image of $T$ in $(K^*)^p$ under the map $f : x \to x^A = (x^{a_1}, \ldots, x^{a_p})$ is also a torus $\tilde{T}$. The character lattice of $\tilde{T}$ is equal to $\tilde{M}$.

**Proof.** The monomial map $f : T \to (K^*)^p$ induces the ring homomorphism

$$f^* : K[y_1^{\pm 1}, \ldots, y_p^{\pm 1}] \to K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \ y_i \mapsto x^{a_i}.$$ 

The spectrum of the image of the ring map $f^*$ is the image $\tilde{T}$ we are interested in. Note that this image is the group algebra $K[M]$. By definition, this is the vector space over $K$ with basis given by elements of $M$ and multiplication induced from addition in $M_T$. The lattice $\tilde{M}$ is isomorphic to the group $\mathbb{Z}^d$ for some integer $d \leq \min(n, p)$. We have $\tilde{T} = \text{Spec } K[M] = (K^*)^d$. □

The $d$-dimensional torus $\tilde{T}$ lives in $(K^*)^p$. We are interested in its Zariski closure in the affine space $K^p$. Such an affine variety is a toric variety.

**Definition 8.3.** An affine toric variety is the closed image of a monomial map $(K^*)^n \to K^p$, $x \mapsto (x^{a_1}, \ldots, x^{a_p})$, where $a_i \in \mathbb{Z}^n$ and $K^* = K \setminus \{0\}$.

We specify a toric variety by an integer matrix $A \in \mathbb{Z}^{n \times p}$. The $p$ columns of $A$ represent characters of the torus $T = (K^*)^n$. Toric geometry relates the combinatorics of these lattice points with the geometry of the toric variety.

**Example 8.4.** (1) Any affine space is a toric variety. The corresponding matrix $A$ is the identity matrix.
(2) The cuspidal cubic curve \( x^3 - y^2 \) is a toric variety. It is the image of the map \( z \mapsto (z^2, z^3) \) that is given by the matrix \( A = (2 \ 3) \).

**Proposition 8.5.** The dimension of the affine toric variety in Definition 8.3 is equal to the rank of the lattice \( \tilde{M} \) that is spanned by \( a_1, \ldots, a_p \) in \( \mathbb{Z}^n \).

**Proof.** We saw in the proof of Proposition 8.2 that the torus \( \tilde{T} \) has dimension \( d = \text{rank}(\tilde{M}) \). The toric variety is the Zariski closure of \( \tilde{T} \). Its has dimension \( d \), since passing to the Zariski closure preserves dimension. \( \square \)

We defined toric varieties as closures in \( K^p \) of subtori of the torus \( (K^*)^p \). In the notation of Proposition 8.2, the toric variety equals \( \text{Spec} \ K[S] \), where \( S \) is the monoid in \( M_T \) generated by the distinguished characters, i.e. the smallest set containing 0, the chosen characters and closed under addition.

**Example 8.6.**

1. The cuspidal curve defined by the equation \( x^3 - y^2 \) equals \( \text{Spec} \ K[z^2, z^3] \). The underlying monoid equals \( \{0, 2, 3, 4, \ldots \} \).
2. The affine line is the closure of the image of the map

\[ K^* \ni x \to x \in K. \]

Here the character lattice is \( M = \mathbb{Z} \), the distinguished character corresponds to \( 1 \in M \) and the monoid equals \( \{0, 1, 2, \ldots \} \).

![Figure 1. The cuspidal cubic curve](image)

There is a fundamental difference between the cuspidal curve and affine line. The monoid for the cuspidal curve has a 'hole', namely the character 1.

**Definition 8.7.** A submonoid \( S \) in a lattice \( M \) is called saturated if and only if for any \( x \in M \) and \( k \in \mathbb{Z}_+ \) the following implication holds:

\[ kx \in S \Rightarrow x \in S. \]

An affine toric variety \( X = \text{Spec} \ K[S] \) for which \( S \) is saturated (in the lattice \( \tilde{M} \) it generates) is called normal. For the algebraic definition of normality see [4] Chapter 5. Nonnormal varieties are always singular. For curves,
the two notions coincide. Example 8.6 shows one normal (i.e. smooth) curve and one nonnormal (i.e. singular) curve. The latter is seen in Figure 1.

We next discuss the prime ideal of the toric variety \( X \). This is computed from the characters that define \( X \). In general, given a variety defined as a Zariski closure of the image of a map, finding the defining equations is a hard problem, known as implicitization. We discussed this in Chapter 4. The implicitization problem greatly simplifies when the variety is toric. The prime ideal \( I_X \) of the toric variety \( X \) lives in the polynomial ring \( K[y_1, \ldots, y_p] \).

**Lemma 8.8.** Let \( X \) be the toric variety defined by \( A = (a_1, \ldots, a_p) \). Then:

1. any relation \( \sum_i b_i a_i = \sum_j c_j a_j \), with positive integral coefficients \( b_i, c_j \in \mathbb{Z}^+ \) provides a binomial \( \prod y_i^{b_i} - \prod y_j^{c_j} \) in the toric ideal \( I_X \);
2. every binomial in the ideal \( I_X \) is of the form described in (1);
3. the toric ideal \( I_X \) is generated by these binomials.

Recall that a binomial is a polynomial with only two terms. Statement (2) is understood up to scaling: we can multiply the binomial by a constant.

**Proof:** Properties (1) and (2) follow from the fact that a polynomial vanishes on the toric variety \( X \) if and only if we obtain zero after substituting \( y_i \) by \( x^{a_i} \). However, such a substitution turns monomials (in variables \( y \)) to monomials (in variables \( x \)). The fact that the monomials in \( x \) cancel is precisely encoded by the integral relations in point (1). Property (3) follows similarly, by induction on number of terms of a polynomial in the ideal of \( X \). The reader is asked to provide a proof in Exercise 2. For an argument using a monomial ordering see \[57\] Lemma 4.1.

**Example 8.9.** Fix \( n = 3, p = 7 \). To specify a toric variety, we must choose characters \( a_1, \ldots, a_7 \in \mathbb{Z}^3 \). Let us take the column vectors of the matrix

\[
A = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{pmatrix}.
\]

The associated toric variety \( X \) is a threefold in \( K^7 \). The toric ideal \( I_X \) equals

\[
\langle y_1y_3 - y_2y_7, y_1y_4 - y_2^2, y_1y_5 - y_6y_7, y_2y_4 - y_3y_7, y_2y_5 - y_7^2, y_2y_6 - y_1y_7, y_3y_5 - y_4y_7, y_3y_6 - y_7^2, y_4y_6 - y_3y_7 \rangle.
\]

Each of these nine binomials vanishes under the substitution \( y_i \mapsto x^{a_i} \). Using the methods in Section 4.2 we can check that \( I_X \) is the desired prime ideal. The toric variety \( X \) has dimension 3 and it lives in the affine space \( K^7 \).

The ideal \( I_X \) is homogeneous. Each of the nine binomials in (8.1) is homogeneous. This comes from the fact that the matrix \( A \) has column sums
(3, 3, 3, 3, 3, 3, 3). Geometrically speaking, the threefold $X$ is a cone in $K^7$. We can therefore also regard $X$ as a surface in the projective space $\mathbb{P}^6$. That surface is nonsingular and it has degree six. This passage from appropriate matrices $A$ to projective toric varieties will be our theme in Section 8.2.

**Theorem 8.10.** The toric ideals $I_X$ are precisely the prime ideals generated by binomials $y^b - y^c$. Every such ideal defines a toric variety $X$ as above.

**Proof.** Let $I$ be a prime ideal generated by a set of binomials $y^{b_i} - y^{c_i}$ in $p$ variables $y_1, \ldots, y_p$. By Hilbert’s Basis Theorem, there is a finite subset of minimal generators. For each such generator, the nonnegative integer vectors $b_i$ and $c_i$ have disjoint support, since $I$ is prime. We write the difference vectors $b_i - c_i$ as the columns of a matrix $B$ that has $p$ rows.

Let $A$ be any integer matrix of format $n \times p$ whose rows span the kernel of $B$ under left multiplication. Here, the kernel is understood as a $\mathbb{Z}$-module (a.k.a. abelian group), so it can be computed using integer linear algebra (e.g. the Hermite normal form algorithm). We claim that the columns of $B$ span the kernel $A$ under right multiplication. This is clear over $\mathbb{Q}$, but it also holds over $\mathbb{Z}$ by our hypothesis that $I$ is a prime ideal. Otherwise, there would exist a vector $b - c$ that is not in the column space of $B$ but some integer multiple $kb - kc$ is in that column span. We pick the smallest possible $k$. The following binomial is in the binomial prime ideal $I$:

$$y^{kb} - y^{kc} = (y^b - y^c) \left( y^{(k-1)b} + y^{(k-2)b}y^c + \cdots + y^{(k-1)c} \right)$$

However, neither of the two factors is in $I$. In characteristic zero this is straightforward, as the second one has coefficients not summing up to zero. In general, one can observe that if the second factor is in $I$, then we must have $y^{(k-1)b} - y^{(k-1-i)b}y^c \in I$ for some positive $i$, which contradicts the choice of $k$. We obtain a contradiction to the hypothesis that $I$ is prime.

We now take $X$ to be the toric variety $X$ in $K^p$ defined by the matrix $A$. The argument above shows that $I = I_X$, which gives the assertion in the theorem. For further details on this proof see [13, Proposition 1.1.11]. □

**Definition 8.11.** A convex polyhedral cone $C$ in $\mathbb{R}^n$ is a subset of elements of the form $\lambda_1 v_1 + \cdots + \lambda_k v_k$ where $v_1, \ldots, v_k \in \mathbb{R}^n$ are fixed and $\lambda_1, \ldots, \lambda_k$ range over $\mathbb{R}_{\geq 0}$. We call $C$ rational if the vectors $v_i$ can be chosen in $\mathbb{Q}^n$. In what follows we refer to rational convex polyhedral cones simply as cones.

**Definition 8.12.** A face $F$ of a cone $C \subset \mathbb{R}^n$ is a subset of the form

$$(8.2) \quad F = \{ c \in C : \ell(c) = 0 \},$$

where $\ell$ is a linear form that is nonnegative on $C$, i.e. $\ell(c) \geq 0$ for all $c \in C$. The dimension of a face of a cone is the dimension of the smallest linear space that contains it. If $\dim C = n = \dim F + 1$, then $\ell$ is uniquely determined,
up to scalar. In this case, $F$ is called a facet and the hyperplane $\ell^\perp$ is the supporting hyperplane of $C$. We note that $\ell = 0$ gives $F = C$. Furthermore, any face of a cone is also a cone, and the relation “is a face of” is transitive.

**Example 8.13.** The orthant $C = \mathbb{R}^n_{\geq 0}$ is a cone. It has $2^n$ faces, ranging from the apex $\{0\}$ to the full cone $C$. There are $\binom{n}{i}$ faces of dimension $i$. Each of the $n$ facets $F_i$ arises by setting one coordinate to zero, so $F_i \simeq \mathbb{R}^{n-1}_{\geq 0}$.

By Proposition 8.2, the toric variety $X$ is the closure in $K^p$ of the torus $\tilde{T} \subset (K^*)^p$. The group $\tilde{T}$ acts both on itself and on $K^p$, and it hence also acts on its closure $X$. The torus orbits on $X$ are the orbits of that action by $\tilde{T}$. We next provide a combinatorial and geometric description of the torus orbits. We assume that $X$ is defined by an integer $n \times p$-matrix $A$ as above. We write $C \subset \mathbb{R}^n$ for the cone that is generated by the $p$ columns $a_i$ of $A$.

**Theorem 8.14.** The torus orbits in $X$ are in bijection with the faces of the cone $C$. The orbit corresponding to a face $F$ is $\{y \in X : y_i \neq 0 \iff a_i \in F\}$. The closure of this orbit is the toric variety $\text{Spec} K[F \cap A]$ whose parametrization is $(x^n : a_i \in F)$. The dimension of this orbit equals $\dim(F)$. Moreover, the inclusion of orbit closures in $X$ corresponds to inclusion of faces of $C$.

**Sketch of proof.** For normal toric varieties, this appears in [13] Section 3.2. However, normality is not needed. From the definition of $F$ being a face, one checks directly that $\{y \in X : y_i \neq 0 \iff a_i \in F\}$ is a torus orbit. Indeed, using the fact that $K$ is algebraically closed, one constructs a point $t$ in the torus that $t \cdot y = p_F$, where $(p_F)_i = 1$ if $a_i \in F$ and zero otherwise. The binomials in $I_X$ ensure that each point of $X$ lies in one of these orbits. □

**Example 8.15.** Let $X$ be the toric threefold in $K^7$ given in Example 8.9. The cone $C$ is spanned in $\mathbb{R}^3$ by the columns of the $3 \times 7$ matrix $A$. It is the cone over a hexagon, so it has $14 = 1 + 6 + 6 + 1$ faces. The variety $X$ is the disjoint union of 14 torus orbits, as follows. The face $F = \{0\}$ corresponds to the origin in $K^7$. The 1-dimensional face $F = \mathbb{R}_{\geq 0}\{a_1\}$ corresponds to the curve $\{(t,0,0,0,0,0,0) \in X : t \in K^*\}$. The 2-dimensional face $F = \mathbb{R}_{\geq 0}\{a_1,a_2\}$ corresponds to the surface $\{(t,u,0,0,0,0,0) \in X : t,u \in K^*\}$. Finally, $F = C$ corresponds to the 3-dimensional torus $\tilde{T} = X \cap (K^*)^7$.

In conclusion, the geometry of $X$ is read off from the cone $C$ representing it.

### 8.2. Varieties from Polytopes

Projective toric varieties are obtained from affine toric varieties that are cones. They can be defined as follows. Let $A = (a_1, a_2, \ldots, a_p)$ be an integer $n \times p$-matrix of rank $n$ that has the vector $(1,1,\ldots,1)$ in its row span. Let $I_A \subset K[y_1,y_2,\ldots,y_p]$ be the prime ideal of polynomial relations among the
Laurent monomials $x^{a_1}, x^{a_2}, \ldots, x^{a_p}$. This is a toric ideal. According to Lemma 8.8, the ideal $I_A$ is generated by the binomials $y^b - y^c$, where $b - c$ is in the kernel of $A$. Such binomials are homogeneous because $(1, 1, \ldots, 1)$ is in the row span of $A$. Hence, $I_A$ defines a projective variety, denoted $X_A$.

**Definition 8.16.** A projective toric variety is any projective variety in $\mathbb{P}^{p-1}$ of the form $X_A = \mathcal{V}(I_A)$, where $I_A$ is a homogeneous toric ideal as above.

**Definition 8.17.** A polytope in $\mathbb{R}^n$ is the convex hull of a finite set of points. A polytope is a lattice polytope if it is the convex hull of points in $\mathbb{Z}^n$. Faces and facets of polytopes are defined as in (8.2) but now $\ell$ is affine-linear.

We write $P = \text{conv}(A)$ for the convex hull of the column vectors $a_i$ in $\mathbb{R}^n$. By construction, $P$ is a polytope of dimension $n - 1$ with $\leq p$ vertices. For instance, in Example 8.9, the polytope $P$ is a regular hexagon. Algebraic geometers know the surface $X_A$ as the blow-up of $\mathbb{P}^2$ at three points.

Given a projective toric variety $X_A$, we associate to it the polytope $P = \text{conv}(A)$. Conversely, any lattice polytope can be coordinatized so that it spans the affine hyperplane $\{y_1 + y_2 + \cdots + y_n = k\}$ for some $k, n \in \mathbb{Z}_+$. We then take $A = P \cap \mathbb{Z}^n$, and we associate the projective toric variety $X_P := X_A$ with the polytope $P$. This variety lives in $\mathbb{P}^{p-1}$ where $p = |A|$. The class of varieties $X_A$ is strictly larger than the class of varieties $X_P$.

The reason is that $A$ can be a proper subset of $\text{conv}(A) \cap \mathbb{Z}^n$, However, the projective toric varieties of most interest to us are $X_P$ for some polytope $P$.

**Example 8.18.** The Veronese variety and the Segre variety from classical algebraic geometry are two prominent examples of projective toric varieties.

We write $\Delta_{n-1}$ for the standard $(n - 1)$-simplex, whose vertices are the unit vectors $e_1, \ldots, e_n$. Fix $k \in \mathbb{Z}_{\geq 0}$ and $P = k\Delta_{n-1}$. Then $A = P \cap \mathbb{Z}^n$ consists of the nonnegative integer vectors with coordinate sum $k$. Hence $p = |A| = \binom{n + k - 1}{k}$. The toric variety $X_P$ is the $k$-th Veronese embedding of $\mathbb{P}^{n-1}$. It has dimension $n - 1$ and degree $k^{n-1}$ in $\mathbb{P}^{p-1}$. Its toric ideal $I_A$ consists of the polynomial relations among all monomials of degree $k$ in $n$ variables. For instance, if $n = k = 3$ then there are ten such monomials:

$$A = \begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \end{pmatrix}.$$  

The ideal defining this Veronese surface in $\mathbb{P}^9$ is generated by 27 quadrics:

$$I_A = \langle y_1 y_4 - y_2^2, y_1 y_5 - y_2 y_3, y_1 y_6 - y_3^2, y_1 y_7 - y_2 y_4, \ldots, y_7 y_{10} - y_8 y_9, y_8 y_{10} - y_9^2 \rangle.$$  

Note that these binomials encode convexity relations among the points in $A$.

Next, fix $n_1, n_2 \in \mathbb{Z}_{\geq 0}$ and set $n = n_1 + n_2$ and $p = n_1 n_2$. Let $P = \Delta_{n_1} \times \Delta_{n_2}$ and write $e_i$ and $e'_j$ for the unit vectors in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively.
Then $A = P \cap \mathbb{Z}^n = \{ e_i + e_j' : 1 \leq i \leq n_1 \text{ and } 1 \leq j \leq n_2 \}$. We here deviate slightly from our hypothesis, as the rank of $A$ is $n-1$, not $n$. The variety $X_P$ is the Segre embedding of the product $\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1}$ into $\mathbb{P}^{p-1}$. The points on $X_P$ are $n_1 \times n_2$ matrices of rank one, up to scaling. The associated toric ideal $I_A$ is generated by the $2 \times 2$ minors of an $n_1 \times n_2$ matrix of unknowns. Each $2 \times 2$ minor corresponds to the square formed by four of the lattice points in $P$. Can you see this for $n_1 = n_2 = 3$, the case in Example 7.18?

In the algebraic geometry literature, it is often assumed that toric varieties are normal. This is motivated by the fact that normal toric varieties admit a nice intrinsic characterization, in terms of fans. In our setting, this hypothesis is generally not needed. Still, we present the relevant definition.

**Definition 8.19.** A lattice polytope $P$ in $\mathbb{R}^n$ is normal if, for any $k \in \mathbb{Z}$ and any $u \in kP \cap \mathbb{Z}^n$, there exist $u_1, \ldots, u_k \in P \cap \mathbb{Z}^n$ such that $u = \sum_{i=1}^{k} u_i$. In this case, $X_P$ is projectively normal in $\mathbb{P}^{p-1}$, i.e. its affine cone is normal.

**Remark 8.20.** In the current literature, the property in Definition 8.19 is called the integer decomposition property (IDP). It is invariant under lattice translation of $P$. If $0 \in P$ and $P \cap \mathbb{Z}^n$ generates $\mathbb{Z}^n$ then all reasonable definitions of normality and IDP coincide. One of these equivalent definitions says that the semigroup generated by $\{1\} \times (P \cap \mathbb{Z}^n) \subset \mathbb{Z}^{n+1}$ is saturated.

To appreciate the subtleties we are alluding to, consider the tetrahedron $P = \text{conv}((0,0,0), (1,1,0), (1,0,1), (0,1,1))$. Then $(1,1,1) \in 2P$, but it is not a lattice sum of two points in $P$. However, note that the lattice points of $P$ do not generate $\mathbb{Z}^3$, but only the sublattice consisting of points with even sum of coordinates. Inside that sublattice, $P$ is just the standard simplex and satisfies the condition in Definition 8.19.

The simplices and products of simplices in Example 8.18 are normal. Hence the Veronese variety and the Segre variety are projectively normal. Exercise 9 gives an example of a 3-dimensional lattice polytope that is not normal. All lattice polytopes of dimension 1 and 2 are normal. If $P$ is normal then the polyhedral fan that characterizes its toric variety $X_P$ intrinsically is the normal fan of $P$. We refer to [13, Section 3.1] for the basic theory.

We now return to the setting where $I_A$ is any homogeneous toric ideal, $X_A \subset \mathbb{P}^{p-1}$ its toric variety, and $P = \text{conv}(A)$ not necessarily normal. Let $T$ denote the subset of $X_A$ consisting of all points with non-zero coordinates. This is a torus of dimension $n-1$. That torus acts on $X_A$ with finitely many orbits. Theorem 8.14 extends essentially verbatim to the projective case.

**Corollary 8.21.** The torus orbits in $X_A$ are in bijection with the faces of the polytope $P$. The orbit corresponding to a face $F$ is $\{ y \in X_A : y_i \neq$
8.2. Varieties from Polytopes

0 \leftrightarrow \mathbf{a}_i \in F \}. The closure of this orbit is the projective toric variety with parametrization \((x^{\mathbf{a}_i} : \mathbf{a}_i \in F)\). The dimension of this orbit equals \(\dim(F)\).

The inclusion of orbit closures in \(X_A\) corresponds to inclusion of faces of \(P\).

**Proof.** We apply Theorem 8.14 to the affine toric variety defined by \(I_A\) in \(K^p\). This is the affine cone over \(X_A \subset \mathbb{P}^{p-1}\). Its orbits correspond to the faces of the cone \(C\) over the polytope \(P\). Note that \(\dim(C) = n = \dim(P) + 1\). Each \(i\)-dimensional face \(F\) of \(P\) corresponds to an \((i+1)\)-dimensional face of \(C\), namely the cone over \(F\). Likewise, each \(i\)-dimensional orbit in \(X_A\) corresponds to an \((i+1)\)-dimensional orbit of the affine cone over \(X_A\). These bijections, for \(i = 0, 1, \ldots, n-1\), establish the desired bijection for \(P\) and \(X_A\). The only face of \(C\) that is missing in \(P\) is the origin \(\{0\}\).

Likewise, the cone point of the affine cone over \(X_A\) disappears in \(X_A\). \(\square\)

**Example 8.22.** Consider the Segre threefold \(X_A = \mathbb{P}^1 \times \mathbb{P}^2\) in \(\mathbb{P}^5\), given by \(n_1 = 2\) and \(n_2 = 3\) in Example 8.18. The toric ideal \(I_A\) is generated by the \(2 \times 2\)-minors of a \(2 \times 3\)-matrix of unknowns, and the polytope \(P = \Delta_1 \times \Delta_2\) is a triangular prism. This 3-dimensional polytope has 21 = 6 + 9 + 5 + 1 faces, one for each of the torus orbits on \(X_A\). For instance, the five 2-dimensional orbits are given by setting one row or column of the \(2 \times 3\)-matrix to zero, and the 0-dimensional orbits in \(X_A\) are the matrices with one non-zero entry.

Corollary 8.21 establishes a combinatorial link between projective toric varieties and their lattice polytopes. In what follows we tighten this to a geometric link. We now fix \(K = \mathbb{C}\), the complex numbers. We seek to argue that the geometry of the polytope coincides with the geometry of the toric variety. The key to this identification is the moment map from \(X_A\) onto \(P\).

We work in the complex projective space \(\mathbb{P}^{p-1}_C\) with its homogeneous coordinates \(y = (y_1 : y_2 : \cdots : y_p)\). The following map onto \(P = \text{conv}(A)\) is defined via the usual Euclidean norm \(|\cdot|\) on the complex plane \(\mathbb{C} \simeq \mathbb{R}^2\):

\[
\begin{align*}
\mathbb{P}^{p-1}_C & \to \mathbb{R}^n, & y \mapsto \frac{1}{\sum_{i=1}^p |y_i|} \sum_{i=1}^p |y_i| \cdot \mathbf{a}_i.
\end{align*}
\]

This map is well-defined because the image is invariant under scaling the vector \(y\), and \(\sum_{i=1}^p |y_i|\) is always positive. Its image lies in the polytope \(P\), since we are taking convex combinations of the points \(\mathbf{a}_i\) in \(\mathbb{R}^n\).

**Definition 8.23.** The algebraic moment map \(\mu_A : X_A \to \mathbb{R}^n\) is defined as the restriction of (8.3) from the ambient space \(\mathbb{P}^{p-1}_C\) to the toric variety \(X_A\). Let \(X_{A,\mathbb{R}}\) be the subset of real points in \(X_A\). Its subset of nonnegative (resp. positive) points is denoted by \(X_{A,\geq 0}\) (resp. \(X_{A,>0}\)). These are semialgebraic sets in the real projective space \(\mathbb{P}^{p-1}_\mathbb{R}\). To be precise, the positive toric variety \(X_{A,>0}\) consists of all positive solutions, up to scale, of the binomial equations in \(I_A\), and similarly for the nonnegative toric variety \(X_{A,\geq 0}\).
The complex projective toric variety $X_A$ maps naturally onto its non-negative part $X_{A,\geq 0}$ under the coordinatewise absolute value map

$$\begin{pmatrix} y_1 : y_2 : \cdots : y_p \end{pmatrix} \mapsto \begin{pmatrix} |y_1| : |y_2| : \cdots : |y_p| \end{pmatrix}.$$  

The fibers of this map are real tori. Specifically, the fiber over each point in $X_{A,>0}$ is homeomorphic to the torus $(\mathbb{S}^1)^{n-1}$. This torus is a subgroup of the complex torus $T \simeq (\mathbb{C}^*)^{n-1}$. We may think of (8.4) as the quotient map $X_A \twoheadrightarrow X_A / (\mathbb{S}^1)^{n-1} \simeq X_{A,\geq 0}$.

See [13, Proposition 12.2.3] for a formal statement for normal toric varieties. The algebraic moment map $\mu_A$ factors through the quotient map (8.4).

**Theorem 8.24.** The restriction of the algebraic moment map $\mu_A$ to the nonnegative toric variety $X_{A,\geq 0}$ is a homeomorphism onto the polytope $P$.

**Proof.** This is found in many sources. A good place to start is Sottile's article [54, Theorem 8.4] on toric methods in geometric modeling. □

**Corollary 8.25.** If the linear system of equations $A\mathbf{y} = \mathbf{b}$ has a nonnegative solution $\mathbf{y} \in \mathbb{R}^p_{\geq 0}$ then it has unique solution $\hat{\mathbf{y}}$ in the toric variety $X_A$.

**Proof.** Here we identify $X_A$ with the affine cone over the projective toric variety defined by the $n \times p$ matrix $A$. The algebraic moment map $\mu_A$ lifts uniquely, by scaling, to a homeomorphism from this affine cone to the cone $C$ over the polytope $P$. The system $A\mathbf{y} = \mathbf{b}$ has a nonnegative solution if and only if $\mathbf{b}$ lies in $C$. In this case, the point $\mathbf{b}$ has a unique preimage $\hat{\mathbf{y}} = \mu_A^{-1}(\mathbf{b}) \in X_{A,\geq 0}$ under the moment map, which is the desired solution. □

**Example 8.26.** Let $A$ be the $(n_1+n_2) \times (n_1 n_2)$ matrix for the polytope $P = \Delta_{n_1-1} \times \Delta_{n_2-1}$ as in Examples 8.18 and 8.22. This matrix $A$ represents the linear map that takes an $n_1 \times n_2$ matrix $\mathbf{y}$ to its vector $\mathbf{b}$ of row and column sums. The polytopes $\{ \mathbf{y} \in \mathbb{R}^p_{\geq 0} : A\mathbf{y} = \mathbf{b} \}$ are known as transportation polytopes. The points in the Segre variety $X_A$ are the $n_1 \times n_2$ matrices $\mathbf{y}$ of rank one. In this case, Corollary 8.25 has the following interpretation: Every transportation polytope contains a unique rank one matrix $\hat{\mathbf{y}}$.

Example 8.26 has important consequences in statistics. We saw this already for $n_1 = n_2 = m$ in Example 2.5. Suppose the nonnegative matrix $\mathbf{y}$ has entries that sum to 1. Then $\mathbf{y}$ is a joint distribution for two random variables that have $n_1$ and $n_2$ states respectively. The nonnegative variety $X_{A,\geq 0}$ is the independence model for these two random variables. The map $A$ computes the sufficient statistics $\mathbf{b} = A\mathbf{y}$, i.e. the column vector of row sums and the row vector of column sums. The product of these vectors is the rank one matrix $\hat{\mathbf{y}} = \mu_A^{-1}(\mathbf{b})$. This is the maximum likelihood estimate for the empirical distribution $\mathbf{y}$ with respect to the independence model.
Example 8.27 ($n_1 = n_2 = 2$). The independence model for two binary random variables is a quadratic surface in $\mathbb{P}^3_{\geq 0} = \Delta_3$. This is the nonnegative part $X_{A,\geq 0}$ of the Segre quadric $X_A = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. That surface meets the boundary of the ambient tetrahedron in four edges that form a 4-cycle. The moment map $\mu_A$ projects the tetrahedron onto a square. The 4-cycle is mapped onto the boundary of the square. The surface $X_{A,>0}$ is mapped bijectively onto the interior of the square. Figure 2 illustrates this scenario.

![Figure 2. The moment map identifies the Segre quadric with a square.](image)

Example 8.27 is an instance of a general construction in algebraic statistics. Projective toric varieties $X_A$ correspond to a class of statistical models, referred to as toric models in [46] and as log-linear models in [55]. The inverse moment map $\mu_A^{-1}$ is the maximum likelihood estimator for the model $X_{A,\geq 0}$. Given a point $b$ in the model polytope $P = \text{conv}(A)$, the estimate $\mu_A^{-1}(b)$ is the Birch point in $X_{A,\geq 0}$. This distribution best explains the data with sufficient statistic $b$. See [46, Proposition 1.9] and [55, Corollary 7.3.9].

Toric varieties are ubiquitous in applications. One explanation for this is the following observation which connects this chapter to the previous one.

**Proposition 8.28.** Let $X$ be an irreducible variety over a field $K$ as in Section 7.3. If $X$ is toric then its tropicalization $\text{Trop}(X)$ is a linear space.

**Proof.** If $X = \mathcal{V}(I_A)$ then every point in $X$ has the form $(x^{a_1}, \ldots, x^{a_p})$. The images of these points under coordinatewise valuation are $u.A$ where $u = \text{val}(x)$ runs over $\mathbb{Q}^p$. This implies that $\text{Trop}(X)$ is the row space of $A$. □

In fact, the converse to this proposition also holds, with a slightly more inclusive definition of toric variety. Informally speaking, toric varieties are precisely those varieties that become linear spaces under taking logarithms.
8. Toric Varieties

8.3. The World is Toric

The occurrence of toric structures in an application can be either obvious or hidden. A typical example for the former is log-linear models in statistics. These are obviously toric, as seen around Example 8.27. In this section we discuss some scenarios where the toric structure is hidden, and it needs to be unearthed, often by a non-trivial choice of coordinates. Our style in this section is extremely informal. We briefly visit four areas where toric varieties arise. Under each header we focus on one concrete instance of a toric variety $X_A \subset \mathbb{P}^{p-1}$. The broader context is discussed alongside that example.

Chemical Reactions. Three chemical species $\sigma_1, \sigma_2, \sigma_3$ can form four chemical complexes $3\sigma_1, 3\sigma_2, 3\sigma_3, \sigma_1 + 2\sigma_2 + \sigma_3$. Each complex can react so as to transform into any other complex. We introduce unknowns $c_1, c_2, c_3$ for the species concentrations and $K_1, K_2, K_3, K_4$ to encode rate constants.

This chemical reaction system is modeled by the toric balancing ideal

$$I_A = \langle 2 \times 2 \text{ minors of } \begin{pmatrix} K_1 & K_2 & K_3 & K_4 \\ c_1^2 & c_2^3 & c_3^3 & c_1 c_2 c_3 \end{pmatrix} \rangle : (c_1 c_2 c_3)\mathbb{P}^\infty.$$

This toric ideal has ten minimal generators, namely the six minors plus $c_1^2 K_2 K_3 - c_2 c_3 K_4^2, c_2^2 K_1 K_3 - c_1 c_3 K_4^2, c_3^2 K_1 K_2 - c_1 c_2 K_4^2, K_1 K_2 K_3 - K_4^3$.

The variety $X_A = \mathcal{V}(I_A)$ is a threefold of degree 13 in $\mathbb{P}^6$. The underlying $4 \times 7$ matrix $A$ is found with the integer linear algebra method in the proof of Theorem 8.10. The polytope $P = \text{conv}(A)$ is a triangular prism. One triangle face is $\Delta_2$ with vertices labeled by $c_1, c_2, c_3$. The other triangle face is $3\Delta_2$ with vertices $K_1, K_2, K_3$ and centroid $K_4$. The underlined cubic generates the moduli ideal, which identifies the toric dynamical systems.

The mathematical theory of chemical reaction network systems with mass action kinetics is an important domain of application for nonlinear algebra. For an introduction we refer to the text book by Dickenstein and Feliu [18]. The term “toric dynamical systems” was coined in the article [14]. In the chemical literature, these are known as complex balancing mass action systems. The toric ideals above were introduced in [14] Section 2].

Gaussian MLE. Let $n = 5, p = 10$, and consider the integer matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

The columns of $A$ are labeled $y_{101}, y_{102}, \ldots, y_{34}$. These are our coordinates for $\mathbb{P}^9$. The polytope $P = \text{conv}(A)$ is the second hypersimplex of dimension 4.
It has f-vector (10, 30, 30, 10) but it is not self-dual. Its toric ideal $I_A$ has 10 quadratic generators $y_{ij}y_{kl} - y_{ik}y_{jl}$, and $X_A$ is a fourfold of degree 11 in $\mathbb{P}^9$.

This toric variety of the second hypersimplex of dimension $m$ arises when studying Gaussian distributions on $\mathbb{R}^m$ with structured covariance matrix $\Sigma$. Consider the model given by prescribing all off-diagonal entries to be equal. Thus, for $m = 4$, we are interested in the linear space of symmetric matrices

$$
\Sigma = \begin{bmatrix}
\sigma_1 & \sigma_0 & \sigma_0 & \sigma_0 \\
\sigma_0 & \sigma_2 & \sigma_0 & \sigma_0 \\
\sigma_0 & \sigma_0 & \sigma_3 & \sigma_0 \\
\sigma_0 & \sigma_0 & \sigma_0 & \sigma_4
\end{bmatrix}.
$$

Given a sample covariance matrix $S$, one seeks to maximize the log-likelihood

$$
\ell(\Sigma) = -\log \det(\Sigma) - \text{trace}(S\Sigma^{-1}) = \log \det(K) - \text{trace}(SK).
$$

This function is convex in the concentration matrix $K = \Sigma^{-1}$. For that reason, we study the variety of matrices $K$ whose inverse has structure [8.6]. This is a non-linear projective variety of dimension $m$ in $\mathbb{P}^{\binom{m}{2}}$. Defining polynomials are obtained by equating off-diagonal entries in the adjoint of $K$. These are complicated expressions with many terms of degree $m - 1$.

We find that this is a toric variety, after the linear change of coordinates

$$
K = \begin{bmatrix}
y_{01} + y_{12} + y_{13} + y_{14} & -y_{12} & -y_{13} & -y_{14} \\
y_{12} & y_{02} + y_{12} + y_{23} + y_{24} & -y_{23} & -y_{24} \\
y_{13} & y_{23} & y_{03} + y_{13} + y_{23} + y_{34} & -y_{34} \\
y_{14} & y_{24} & y_{34} & y_{04} + y_{14} + y_{24} + y_{34}
\end{bmatrix}.
$$

For any $m$, this is the reduced Laplacian matrix of the complete graph on $m + 1$ nodes. It follows from [61, Theorem 1.2] that the variety of matrices $K$ whose inverse is constant outside the diagonal equals the toric variety of the second hypersimplex. Using the coordinates $y_{ij}$, the inverse of the matrix $\Sigma$ in [8.6] satisfies the ten quadratic binomials in $I_A$. The article [61] establishes this toric structure for a larger class of Gaussian models, one for each rooted tree, thus contributing to likelihood inference for such models.

**Phylogenetics.** Group-based models in phylogenetics are varieties that become toric after a linear change of coordinates. The nonlinear algebra of this transformation was pioneered in [59]. For the relevant background from molecular biology we refer to [46, Chapter 4]. The following case study is taken from [59, Example 3]. The *Jukes-Cantor model* for the claw tree $K_{1,3}$ is a model for three binary random variables. Its eight joint probabilities are

$$
p_{000} = \pi_0 \alpha_0 \beta_0 \gamma_0 + \pi_1 \alpha_1 \beta_1 \gamma_1, \quad p_{001} = \pi_0 \alpha_0 \beta_0 \gamma_1 + \pi_1 \alpha_1 \beta_1 \gamma_0, \\
p_{010} = \pi_0 \alpha_0 \beta_1 \gamma_0 + \pi_1 \alpha_1 \beta_0 \gamma_1, \quad p_{011} = \pi_0 \alpha_0 \beta_1 \gamma_1 + \pi_1 \alpha_1 \beta_0 \gamma_0, \\
p_{100} = \pi_0 \alpha_1 \beta_0 \gamma_0 + \pi_1 \alpha_0 \beta_1 \gamma_1, \quad p_{101} = \pi_0 \alpha_1 \beta_0 \gamma_1 + \pi_1 \alpha_0 \beta_1 \gamma_0, \\
p_{110} = \pi_0 \alpha_1 \beta_1 \gamma_0 + \pi_1 \alpha_0 \beta_0 \gamma_1, \quad p_{111} = \pi_0 \alpha_1 \beta_1 \gamma_1 + \pi_1 \alpha_0 \beta_0 \gamma_0.
$$
Here $\pi_0$ and $\pi_1 = 1 - \pi_0$ give the root distribution. The other model parameters $\alpha_0 = 1 - \alpha_1$, $\beta_0 = 1 - \beta_1$ and $\gamma_0 = 1 - \gamma_1$ are the transition probabilities from the root to the three leaves. Since all parameters are nonnegative, the model is a 4-dimensional semialgebraic subset of the probability simplex $\Delta_7$. The Fourier transform gives a change of coordinates in the parameter space,

$$\pi_0 = \frac{1}{2}(r_0 + r_1), \quad \pi_1 = \frac{1}{2}(r_0 - r_1), \quad \alpha_0 = \frac{1}{2}(a_0 + a_1), \quad \alpha_1 = \frac{1}{2}(a_0 - a_1),$$

$$\beta_0 = \frac{1}{2}(b_0 + b_1), \quad \beta_1 = \frac{1}{2}(b_0 - b_1), \quad \gamma_0 = \frac{1}{2}(c_0 + c_1), \quad \gamma_1 = \frac{1}{2}(c_0 - c_1),$$

and it also gives a linear change of coordinates in the probability space:

$$P_{ijk} = \sum_{r=0}^{1} \sum_{s=0}^{1} \sum_{t=0}^{1} (-1)^{ir+js+kt} \cdot y_{rst}.$$ 

After these coordinate changes, the parametrization is now toric:

$$y_{000} = r_0 a_0 b_0 c_0, \quad y_{001} = r_1 a_0 b_0 c_1, \quad y_{010} = r_1 a_0 b_1 c_0, \quad y_{011} = r_0 a_0 b_1 c_1, \quad y_{100} = r_1 a_1 b_0 c_0, \quad y_{101} = r_0 a_1 b_0 c_1, \quad y_{110} = r_0 a_1 b_1 c_0, \quad y_{111} = r_1 a_1 b_1 c_1.$$ 

This corresponds to a matrix $A \in \{0,1\}^{8 \times 8}$ of rank 5. The toric ideal equals

$$I_A = \langle y_{001} y_{110} - y_{000} y_{111}, \quad y_{010} y_{101} - y_{000} y_{111}, \quad y_{100} y_{110} - y_{000} y_{111} \rangle.$$ 

Hence $X_A$ is a complete intersection of codimension 3 and degree 8 in $\mathbb{P}^7$. The study of such phylogenetics models is an active area of research.

**Paths and Signatures.** Let $n = 6$, $p = 19$ and consider the monomial map

$$y_{ijk} = a_i a_j a_k \quad \text{for} \quad 1 \leq i \leq j \leq k \leq 3,$$

$$z_{k;ij} = a_k b_{ij} \quad \text{for} \quad k = 1, 2, 3 \text{ and } 1 \leq i < j \leq 3.$$ 

This defines a toric variety $X_A$ of dimension 5 and degree 24 in $\mathbb{P}^{18}$. The matrix $A \in \{0,1\}^{6 \times 19}$ has rows indexed by $a_1, a_2, a_3, b_{12}, b_{13}, b_{23}$ and 19 = 10 + 9 columns indexed by $y_{111}, y_{112}, \ldots, y_{333}$ and $z_{1;12}, z_{1;13}, \ldots, z_{3;23}$. The toric ideal $I_A$ is generated by 81 binomial quadrics, namely the $2 \times 2$-minors of

$$\begin{pmatrix}
  y_{111} & y_{112} & y_{113} & y_{112} & y_{113} & y_{112} & y_{113} & z_{1;12} & z_{1;13} & z_{1;23} \\
  y_{112} & y_{112} & y_{113} & y_{112} & y_{113} & y_{112} & y_{113} & z_{2;12} & z_{2;13} & z_{2;23} \\
  y_{113} & y_{123} & y_{133} & y_{223} & y_{333} & z_{3;12} & z_{3;13} & z_{3;23}
\end{pmatrix}. \quad (8.7)$$

Let $\tilde{X}_A$ be the join of $X_A$ with $\mathbb{P}^7$. This is a 13-dimensional toric variety of degree 24 in $\mathbb{P}^{26}$. It is defined by the same ideal $I_A$ but now in 27 variables. We replace these by the entries of a $3 \times 3 \times 3$ tensor $\sigma = (\sigma_{ijk})$ as follows:

$$y_{ijk} = \sigma_{kij} + \sigma_{ikj} + \sigma_{ijk},$$

$$z_{k;ij} = \frac{1}{2}(\sigma_{kij} + \sigma_{ikj} + \sigma_{ijk}) - \frac{1}{2}(\sigma_{kji} + \sigma_{jki} + \sigma_{jik}). \quad (8.8)$$

The resulting variety $\mathcal{U}_{3,3}$ is the universal variety of third-order signature tensors of arbitrary paths in $\mathbb{R}^3$. Such tensors play an important role in stochastic analysis, especially in the Hairer-Lyons theory of rough paths.
A natural generalization of the universal variety is the \textit{rough Veronese variety} which was studied and shown to be toric by Colmenarejo et al. in [10]. This variety is a variant of the classical Veronese variety, but adapted to the study of rough paths. For an introduction to this theory we see [10, Section 1] and the references therein. Returning to the example above, in the notation of [10, Section 2], we have $\mathcal{U}_{3,3} = \mathcal{R}_{3,3,3} = \bar{X}_A \subset \mathbb{P}^{26}$ and $\mathcal{R}_{3,3,2} = X_A \subset \mathbb{P}^{18}$. The equations defining these varieties are obtained by substituting (8.8) into (8.7). This $3 \times 10$ matrix has rank $\leq 1$ for the signatures of all paths in $\mathbb{R}^a$.

\section*{Exercises}

(1) Prove that every character $\chi$ of the torus $T = (K^*)^n$ is given by a Laurent monomial $x = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$, for some integer vector $b$ in $\mathbb{Z}^n$.

(2) Show that every polynomial in the ideal $I_A$ of an affine toric variety is a $K$-linear combination of binomials. This gives item (3) in Lemma 8.8.

(3) Describe the ideal of the Segre variety $\mathbb{P}^{a_1-1} \times \cdots \times \mathbb{P}^{a_s-1}$ inside $\mathbb{P}^{a_1 \cdots a_s-1}$. What is the degree of this toric variety? Describe its lattice polytope $P$.

(4) There is a natural bijection between (convex, rational, polyhedral) cones in $\mathbb{R}^d$ and finitely generated saturated monoids in $\mathbb{Z}^d$. Prove this fact.

(5) Determine the toric ideal $I_A$ and the toric variety $X_A$ for the matrix

$$A = \begin{pmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 2 & 3 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}.$$ 

(6) Show that the 3-dimensional lattice polytope $P = \text{conv}(A)$, where $A$ is as in Exercise 5 is not normal. Draw this polytope and determine its $f$-vector.

(7) Prove that every 2-dimensional lattice polytope is normal.

(8) Prove the following fact which generalizes the previous exercise: for any $k$-dimensional lattice polytope $P$, the scaled polytope $(k-1)P$ is normal.

(9) Determine the number of lattice points in $kP$ where $P$ is the polytope in Exercises 5 and 6. Show that this number is a cubic polynomial in $k$.

(10) Prove the following theorem due to Mumford in the case of toric varieties. Let $X$ be a projective toric variety. For $r$ large enough the $r$-th Veronese reembedding $v_r(X)$ of $X$ is defined by quadratic equations.

(11) Compute an explicit Gröbner basis for the toric ideal $I_A$, where $A$ is the matrix in Example 8.5. Is your initial monomial ideal in($I_A$) radical?
(12) Compute the inverse of the matrix $\Sigma$ in 8.6 and verify that its entries satisfy the ten quadratic binomials given by the second hypersimplex.

(13) Let $A$ be the $3 \times 7$-matrix in Example 8.9. Can you give a formula for the inverse moment map $\mu_A^{-1}$? Is there an expression in radicals?

(14) (a) Compute the number of points of a projective toric variety $X_P$ over a finite field, in terms of the $f$-vector of the associated polytope $P$.
   
   (b) * Assuming that $X_P$ is smooth and $K = \mathbb{C}$, use the Weil conjectures (which are theorems, thanks to Grothendieck and Deligne), to give a formula for Betti numbers of $X_P$, again in terms of the $f$-vector.

(15) Give an example of a toric threefold $X_A$ in $\mathbb{P}^6$ that has degree 11. Draw the polytope $P = \text{conv}(A)$. Can you arrange for $X_A$ to be smooth?

(16) Verify Theorem 8.24 for the case when $P$ is the triangular prism. This corresponds to the Segre variety with $n_1 = 2, n_2 = 3$ in Example 8.18.

(17) Section 8.3 discusses four toric varieties that arise in applications. Pick two of them and study the corresponding polytopes. Find their f-vectors.
Tensors

“The name TensorFlow derives from the operations that neural networks perform on multidimensional data arrays, which are referred to as tensors.”

Wikipedia

Tensors are ubiquitous in many different branches of modern mathematics. They are higher dimensional analogs of matrices. Just as matrices are basic objects in linear algebra, tensors are fundamental for nonlinear algebra. One reason they appear so late in this book is that we already saw them in disguise: homogeneous polynomials are symmetric tensors. In this chapter we show that basic attributes of matrices, like eigenvectors and rank, can be defined also for tensors. However, their behavior is far more interesting now. We also discuss applications of tensors, with focus on a central open algorithmic problem: how fast can one multiply two matrices? As always, linear algebra is our door to nonlinear algebra. Further, the new nonlinear tools will be applied to revisit fundamental questions in linear algebra.

9.1. Eigenvectors

In this section we extend the familiar concepts of eigenvectors, rank and singular values from matrices to the setting of tensors. We start by reviewing some basics of linear algebra, beginning with the study of symmetric matrices. Recall that symmetric matrices uniquely represent quadratic forms.

For instance, consider the following quadratic form in three variables:

\[
Q = 2x^2 + 7y^2 + 23z^2 + 6xy + 10xz + 22yz. \tag{9.1}
\]
This quadratic form is represented by a symmetric $3 \times 3$-matrix as follows:

\begin{equation}
Q = \begin{pmatrix} x & y & z \\ 2 & 3 & 5 \\ 3 & 7 & 11 \\ 5 & 11 & 23 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{equation}

The gradient of the quadratic form $Q$ is the vector of its partial derivatives. Thus, the gradient is a vector of linear forms. It defines a linear map from $K^3$ to itself. Up to multiplication by 2, this is the linear map one usually associates with a square matrix. For the quadratic form in (9.1) we have

\[
\nabla Q = \begin{pmatrix} \frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial y} \\ \frac{\partial Q}{\partial z} \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \\ 5 & 11 & 23 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

In this section, the field $K$ is usually $\mathbb{R}$ or $\mathbb{C}$. We call $v \in K^n \setminus \{0\}$ an eigenvector of $Q$ if $v$ is mapped to a scalar multiple of $v$ by the gradient map:

\[
(\nabla Q)(v) = \lambda \cdot v \quad \text{for some } \lambda \in K.
\]

Just like in the earlier chapters, it is convenient to replace the $n$-dimensional affine space $K^n$ with the $(n - 1)$-dimensional projective space $\mathbb{P}^{n-1}$. Thus, two nonzero vectors are identified if they are parallel. From a nonzero quadratic form $Q$ we obtain a rational self-map of the projective space:

\begin{equation}
\nabla Q : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}.
\end{equation}

The dashed arrow means that this map may not be defined everywhere. Algebraic geometers call this a rational map. If $Q$ is rank-deficient then the linear map has a kernel. These are the points where the gradient $\nabla Q$ vanishes. They are the base points of the map (9.3). If $Q$ has full rank then $\nabla Q$ is a regular map $\mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$, i.e. it is defined on all of $\mathbb{P}^{n-1}$. We conclude our discussion with the following remark on the gradient map:

**Remark 9.1.** The eigenvectors of $Q$ are the fixed points ($\lambda \neq 0$) and base points ($\lambda = 0$) of the gradient map $\nabla Q$ in (9.3). These points $v$ live in $\mathbb{P}^{n-1}$.

Symmetric $n \times n$ matrices often appear in statistics. Consider $n$ real-valued random variables $X_1, \ldots, X_n$. Their covariance matrix is the matrix $\Sigma$ whose $(i, j)$ entry is $\text{cov}[X_i, X_j] = E[(X_i - E[X_i])(X_j - E[X_j])]$. We note that $\Sigma$ is positive semidefinite, i.e. all its eigenvalues are nonnegative.

An $n \times n$-matrix usually has $n$ linearly independent eigenvectors, provided the underlying field $K$ is algebraically closed. If the matrix is real and symmetric, then its eigenvectors have real coordinates and are orthogonal. For a rectangular matrix, one considers pairs of singular vectors, defined below; one on the left and one on the right. The number of these singular vector pairs is equal to the smaller of the two matrix dimensions.
Eigenvectors and singular vectors are familiar from linear algebra, where they are taught in concert with *eigenvalues* and *singular values*. Numerical linear algebra is the foundation of applied mathematics and scientific computing. Specifically, the concept of eigenvectors, and numerical algorithms for computing them, became a key technology during the 20th century.

Singular vectors are associated to rectangular matrices. We review their definition through the lens of Remark 9.1. We begin with the observation that each rectangular matrix represents a bilinear form, e.g.

\[(9.4) \quad B = 2ux + 3uy + 5uz + 3vx + 7vy + 11vz = (u \ v) \begin{pmatrix} 2 & 3 & 5 \\ 3 & 7 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\]

The gradient of the bilinear form defines an endomorphism on the direct sum of and \(m\)-dimensional and \(n\)-dimensional space. This fuses left multiplication and right multiplication by our matrix into a single linear map.

For the example (9.4), the gradient is the following vector of linear forms

\[(9.5) \quad \nabla B = \left( \begin{pmatrix} \frac{\partial B}{\partial u} & \frac{\partial B}{\partial v} \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} \\ \frac{\partial B}{\partial z} \end{pmatrix} \right).\]

The associated endomorphism has the form \(\nabla B : K^3 \oplus K^2 \to K^2 \oplus K^3\). This gradient map takes the pair \(((x, y, z), (u, v))\) to the pair in (9.5), i.e. to

\[((2x + 3y + 5z, 3x + 7y + 11z), (2u + 3v, 3u + 7v, 5u + 11v)).\]

More generally, let \(B\) be an \(m \times n\)-matrix over \(K\). Consider the equations

\[(9.6) \quad Bx = \lambda y \quad \text{and} \quad B^t y = \lambda x,\]

where \(\lambda\) is a scalar, \(x\) is a nonzero vector in \(K^n\), and \(y\) is a nonzero vector in \(K^m\). These are our unknowns. Given a solution to (9.6), we see that \(x\) is an eigenvector of \(B^t B\), \(y\) is an eigenvector of \(BB^t\), and \(\lambda^2\) is a common eigenvalue of these two symmetric matrices. Assuming \(K = \mathbb{R}\), its nonnegative square root \(\lambda \geq 0\) is a *singular value* of \(B\). Associated to \(\lambda\) are the *right singular vector* \(x\) and the *left singular vector* \(y\). In analogy to Remark 9.1, the process of solving (9.6) has the following dynamical interpretation:

**Remark 9.2.** The singular vector pairs \((x, y)\) of a rectangular matrix \(B\) are the fixed points of the gradient map \(\nabla B\) of the associated bilinear form. Up to scaling, this is a self-map on a product of projective spaces:

\[\nabla B : \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \to \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}\]

\[((x, y)) \mapsto \left( \begin{pmatrix} \frac{\partial B}{\partial x_1}, \ldots, \frac{\partial B}{\partial x_n} \\ \frac{\partial B}{\partial y_1}, \ldots, \frac{\partial B}{\partial y_m} \end{pmatrix} \right).\]

To be precise, each pair \((x, y)\) of singular vectors gives rise either to a base point or a fixed point of the rational map \(\nabla B\) between the product of projective spaces. Conversely, any base point can be lifted to a pair
of singular vectors. However, for a fixed point \((x, y)\), we must choose a compatible lift. To do this, we can take an arbitrary vector representative of \(x\) and the corresponding representative of \(y\).

Working over \(\mathbb{R}\), we have the following orthogonal decompositions:

\[
Q = O \cdot \text{diag} \cdot O^t \quad \text{and} \quad B = O_1 \cdot \text{diag} \cdot O_2.
\]

Here \(\text{diag}\) represents diagonal matrices, the latter one being rectangular. The matrices \(O, O_1\) and \(O_2\) are orthogonal, i.e. \(OO^t = \text{Id}, O_1O_1^t = \text{Id}\) and \(O_2O_2^t = \text{Id}\). The entries of \(\text{diag}\) are respectively the eigenvalues of \(Q\) and the singular values of \(B\). The columns of \(O\) are the eigenvectors of \(Q\). The columns of \(O_1\) and rows of \(O_2\) give all pairs of singular vectors of \(B\). Note that \(O_1\) and \(O_2\) are both square matrices, but of different sizes. The possibly unmatched rows of \(O_2\) are in the kernel of \(B\). The formulas above are known as the spectral decomposition and the singular value decomposition.

When \(K = \mathbb{C}\) there is a version of the spectral decomposition for Hermitian matrices \(Q\), i.e. \(Q\) equal to its conjugate transpose: \(Q_{ij} = \overline{Q_{ji}}\). The matrix \(O\) is now unitary, i.e. its inverse is equal to its conjugate transpose. The eigenvalues of \(Q\) remain real. Analogously, for the singular value decomposition one needs to replace the orthogonal matrices by unitary matrices. The singular values are nonnegative also in this case.

We summarize our brief review of linear algebra in the following points:

- Symmetric matrices \(Q\) represent quadratic forms.
- Rectangular matrices \(B\) represent bilinear forms.
- Their gradients \(\nabla Q\) and \(\nabla B\) specify the linear maps one usually identifies with the matrices \(Q\) and \(B \oplus B^T\).
- The eigenvectors and singular vectors are fixed points of these maps.
- These fixed points are computed via spectral and singular value decompositions:

\[
Q = O \cdot \text{diag} \cdot O^t \quad \text{and} \quad B = O_1 \cdot \text{diag} \cdot O_2.
\]

In the age of Big Data, the role of matrices is increasingly played by tensors, that is, multidimensional arrays of numbers. Principal component analysis tells us that the eigenvectors of a covariance matrix \(Q = BB^t\) give directions in which the data \(B\) is most spread. One hopes to identify similar features for tensor data. This has encouraged engineers and scientists to spice up their linear algebra tool box with a pinch of algebraic geometry.

The spectral theory of tensors is the theme of the following discussion. This theory was pioneered around 2005 by Lek-Heng Lim and Liqun Qi. We refer to the textbook \([48]\) for background and context. Our aim is to generalize familiar notions, such as rank, eigenvectors and singular vectors, from
matrices to tensors. Specifically, we address the following two questions. The answers to these two questions are provided in Examples 9.7 and 9.13.

**Question 9.3.** How many eigenvectors does a $3 \times 3 \times 3$-tensor have?

**Question 9.4.** How many singular vector triples does a $3 \times 3 \times 3$-tensor have?

A tensor is a $d$-dimensional array $T = (t_{i_1 i_2 \ldots i_d})$. Here the entries $t_{i_1 i_2 \ldots i_d}$ are elements in the ground field $K$. The set of all tensors of format $n_1 \times n_2 \times \cdots \times n_d$ forms a vector space of dimension $n_1 n_2 \cdots n_d$ over $K$. For $d = 1, 2$ we get vectors and matrices. A tensor has rank 1 if it is the outer product of $d$ vectors, written $T = u \otimes v \otimes \cdots \otimes w$, or, in coordinates,

$$t_{i_1 i_2 \ldots i_d} = u_{i_1} v_{i_2} \cdots w_{i_d}.$$  

The problem of tensor decomposition is the following. We wish to express a given tensor $T$ as a sum of rank 1 tensors, using as few summands as possible. That minimal number of rank 1 summands needed to represent $T$ is the rank of $T$. We note that the rank of any tensor is always finite. We will discuss this topic in detail in the next section.

An $n \times n \times \cdots \times n$-tensor $T = (t_{i_1 i_2 \ldots i_d})$ is called symmetric if it is unchanged under permuting the indices. The space $\text{Sym}_d(\mathbb{R}^n)$ of such symmetric tensors has dimension $\binom{n+d-1}{d}$. It is identified with the space of homogeneous polynomials of degree $d$ in $n$ variables, written as

$$T = \sum_{i_1, \ldots, i_d=1}^n t_{i_1 i_2 \ldots i_d} x_{i_1} x_{i_2} \cdots x_{i_d}.  \quad (9.7)$$

![Figure 1](image_url)

**Figure 1.** A symmetric $3 \times 3 \times 3$ tensor represents a cubic curve in $\mathbb{P}^2$.

**Example 9.5.** A tensor $T$ of format $3 \times 3 \times 3$ has 27 entries. If $T$ is a symmetric tensor then it has at most ten distinct entries, one for each coefficient of the associated cubic polynomial in three variables. This polynomial defines a cubic curve in the projective plane $\mathbb{P}^2$, as indicated in Figure 1.
Symmetric tensor decomposition writes $T$ as sum of powers of linear forms:

\[(9.8) \quad T = \sum_{j=1}^{r} \lambda_j v_j^d = \sum_{j=1}^{r} \lambda_j (v_{1j}x_1 + v_{2j}x_2 + \cdots + v_{nj}x_n)^d,\]

where $v_j = (v_{1j}, \ldots, v_{nj})$ and $\lambda_j$ are scalars. As before, the gradient of $T$ defines a map $\nabla T : K^n \to K^n$. A vector $v \in K^n$ is an eigenvector of $T$ if $(\nabla T)(v) = \lambda \cdot v$ for some $\lambda \in K$.

Eigenvectors of tensors arise naturally in optimization. Consider the problem of maximizing a real homogeneous polynomial $T$ over the unit sphere in $\mathbb{R}^n$. If $\lambda$ denotes a Lagrange multiplier, then one sees that the eigenvectors of $T$ are the critical points of this optimization problem. One can check the values of $T$ at these points to find global maxima and minima.

We find it convenient to replace $K^n$ by the projective space $\mathbb{P}^{n-1}$. The gradient map is then a rational map from this projective space to itself:

$$\nabla T : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}.$$ 

The eigenvectors of $T$ are fixed points ($\lambda \neq 0$) and base points ($\lambda = 0$) of $\nabla T$. Thus the spectral theory of tensors is closely related to the study of dynamical systems on $\mathbb{P}^{n-1}$. The matrix case ($d = 2$) appeared in (9.3).

By the Spectral Theorem, a real quadratic form $T$ has a real decomposition (9.8) with $r$ being the rank, the $\lambda_j$ are the eigenvalues of $T$, and the eigenvectors $v_j = (v_{1j}, v_{2j}, \ldots, v_{nj})$ are orthonormal. We can compute this numerically by power iteration, namely, by applying $\nabla T$ until an approximate fixed point is reached. This will be a dominant eigenvector, and the other eigenvectors are then found by an appropriate inductive scheme.

For $d \geq 3$, one can still use the power iteration to compute eigenvectors of $T$. However, the eigenvectors are usually not the vectors $v_i$ in the low rank decomposition (9.8). One exception arises when the symmetric tensor is odecor, or orthogonally decomposable. This means that $T$ has the form (9.8), where $r = n$ and $\{v_1, v_2, \ldots, v_r\}$ is an orthogonal basis of $\mathbb{R}^n$. These basis vectors are the attractors of the dynamical system $\nabla T$, provided $\lambda_j > 0$.

The following gives a count of the eigenvectors of a symmetric tensor.

**Theorem 9.6** (Cartwright-Sturmfels [9]). If $K$ is algebraically closed, then the number of eigenvectors of a general tensor $T \in \text{Sym}_d(K^n)$ equals

$$\frac{(d-1)^n - 1}{d-2} = \sum_{i=0}^{n-1} (d-1)^i.$$ 

**Example 9.7** ($n = d = 3$). The Fermat cubic $T = x^3 + y^3 + z^3$ is an odecor tensor. Its gradient map is the regular map that squares each coordinate: $\nabla T : \mathbb{P}^2 \to \mathbb{P}^2$, $(x : y : z) \mapsto (x^2 : y^2 : z^2)$. This dynamical system has
Theorem 9.9 \( (Abo-Seigal-Sturmfels) \) tensors, the eigendiscriminant is a polynomial of degree 24 in 27 unknowns.

Example 9.10 \( \text{Example 9.10} \) is the discriminant of the characteristic polynomial of an \( n \times n \)-matrix \( \text{Example 9.8} \). Let \( \nabla T \) be a product of linear forms in \( x, y, z \), defining \( d \) lines in \( \mathbb{P}^n \). The \( \binom{d}{2} \) vertices of the line arrangement are base points of \( \nabla T \). One can show that each of the \( \binom{d}{2} + 1 \) regions contains one fixed point. This accounts for all \( 1 + (d-1) + (d-1)^2 \) eigenvectors, which are hence real.

Example 9.8. Let \( d = 4 \) and fix the quartic \( T = xyz(x + y + z) \), which is a symmetric \( 3 \times 3 \times 3 \times 3 \) tensor. Its curve in \( \mathbb{P}^2 \) is an arrangement of four lines. All 13 = 6 + 7 eigenvectors of \( T \) are real. The 6 vertices of the arrangement are the base points of \( \nabla T \). Each of the 7 regions contains one fixed point.

For special tensors \( T \), two of the eigenvectors in Theorem 9.6 may coincide. This corresponds to vanishing of the eigendiscriminant, a big polynomial in the \( t_{11,12,\ldots,i_d} \). In the matrix case \( (d = 2) \), this is the discriminant of the characteristic polynomial of an \( n \times n \)-matrix \( \text{[58] \S7.5} \). For \( 3 \times 3 \times 3 \) tensors, the eigendiscriminant is a polynomial of degree 24 in 27 unknowns.

Theorem 9.9 \( (Abo-Seigal-Sturmfels) \). The eigendiscriminant is an irreducible homogeneous polynomial of degree \( n(n-1)(d-1)^{n-1} \) in the \( t_{11,12,\ldots,i_d} \). Example 9.10 \( (n = 2) \). The eigendiscriminant of a binary form \( T(x, y) \) of degree \( d \) is the discriminant of \( x \frac{\partial T}{\partial y} - y \frac{\partial T}{\partial x} \), so it has degree \( 2d - 2 \) in \( T \).

Singular value decomposition is a central notion in linear algebra and its applications. Remark 9.2 casts the singular vector pairs of a matrix as fixed points of a self-map of a product of two projective spaces. Consider now a \( d \)-dimensional tensor \( T \) in \( K^{n_1 \times \cdots \times n_d} \). It corresponds to a multilinear form. The singular vector tuples of \( T \) are the fixed points of the gradient map

\[
\nabla T : \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_d-1} \longrightarrow \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_d-1}.
\]

Example 9.11. The trilinear form \( T = x_1 y_1 z_1 + x_2 y_2 z_2 \) is interpreted as a \( 2 \times 2 \times 2 \) tensor. The gradient \( \nabla T \) of this trilinear form is the rational map

\[
\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,
\]

\[
((x_1 : x_2), (y_1 : y_2), (z_1 : z_2)) \mapsto ((y_1 z_1 : y_2 z_2), (x_1 z_1 : x_2 z_2), (x_1 y_1 : x_2 y_2)).
\]

This map has six fixed points, namely \( (1:0), (1:0) \), \( (0:1), (0:1) \), \( (1:1), (1:1), (1:1) \), \( (1:1), (1:1), (1:1) \), \( (1:1), (1:1), (1:1) \), \( (1:1), (1:1), (1:1) \), \( (1:1), (1:1), (1:1) \). These are the singular vector triples of the tensor \( T \).
Here is a formula for the expected number of singular vector tuples.

**Theorem 9.12** (Friedland and Ottaviani [22]). For a general $n_1 \times \cdots \times n_d$-tensor $T$ over an algebraically closed field $K$, the number of singular vector tuples is the coefficient of the monomial $z_1^{n_1-1} \cdots z_d^{n_d-1}$ in the polynomial

$$\prod_{i=1}^{d} \frac{(\hat{z}_i)^{n_i} - z_i^{n_i}}{\hat{z}_i - z_i},$$

where $\hat{z}_i = z_1 + \cdots + z_{i-1} + z_{i+1} + \cdots + z_d$.

We finish our study of spectral theory of tensors by answering Question 2.

**Example 9.13.** Let $d=n_1=n_2=n_3=3$. The polynomial in Theorem 9.12 is

$$(\hat{z}_1^2 + \hat{z}_1 z_1 z_1^2)(\hat{z}_2^2 + \hat{z}_2 z_2 z_2^2)(\hat{z}_3^2 + \hat{z}_3 z_3 z_3^2) = \cdots + 37 z_1^2 z_2^2 z_3^2 + \cdots$$

Therefore, a general $3 \times 3 \times 3$-tensor has exactly 37 triples of singular vectors. Likewise, a general $3 \times 3 \times 3 \times 3$-tensor has 997 quadruples of singular vectors.

### 9.2. Tensor Rank

There are many ways to define the rank of an $a \times b$ matrix $M$ over a field $K$:

1. the smallest integer $r$ such that all $(r+1) \times (r+1)$ minors vanish,
2. the dimension of the image of the induced linear map $K^a \to K^b$,
3. the dimension of the image of the induced linear map $K^b \to K^a$,
4. the smallest $r$ such that $M = UW$ where $U \in K^{a \times r}$ and $W \in K^{r \times b}$.

The first point implies that matrices of rank at most $r$ form a variety. The last point implies that a matrix of rank $r$ is a sum of $r$ matrices of rank one. This is also true for symmetric matrices: a symmetric matrix of rank $r$ is a sum of $r$ symmetric rank one matrices. Another fact is that a real matrix of rank $r$ has also rank $r$ when viewed over $\mathbb{C}$. This seems obvious, but a priori, it is not clear why there is no shorter decomposition into rank one matrices with entries in $\mathbb{C}$. Our aim is to study these issues for arbitrary tensors.

In this section we work in a tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_d$ of finite-dimensional vector spaces $V_i$ over a field $K$. This is the vector space of all tensors of format $(\dim V_1) \times \cdots \times (\dim V_d)$. By fixing a basis, we may identify $V_i \simeq K^{n_i}$, and thus $V_1 \otimes V_2 \otimes \cdots \otimes V_d \simeq K^{n_1 \times n_2 \times \cdots \times n_d}$. A nonzero tensor $T$ in this space has rank one if it is the outer product of $d$ vectors, i.e. $T = u \otimes v \otimes \cdots \otimes w$. In coordinates, this means that the entries of $T$ factor as

$$t_{i_1 i_2 \cdots i_d} = u_{i_1} v_{i_2} \cdots w_{i_d}.$$

Tensors of rank at most one form an affine variety. It is the affine cone over the Segre variety $\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_d-1}$ in $\mathbb{P}^{n_1 \cdots n_d-1}$. In fact, from Chapter 2 and 8 we know the equations of this variety. They are binomial
9.2. Tensor Rank

quadrics, namely the $2 \times 2$ minors of all flattenings of $T$. By this we mean representations of the tensor $T$ as a matrix. We shall now explain this.

To begin with, the flattenings have the following invariant description. Let $I$ be any nonempty, proper subset of $[d] = \{1, 2, \ldots, d\}$. The corresponding flattening is the linear map defined by $T$ as follows:

\begin{equation}
K^{\prod_{i \in I} n_i} = \bigotimes_{i \in I} V_i^* \rightarrow \bigotimes_{j \in [d] \setminus I} V_j = K^{\prod_{j \in [d] \setminus I} n_j}
\end{equation}

\[e_i^* \mapsto \sum_j t_{ij} e_j.\]

Here $e_i^*$ is a fixed basis vector of $K^{\prod_{i \in I} n_i} = \bigotimes_{i \in I} V_i^*$, $e_j$ is any basis vector of $\bigotimes_{j \in [d] \setminus I} V_j = K^{\prod_{j \in [d] \setminus I} n_j}$ and $t_{ij}$ is the entry of $T$ corresponding to given $i$ and $j$. Thus a tensor $T$ has rank one if and only if all $2^d - 2$ flattenings of $T$ are matrices of rank one. In fact, it is enough to check $d$ flattening corresponding to $|I| = 1$. There is a similar result for tensors of rank two, due to Landsberg–Manivel and Raicu, but for all higher ranks only one direction is true: the rank of $T$ is bounded below by that of any flattening.

Example 9.14. A tensor $T = (t_{ijk}) \in V_1 \otimes V_2 \otimes V_3$ induces the linear map

\begin{equation}
V_i^* \rightarrow V_2 \otimes V_3, \quad e_i^* \mapsto (t_{ijk})_{j,k} = \sum_{j,k} t_{ijk} \cdot f_j \otimes g_k,
\end{equation}

where $(e_i)$, $(f_j)$, $(g_k)$ are respectively bases of $V_1$, $V_2$, $V_3$. This is the case $I = \{1\}$ in (9.9). We think of (9.10) as an $n_1 \times n_2 n_3$ matrix. The transpose of this matrix is the $n_2 n_3 \times n_1$ matrix that corresponds to $|I| = 1$. Thus, a three-way tensor has three distinct flattenings, up to transposing.

We conclude that rank one tensors behave in a very nice way. However, arbitrary tensors exhibit rather strange properties. Recall that the rank of a tensor $T$ is the minimal $r$ such that $T$ is the sum of $r$ rank one tensors. For instance, the three-way tensors of rank $\leq 2$ are the tensors of the form

\begin{equation}
T = a \otimes b \otimes c + d \otimes e \otimes f.
\end{equation}

We shall see that the set of these tensors is not Zariski closed in $K^{n_1 \times n_2 \times n_3}$.

Example 9.15. Let $d = 3$ and $V_1 = V_2 = V_3 = \mathbb{C}^2$ with basis $\{e_0, e_1\}$. The following $2 \times 2 \times 2$ tensor is known in quantum physics as the $W$-state:

\begin{equation}
W = e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0.
\end{equation}

This representation shows that $W$ has rank at most three. In fact, $\text{rk} W = 3$, as the reader is asked to prove in Exercise 9. To do so, equate $W$ with $T$ in (9.11). This gives an inconsistent system of 8 cubic equations in the 12 unknown coordinates $a_0, a_1, b_0, \ldots, f_1$ of the vectors $a, b, \ldots, f$ in (9.11).
However, there exist rank two tensors arbitrarily close to \( W \). We have
\[
\frac{1}{\epsilon} \left( (e_0 + \epsilon e_1) \otimes (e_0 + \epsilon e_1) \otimes (e_0 + \epsilon e_1) - e_0 \otimes e_0 \otimes e_0 \right) =
\]
\[
W + \epsilon (e_1 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_1) + \epsilon^2 e_1 \otimes e_1 \otimes e_1.
\]
This is an identity for all \( \epsilon \neq 0 \). In particular, we have
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( (e_0 + \epsilon e_1) \otimes (e_0 + \epsilon e_1) \otimes (e_0 + \epsilon e_1) - e_0 \otimes e_0 \otimes e_0 \right) = W.
\]
We conclude that the \( W \)-state is a tensor of rank three, but it can be approximated with arbitrary precision by a sequence of tensors of rank two.

**Definition 9.16.** The border rank \( \text{brk}(T) \) of a complex tensor \( T \) is the smallest \( r \) such that \( T \) lies in the closure of the set of tensors of rank \( r \).

The notion of border rank requires a topology on the space of tensors. The geometric locus of tensors of border rank \( \leq r \) is the closure of the locus of tensors of rank \( \leq r \). Over the complex numbers, by Chevalley’s Theorem 4.18, it does not matter if we take Euclidean or Zariski topology: the closures coincide. However, the situation is different over the real numbers. To prove this, we shall use the hyperdeterminant, denoted \( \text{Det} \), from Example 4.10.

**Lemma 9.17.** The hyperdeterminant of the rank two tensor in (9.11) equals
\[
(9.13) \quad \text{Det}(T) = (a_0 d_1 - a_1 d_0)^2 (b_0 c_1 - b_1 c_0)^2 (c_0 f_1 - c_1 f_0)^2.
\]

**Proof.** We have an explicit formula for \( \text{Det}(T) \) as a homogeneous polynomial of degree 4 in the eight tensor entries \( t_{ijk} \). If we substitute \( t_{ijk} = a_i b_j c_k + d_i e_j f_k \) and factor, then we obtain the above product of degree 12. \( \square \)

**Corollary 9.18.** Let \( T \in \mathbb{R}^{2 \times 2 \times 2} \). If \( T \) has real rank \( \leq 2 \) then \( \text{Det}(T) \geq 0 \).

**Proof.** Our hypothesis says that \( T \) has a representation (9.11) over \( \mathbb{R} \). The expression (9.13) for \( \text{Det}(T) \) is a square in \( \mathbb{R} \). It is hence nonnegative. \( \square \)

**Example 9.19.** Let \( i = \sqrt{-1} \). The following tensor has rank two in \( \mathbb{C}^{2 \times 2 \times 2} \):
\[
T = \frac{1}{2} \left( (e_1 + i e_0) \otimes (e_1 + i e_0) + (e_1 - i e_0) \otimes (e_1 - i e_0) \right) = e_1 \otimes e_1 \otimes e_1 - W.
\]

Note that this tensor is real. By substituting its coefficients into the formula for the hyperdeterminant in Example 4.10 we find that \( \text{Det}(T) = -4 < 0 \). Corollary 9.18 implies that the real rank of \( T \) is \( \geq 3 \). Thus the tensor \( T \) has the property that its complex rank is strictly smaller than its real rank.

In Exercise 10, the reader is asked to prove that the set of rank two tensors is Zariski dense in the space of \( 2 \times 2 \times 2 \) tensors. This holds for any infinite field \( K \). If \( K = \mathbb{C} \) then they are also dense in the Euclidean
topology. In fact, a tensor $T$ has complex rank $\leq 2$ if $\text{Det}(T) \neq 0$. Note that the $W$-state has rank 3 and it satisfies $\text{Det}(W) = 0$. If $\text{Det}(T) = 0$ and all flattenings have rank two then $T$ has always complex rank three. If exactly one flattening has rank one, then $T$ has complex rank two. If two flattenings have rank one, then all three have rank one and $T$ has complex rank one. If $K = \mathbb{R}$ then we must distinguish the two cases $\text{Det}(T) > 0$ and $\text{Det}(T) < 0$. In the former case, $T$ has real rank two. In the latter case, $T$ has real rank three.

Our discussion has the following interpretation in projective geometry. Tensors of rank one form the Segre threefold $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^7$. Exercise 11 says that the secant variety of $X$, described in Definition 9.20 below, fills $\mathbb{P}^7$. However, the tangential variety of $X$, which is the union of all tangent spaces to $X$, has dimension six. It is the hypersurface $\{\text{Det}(T) = 0\}$ in $\mathbb{P}^7$. The $W$-state is a point on that hypersurface. The line $\{\lambda e_0 \otimes e_0 \otimes e_0 + W : \lambda \in K\}$ crosses this hypersurface transversally. If $K = \mathbb{R}$ then the real rank depends only on the sign of $\lambda$.

To conclude, unlike in the case of matrices or rank one tensors, we have:

- Tensors of rank at most $r$ may form a non-closed set.
- A real tensor can have smaller rank when seen as a complex tensor.
- Real tensors of bounded real border rank form semialgebraic sets.

We have described rank one tensors as the Segre product of projective spaces. It is natural to ask for a geometric description of tensors of rank at most $r$.

**Definition 9.20 (Secant Variety).** Let $X$ be any projective variety in $\mathbb{P}^n$. The $k$-th secant variety of $X$ is the closure of the set of $k$-secant planes to $X$:

\[
\sigma_k(X) := \bigcup_{p_1, \ldots, p_k \in X} \langle p_1, \ldots, p_k \rangle.
\]

Note that $X = \sigma_1(X) \subset \sigma_2(X) \subset \cdots \subset \sigma_{\dim(X)}(X) = \langle X \rangle$. These containments are strict until $\sigma_r(X)$ equals the linear span $\langle X \rangle$ of the variety $X$.

If $X$ is the Segre variety, then the union in (9.14) is the set of tensors of rank $\leq r$. Its closure $\sigma_r(X)$ is the set of tensors of border rank $\leq r$. It is a major open problem to determine the ideal of $\sigma_r(X)$. This would provide a test for a tensor to have border rank $r$. The simplest equations for $\sigma_r(X)$ are the $(r+1) \times (r+1)$ minors of the various flattenings, as in (9.9). These have degree $r+1$. No polynomials of degree $\leq r$ vanish on $\sigma_r(X)$.

If $X$ is the Veronese variety then we obtain the notion of symmetric rank. A symmetric tensor $T$ is in $X$ if the following equivalent conditions hold:

1. The rank of $T$ as a tensor is equal to one.
2. $T = \lambda \mathbf{v} \otimes \mathbf{v} \otimes \cdots \otimes \mathbf{v}$ for some vector $\mathbf{v}$ and scalar $\lambda$. 
(3) $T$, as a polynomial, is a power of a linear form times a constant.

Given a symmetric tensor $T$, the symmetric border rank of $T$ is the smallest positive integer $r$ such that $T \in \sigma_r(X)$. The rank of $T$ is a lower bound for the symmetric rank of $T$, and ditto for border rank.

It was a longstanding question, known as Comon’s Conjecture, whether the rank of a symmetric tensor is always equal to its symmetric rank. It turns out that the answer is no. A counterexample was constructed by Shitov in [52]. The border rank analogue of Comon’s Conjecture remains open.

It is easy to prove that general tensors have high rank and border rank, but it is extremely hard to find explicit examples. In particular, it is not known how to provide examples of complex $n \times n \times n$ tensors $T$ of rank greater than $3n$. Constructions of tensors of rank close to $3n$ can be found in [2] and of border rank above $2n$ in [36]. By Exercise [12] a general tensor in $K^{n \times n \times n}$ has border rank quadratic in $n$. We offer a case study on tensor ranks in the case visualized in Figure 1.

Example 9.21 ($3 \times 3 \times 3$ tensors). Fix an algebraically closed field $K$. Tensors of format $3 \times 3 \times 3$ are points in the projective space $\mathbb{P}^2$. The 6-dimensional Segre variety $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ consists of all tensors of rank 1. Tensors of border rank at most 2 form the secant variety $\sigma_2(X)$, which has dimension 13. Its ideal is generated by the $3 \times 3$-minors of the three flattenings. These flattenings are $3 \times 9$-matrices, like $[A \mid B \mid C]$ where $A = (t_{ij1})$, $B = (t_{ij2})$, $C = (t_{ij3})$ are matrices obtained as the slices in our tensor. These $3 \times 3$-minors span a space of cubics that has dimension 222. With another computation one also verifies that $\sigma_2(X)$ has degree 783.

The variety $\sigma_3(X)$ of border rank at most 3 tensors has dimension 20. Its ideal is generated by a collection of quartic polynomials, namely the entries of the $3 \times 3$-matrices $A \cdot \text{adj}(B) \cdot C - C \cdot \text{adj}(B) \cdot A$, where we allow all possible ways of slicing the tensor. What is the degree of this variety?

Finally, there is the variety $\sigma_4(X)$ of $3 \times 3 \times 3$ tensors of border rank $\leq 4$. This is a hypersurface of degree 9 in $\mathbb{P}^2$. Its defining polynomial is known as the Strassen invariant. The Strassen invariant can be computed as

$$\det(B)^2 \cdot \det(A \cdot B^{-1}C - C \cdot B^{-1}A).$$

The expression has 9216 terms and it is independent of the choice of slicing. The fifth secant variety $\sigma_5(X)$ is equal to $\mathbb{P}^2$. In other words, the set of tensors of rank $\leq 5$ is dense in the space of all $3 \times 3 \times 3$ tensors.

We refer to the book by Landsberg [35] for further details and much more information on ideals defining the varieties of tensor of bounded rank.

We now restrict the rank stratification to the space of symmetric tensors.
Example 9.22 (Ternary Cubics). Symmetric $3 \times 3 \times 3$ tensors $T$ are ternary cubics, that is, homogeneous polynomials of degree three in three variables. We regard them as points in $\mathbb{P}^9 = \mathbb{P}({\text{Sym}}_3(K^3))$. Their ranks coincide with their symmetric ranks, i.e., Comon’s Conjecture is true in this tiny case \cite{49}.

The three flattenings $[A | B | C]$ in Example 9.21 are now all equal. After removing redundant columns, this becomes a $3 \times 6$ matrix, known as Hankel matrix or catalecticant. The ideal of $2 \times 2$-minors of the Hankel matrix is generated by the 27 binomial quadrics seen for $A = 3\Delta_2$ in Example 8.18. Its variety is the Veronese surface $X \simeq \mathbb{P}^2$ whose points in $\mathbb{P}^9$ are the cubics of rank 1. The secant variety $\sigma_2(X)$ has dimension 5, its points are cubics of border rank $\leq 2$, and it is defined by the $3 \times 3$ minors of the Hankel matrix. Finally, the variety $\sigma_3(X)$ of border rank $\leq 3$ cubics is a quartic hypersurface in $\mathbb{P}^9$. Its defining polynomial is the classical Aronhold invariant. This has 25 terms and it can be obtained by specializing any of the entries of

\begin{equation}
A \cdot \text{adj}(B) \cdot C - C \cdot \text{adj}(B) \cdot A.
\end{equation}

We have already discussed the distinction between complex rank and real rank. A further refinement of the latter is the notion of nonnegative rank. It is very important in applications, e.g., in statistics, where one deals with probabilities. A tensor $T = (t_{i_1i_2\cdots i_d})$ is called nonnegative if its entries $t_{i_1i_2\cdots i_d}$ are all nonnegative. A nonnegative rank of a nonnegative tensor $T$ is the minimal number $r$ of nonnegative rank one tensors that sum up to $T$. In generally, nonnegative rank is larger than real rank, even for matrices.

9.3. Matrix Multiplication

The multiplication of two matrices is a bilinear operation. In this section we identify this operation with a very special tensor. We will use this to explain how tensors may be regarded as computational problems, tensor decompositions as algorithms, and tensor rank as a complexity measure.

Determining the rank of a tensor is an important computational problem in non-linear algebra. In general one cannot hope for an efficient solution, as it is NP-hard \cite{29}. However, special cases are of particular interest. The most well-known and important one is the matrix multiplication tensor.

Let $\text{Mat}_{a,b} \simeq K^{a \times b}$ be the space of $a \times b$ matrices over a field $K$. The operation of matrix multiplication is a bilinear map $\text{Mat}_{a,b} \times \text{Mat}_{b,c} \to \text{Mat}_{a,c}$. Bilinear maps from a Cartesian product of vector spaces are in bijection with linear maps from the tensor product of those spaces, cf. Exercise 14. Hence, matrix multiplication is an element of the vector space

$$\text{Hom}(\text{Mat}_{a,b} \otimes \text{Mat}_{b,c}, \text{Mat}_{a,c}) \simeq \text{Mat}_{a,b}^* \otimes \text{Mat}_{b,c}^* \otimes \text{Mat}_{a,c}.$$

This is a canonical isomorphism. We write $M_{a,b,c}$ for the element in the tensor space on the right hand side. This special three-way tensor is the matrix multiplication tensor. To simplify notation we write $M_n := M_{n,n,n}$ for the tensor that represents the multiplication of two square matrices.

Let $\{e_{ij}\}$, $\{f_{jk}\}$ and $\{g_{ik}\}$ be the standard bases of the spaces $\text{Mat}_{a,b}^*$, $\text{Mat}_{b,c}^*$ and $\text{Mat}_{a,c}^*$. Thus $g_{ik}$ is the $a \times c$ matrix whose entries are zero except for a one in row $i$ and column $k$. The other two bases are dual to such matrix units. The matrix multiplication tensor has the following representation:

\begin{equation}
M_{a,b,c} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} e_{ij} \otimes f_{jk} \otimes g_{ik}.
\end{equation}

Another presentation is suggested in Exercise 13.

**Example 9.23.** Consider the tensor $M_2$ that represents multiplication of $2 \times 2$ matrices. Fixing the ordered basis $(e_{00}, e_{01}, e_{10}, e_{11})$ for $\text{Mat}_{2,2} \simeq K^4$, we can write $M_2$ explicitly as a $4 \times 4 \times 4$ tensor with entries in $\{0, 1\}$. Among the 64 entries in this tensor, there are precisely eight ones and 56 zeros.

The rank one decomposition of the tensor $M_{a,b,c}$ given in (9.16) can be interpreted as an algorithm for computing the product of the two matrices:

- To carry out matrix multiplication, one needs to add $abc$ partial results labelled by $(i, j, k)$ in $\{1, \ldots, a\} \times \{1, \ldots, b\} \times \{1, \ldots, c\}$.

- In step $(i, j, k)$ one performs one multiplication. Namely, one multiplies the $(i, j)$-th entry of the first matrix with the $(j, k)$-th entry of the second. The result is stored in the $(i, k)$-th entry of the third.

This is the familiar classical algorithm for multiplying two matrices. It performs $abc - 1$ additions and $abc$ multiplications. For $a = b = c = n$, its running time is $O(n^3)$. We note that the number of multiplications is exactly equal to the number of rank one tensors appearing in the decomposition.

What if we present $M_{a,b,c}$ in a different way? Could it be that the number of multiplications we need is smaller than $abc$? Equivalently, is the rank of $M_{a,b,c}$ smaller than $abc$? Half a century ago, Volker Strassen set out on the quest to prove that this is not possible. He quickly realized that the case of arbitrary $a, b, c$ is extremely hard and focused on the first nontrivial case
$a = b = c = 2$. For that tensor, he discovered a most surprising formula:

$$
M_2 = (e_{11} + e_{22}) \otimes (f_{11} + f_{22}) \otimes (g_{11} + g_{22}) \\
+ (e_{21} + e_{22}) \otimes f_{11} \otimes (g_{21} - g_{22}) \\
+ e_{11} \otimes (f_{12} - f_{22}) \otimes (g_{12} + g_{22}) \\
+ e_{22} \otimes (f_{21} - f_{11}) \otimes (g_{11} + g_{21}) \\
+ (e_{11} + e_{12}) \otimes f_{22} \otimes (g_{12} - g_{11}) \\
+ (e_{21} - e_{11}) \otimes (f_{11} + f_{12}) \otimes g_{22} \\
+ (e_{12} - e_{22}) \otimes (f_{21} + f_{22}) \otimes g_{11}.
$$

(9.17)

Thus the rank of the matrix multiplication tensor $M_2$ is less than 8. In fact, it is known that the rank and border rank of $M_2$ are both exactly 7. The latter is a highly nontrivial statement. We are not aware of any easy proof.

Why could such a decomposition be interesting? It furnishes an algorithm to multiply $2 \times 2$ matrices that adds seven partial results. We describe only the first two, as we are sure the reader can reconstruct the other five:

1. Add the $(1, 1)$ entry to the $(2, 2)$ entry of the first matrix and multiply by the sum of the $(1, 1)$ and $(2, 2)$ entries of the second matrix. Retain this in the $(1, 1)$ and $(2, 2)$ entries of the first partial result.

2. Add the $(2, 1)$ entry of the first matrix to the $(2, 2)$ entry and multiply by the $(1, 1)$ entry of the second matrix. Put the result in the $(2, 1)$ entry and negated $(2, 2)$ entry of the second partial result.

Computing each partial result requires only one multiplication. Although we improved on the number of multiplications, we increased the number of additions (and subtractions) to 21. Why should this be exciting? The reason is that multiplication of $2 \times 2$ matrices is not our final aim.

We would like to multiply very large matrices. Consider two $512 \times 512$ matrices. How do we multiply them? We may regard our matrices as $2 \times 2$ matrices with entries that are $256 \times 256$ matrices and apply Strassen’s Algorithm! We will have to add a lot of $256 \times 256$ matrices, but we only need to perform seven multiplications of such matrices. Further, these multiplications may be done recursively applying the same algorithm, reducing to multiplication of $128 \times 128$ matrices, etc. Anyone who tried multiplying or adding very large matrices knows that it is beneficial to trade multiplication even for many additions. This is in fact a theorem: the complexity of the (optimal) algorithm to multiply matrices is governed by the rank of $M_n$.

The asymptotics of these quantities is measured by the constant

$$
\omega = \inf \{ \tau : \text{the complexity of multiplying two } n \times n \text{ matrices is } O(n^\tau) \}
= \inf \{ \tau : \text{rank of } M_n = O(n^\tau) \}.
$$
This quantity is known as the exponent of matrix multiplication. The naive algorithm shows that $\omega \leq 3$. However Strassen’s Algorithm, as described above, gives $\omega \leq \log_2 7$. As matrices are of size $n^2$, we also know that $\omega \geq 2$.

The central conjecture in this field says that the lower bound is attained:

**Conjecture 9.24.** The constant $\omega$ is equal to two.

The conjecture would imply that it is not much harder to multiply very large matrices than to add them (or even output the result)! At this point we note that our story is really relevant for scientific computing. Strassen’s Algorithm is implemented and used in practice to multiply large matrices.

A careful reader might now have an idea how to proceed with a proof of Conjecture 9.24. As Strassen looked at $2 \times 2$ matrices, we should focus on larger matrices, say $3 \times 3$. The disappointing fact is that despite many attempts, no one knows either the rank or the border rank of the $9 \times 9 \times 9$ tensor $M_3$. For the current best estimates we refer to [37, 38, 39, 53].

For each fixed $n$, deciding if the rank (resp. border rank) of $M_n$ is $\leq r$ means deciding if $M_n$ belongs to the image (resp. closed image) of a particular polynomial map. Thus, the methods of Chapter 3 apply. However, as tensor spaces are high-dimensional, this is impossible to carry out on a computer, even for $n = 3$. What one can use instead is representation theory, as described in Chapter 10. The optimal estimates for $\omega$ are beyond the scope of this book. Currently we know $2 \leq \omega < 2.38$. It is fascinating that the upper bounds are based on border rank and nonconstructive methods: one proves the existence of an algorithm without explicitly providing it.

In general, we lack methods to show that a tensor has high rank or high border rank. To prove that $\omega > 2$ we would need to show that the rank of the tensor $M_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$ grows superlinearly with the dimension $n^2$ of the space of matrices. However, we currently cannot even prove that any (explicit) given tensor has rank greater than $3n^2$. Some methods to obtain bounds for the rank of the tensor will be discussed in Section 10.3.

---

**Exercises**

1. Fix the quadratic form $Q$ in (9.1). Compute all the maxima and minima of $Q$ on the unit 2-sphere. Find all fixed points of the gradient map $\nabla Q : \mathbb{P}^2 \to \mathbb{P}^2$. How are these two questions related?

2. Compute all fixed points of the map $\nabla B : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2 \times \mathbb{P}^1$ that is given by the bilinear form $B$ in (9.4). What are the singular vectors?
Exercises

(3) Consider the $3 \times 3 \times 2 \times 2$ tensor defined by the multilinear form $T = x_1 y_1 z_1 w_1 + x_2 y_2 z_2 w_2$. Determine all quadruples of singular vectors of $T$.

(4) For $n = 2, 3, 4$, pick random symmetric tensors of formats $n \times n \times n$ and $n \times n \times n \times n$ with entries in $\mathbb{R}$. Compute all eigenvectors of your tensors.


(6) Find an explicit real $3 \times 3 \times 3$ tensor with precisely 13 real eigenvectors.

(7) Find the number of singular vector tuples for your tensors in Exercise 4.

(8) Compute the eigendiscriminants for symmetric tensors of format $2 \times 2$ and $2 \times 2 \times 2$ and $2 \times 2 \times 2 \times 2$. Write them explicitly as homogeneous polynomials in these entries of an unknown tensor.

(9) Prove that the rank of the $W$-state equals three. Hint: Show that the polynomial system $W = T$ described in Example 9.15 has no solution.

(10) Show that the Zariski closure of the set of tensors of rank two in $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ is the whole space. Hint: Use the Jacobian of the parametrization.

(11) Find the equation of the tangential variety to $P^1 \times P^1 \times P^2$ in $P^{11}$.

(12) Prove that in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$:
   (a) there exists a tensor of border rank at least $\frac{1}{3} n^2$,
   (b) every tensor has rank at most $n^2$.

(13) Linear maps from $V_1$ to $V_2$ are identified with tensors in $V_1^* \otimes V_2$. The composition of linear maps in $V_1 \rightarrow V_2 \rightarrow V_3$ may be regarded as a map
   $$(V_1^* \otimes V_2) \times (V_2^* \otimes V_3) \rightarrow (V_1^* \otimes V_3).$$
   Hence, the matrix multiplication tensor $M_{\dim V_1, \dim V_2, \dim V_3}$ belongs to
   $$(V_1^* \otimes V_2)^* \otimes (V_2^* \otimes V_3)^* \otimes (V_1^* \otimes V_3) = (V_1 \otimes V_1^*) \otimes (V_2 \otimes V_2^*) \otimes (V_3 \otimes V_3^*).$$
   (a) How to interpret $M_{\dim V_1, \dim V_2, \dim V_3}$ as an element of the last space?
      Do not refer to the basis, but only linear maps $V_i \rightarrow V_i$. Hint: The identity map is a distinguished element in $V_i^* \otimes V_i$.
   (b) Provide a natural isomorphism $\text{Mat}_{a,b}^* \simeq \text{Mat}_{b,a}$.
   (c) The tensor $M_{a,b,c}$ can also be identified with a trilinear map
      $$\text{Mat}_{a,b} \times \text{Mat}_{b,c} \times \text{Mat}_{c,a} \rightarrow K.$$ 
      Describe this trilinear map without referring to coordinates.

(14) Show that the following four vector spaces are naturally isomorphic:
   • $V_1 \otimes \cdots \otimes V_k$,
   • $k$-linear maps $V_1^* \times \cdots \times V_k^* \rightarrow \mathbb{C}$,
   • $(k-1)$-linear maps $V_1^* \times \cdots \times V_{k-1}^* \rightarrow V_k$,
   • linear maps $V_1^* \otimes \cdots \otimes V_{k-1}^* \rightarrow V_k$.

(15) The matrix multiplication tensor $M_{2,2,3}$ has format $4 \times 6 \times 6$. Write this tensor explicitly in coordinates. What do you know about its rank?
(16) Expand the Aronhold invariant and the Strassen invariant in monomials.
(17) Compute the ideal of the secant variety $\sigma_2(X)$ where $X = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$
    is the Segre variety in $\mathbb{P}^{17}$. How about the same question for $\sigma_3(X)$?
(18) How can you test whether a complex $4 \times 4 \times 4$ tensor has rank $\leq 4$?
(19) How many singular vector triples does a general $3 \times 4 \times 5$ tensor have?
Chapter 10

Representation Theory

“Reality favors symmetry”, Jorge Luis Borges

Symmetry is the key to many applications and computations. While this is true across the mathematical sciences, it is especially pertinent in nonlinear algebra. In its most basic form, symmetry is expressed via the action of a group acting linearly on a vector space. The study of such actions is the subject of representation theory. For instance, the symmetric group on three letters acts on the plane by the rotations and reflections that fix a regular triangle with centroid at the origin. The map that takes each group element to its associated matrix is the representation of the group. The matrix representations of the groups we study here can be simultaneously block-diagonalized. The blocks are irreducible representations. Identifying these blocks is tantamount for exploiting symmetry in explicit computations. Our objective in this chapter is to offer a first glimpse on representation theory.

10.1. Groups, Representations and Characters

The most important groups we study in this chapter are:

- \(\text{GL}(V) = \text{GL}(\dim V)\) — the group of linear automorphisms of a finite-dimensional vector space \(V\). This group has the structure of an algebraic variety, given by Exercise 8 in Chapter 2.
- \(\text{SL}(V) = \text{SL}(\dim V)\) — the group of volume and orientation preserving linear automorphisms of \(V\), with the structure of an algebraic variety given by the polynomial equation \(\det A = 1\);
• $S_n$ — the group of permutations of a set with $n$ elements. This is an algebraic variety consisting of $n!$ distinct points in $\text{GL}(n)$, namely the $n \times n$ permutation matrices.

The groups that we consider have two structures: of an abstract group and of an algebraic variety. We note that basic group operations, like taking the inverse or acting by group elements, are in fact morphisms of algebraic varieties. We call such groups algebraic. Thus, we restrict our attention to algebraic groups and morphisms between them. These are maps that are both group morphisms and morphisms of algebraic varieties. In what follows, we work over an algebraically closed field $K$ of characteristic zero.

In general, the following strategy to study a mathematical object can be very powerful. We examine all maps from (resp. to) this object into (resp. from) another basic object that we know well. This general approach could be seen as motivation to study homotopy, homology or the theory of embeddings. For groups, we obtain the following central definition.

**Definition 10.1.** A representation of a group $G$ is a morphism $G \to \text{GL}(V)$. We will always assume that $V$ is finite dimensional.

Given a representation $\rho : G \to \text{GL}(V)$, every element $g \in G$ induces a linear map $\rho(g) : V \to V$. It is useful to think about a representation as a map $G \times V \to V$ with the notation

$$gv := \rho(g)(v) \in V.$$

The following compatibilities hold for all $\lambda \in K$, $v, v_1, v_2 \in V$, $g, g_1, g_2 \in G$:

$$(g_1 g_2)v = g_1(g_2v) \quad \text{and} \quad g(\lambda v_1 + v_2) = \lambda gv_1 + gv_2,$$

If this holds, then we say that the group $G$ acts on the vector space $V$. Often, the action is clear from the context, and we just call $V$ a representation of $G$.

**Example 10.2.** Consider the regular triangle in $\mathbb{R}^2$ with vertices $(0,1)$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$. The group $S_3$ permutes the vertices of this triangle. This induces a representation $S_3 \to \text{GL}(2)$. Explicitly, we have

$$(12) \mapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad (23) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These two transpositions generate $S_3$. This example generalizes to higher dimensions by the action of $S_n$ on a regular $(n-1)$-dimensional simplex.

**Example 10.3.** The groups $\text{GL}(n)$ and $\text{SL}(n)$ act (by linear change of coordinates) on the space $V = K[x_1, \ldots, x_n]_k \simeq K^{(n+k-1)}$ of homogeneous polynomials of degree $k$ in $n$ variables. Using the monomial basis on $V$, the representation $\rho$ maps a small matrix, of size $n \times n$, to a large matrix, with rows and columns indexed by monomials of degree $k$. Its entries are
homogeneous polynomials of degree \( k \) in the entries of the small matrix. We recommend working this out for \( n = k = 2 \). This representation \( \rho \) of \( \text{GL}(n) \) plays an important role in Invariant Theory, our topic in Chapter 11.

A morphism \( f \) between representations \( \rho_1 : G \to \text{GL}(V_1) \) and \( \rho_2 : G \to \text{GL}(V_2) \) is a linear map \( f : V_1 \to V_2 \) that is compatible with the group action:

\[
f(\rho_1(g)(v)) = \rho_2(g)(f(v)) \quad \text{for all } g \in G \text{ and } v \in V_1.\]

This can also be written as \( f(gv) = gf(v) \). The kernel and cokernel of \( f \) is also a representation of \( G \)—cf. Exercise 3.

Our first aim is to describe the basic building blocks of representations. A subrepresentation of a representation \( V \) of a group \( G \) is a linear subspace \( W \subset V \) such that the action of \( G \) restricts to \( W \), i.e. \( gw \in W \) for all \( w \in W \) and \( g \in G \).

For any representation \( V \), the subspaces \( \{0\} \) and \( V \) are subrepresentations.

**Definition 10.4.** A representation \( V \) is called irreducible if and only if \( 0 \) and \( V \) are its only subrepresentations.

We next show that the only morphism between nonisomorphic irreducible representations is trivial.

**Lemma 10.5 (Schur’s Lemma).** Let \( V_1 \) and \( V_2 \) be irreducible representations of a group \( G \). If \( f : V_1 \to V_2 \) is a morphism of representations then either \( f \) is an isomorphism or \( f = 0 \). Further, any two such isomorphisms between \( V_1 \) and \( V_2 \) differ by a scalar multiple.

**Proof.** Both \( \ker f \) and \( \text{im } f \) are representations. As \( V_1 \) is irreducible, either \( \ker f = V_1 \) or \( f \) is injective. In the latter case, \( \text{im } f \simeq V_1 \) is a subrepresentation of \( V_2 \), hence \( f \) is also surjective, i.e. it is an isomorphism. For the last part, consider two isomorphisms \( f_1 \) and \( f_2 \). We may assume that \( f_1 \) is the identity on \( V_1 \). If \( v \) is the eigenvector of \( f_2 \) with eigenvalue \( \lambda \in K \) then

\[
f_2(v) = \lambda v = \lambda f_1(v).\]

Consider the morphism of representations \( f := f_2 - \lambda f_1 \). Clearly, \( v \in \ker f \). Hence, by the first part, \( f_2 - \lambda f_1 \) is the zero map, and hence \( f_2 = \lambda f_1 \). \( \square \)

**Theorem 10.6 (Maschke’s Theorem).** Let \( V \) be a representation of a finite group \( G \). There exists a direct sum decomposition

\[
V = \bigoplus V_i,
\]

where each \( V_i \) is an irreducible representation of \( G \).
Remark 10.7. We recall our assumption that the field has characteristic zero and is algebraically closed. This makes representation theory well-behaved. If the characteristic is finite, the situation is much more complicated.

Proof of Theorem 10.6. By induction on the dimension, it is enough to prove the following statement: if $W$ is a subrepresentation of $V$, then there exists a subrepresentation $W'$ such that $V = W \oplus W'$.

We fix a representation $\rho : G \to \text{GL}(V)$. Let $\pi : V \to W$ be any linear (surjective) projection. We define a linear map $\tilde{\pi} : V \to W$ by

$$\tilde{\pi}(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)_W \circ \pi \circ \rho(g)^{-1}.$$ 

We note that $\tilde{\pi}$ is a morphism of representations and $V = W \oplus \ker \tilde{\pi}$. □

Remark 10.8. The existence of a decomposition into irreducible components holds not only for finite groups. It also holds for $\text{GL}(n)$ and $\text{SL}(n)$. One possible proof is similar to the one above and is known as the unitarian trick. It was introduced by Hurwitz and generalized by Weyl.

A representation of an arbitrary group that allows such a decomposition is called semi-simple or completely reducible. If all representations of $G$ have this property then the group $G$ is called reductive. This property will be essential in our discussion of invariant theory in Chapter 11. The group $G$ being reductive is equivalent to the existence of a Reynolds operator as in Lemma 11.3. Remark 10.8 says that $\text{GL}(n)$ and $\text{SL}(n)$ are reductive groups.

Example 10.9. The additive group $G = (K, +)$ is not reductive. Indeed, let us consider the following representation:

$$G \ni a \to \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \text{GL}(2).$$

The subspace of $K^2$ spanned by the second basis vector is invariant under the action. However, it does not allow an invariant complement (cf. Exercise 13).

The decomposition into irreducible representations in Maschke’s Theorem is not unique. The following example makes this clear.

Example 10.10. Any group $G$ acts on any vector space $V$ trivially by $gv = v$. Any subspace of $V$ is a subrepresentation. The irreducible subrepresentations are the one-dimensional subspaces of $V$. Hence, any decomposition into one-dimensional subspaces $V = \bigoplus_{i=1}^{\dim V} K^1$ is a decomposition into irreducible representations, but there is no distinguished one.

As we will see, the reason for nonuniqueness is the fact that distinct $V_i$’s appearing in the decomposition may be isomorphic. Let us therefore group
the isomorphic $V_i$’s together. Then we obtain the direct sum
\[(10.1) \quad V = \bigoplus V_j^{\times a_j},\]
where $V_{j_0} \simeq V_{j_1}$ if and only if $j_0 = j_1$. The subrepresentations $V_j^{\times a_j}$ are called isotypic components. The number $a_j$ is the multiplicity of the irreducible representation $V_j$ in $V$.

**Corollary 10.11** (to Schur’s Lemma). The isotypic components and multiplicities of a semi-simple representation $V$ are well defined, i.e. do not depend on the choice of the decomposition into irreducible representations.

**Proof.** Consider two decompositions of a semi-simple representation:
\[V = \bigoplus_j V_j^{\times a_j} = \bigoplus_k V_k^{\times b_k}.\]
Allowing $a_j, b_k$ to be equal to zero, we may assume that all irreducible representations occur and that the indexing in both sums $\bigoplus$ is the same.

First we prove that for a given irreducible representation $V_i$ we have $a_i = b_i$. The restriction of identity gives us an injective map:
\[m : V_i^{\times a_i} \to \bigoplus_k V_k^{\times b_k}.\]
By Schur’s Lemma, the composition of $m$ with the projection
\[\pi_s : \bigoplus_k V_k^{\times b_k} \to V_s^{\times b_s}\]
equals zero, unless $s = i$. Hence, $\text{im}(m) \subseteq V_i^{\times b_i}$. In particular, by a dimension count, $a_i \leq b_i$. Analogously $b_i \leq a_i$, i.e. the multiplicities do not depend on the decomposition. Further, the composition $\pi_s \circ m$ is an isomorphism if $s = i$ and it is zero if $s \neq i$. It follows that $\text{im}(m) = V_i^{\times b_i}$. Thus, the identity maps isotypic components to (the same) isotypic components. \(\square\)

Our next aim is to understand the irreducible representations of a given group $G$. The following definition provides us with the most important tool.

**Definition 10.12** (Character). Let $\rho : G \to \text{GL}(V)$ be a representation. The character $\chi_\rho = \chi_V$ of $\rho$ is the function $G \to K$ obtained by composing $\rho$ with the trace function $\text{Tr}$:
\[\chi_\rho(g) = \text{Tr}(\rho(g)).\]
See Remark 10.14 for the relationship to the characters in Chapter 8.

Properties of the trace function imply the following facts about characters:
- If $V = \bigoplus V_i$ then $\chi_V = \sum \chi_{V_i}$.
- If $g_1, g_2 \in G$ are conjugate then $\chi(g_1) = \chi(g_2)$ for any character $\chi$. 

• If $V_1, V_2$ are representations with characters $\chi_1, \chi_2$ then their tensor product $V_1 \otimes V_2$ is a representation with character $\chi_1 \chi_2$.

• We have $\chi_V(e) = \dim V$, where $e \in G$ is the identity element.

We now set $K = \mathbb{C}$. For a finite group $G$, we fix the following scalar product on the finite-dimensional vector space $\mathbb{C}^G$ of functions from $G$ to $\mathbb{C}$:

\[
\langle \chi_1, \chi_2 \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g).
\]

We learn in Serre’s book [50] Chapter 2] that the characters of all irreducible representations of $G$ are orthonormal with respect to this scalar product. In particular, the characters of irreducible representations are linearly independent elements in $\mathbb{C}^G$. Hence, we can find the multiplicities $a_j$ in the isotypic decomposition $V = \bigoplus_j V^{a_j}$ by decomposing the character:

$$\chi_V = \sum_j a_j \chi_j.$$ 

For any finite group $G$ there are finitely many irreducible representations. The sum of squares of their dimensions equals the order of the group [50] Chapter 2.5, Corollary 2]. A class function is a function $G \to \mathbb{C}$ that is constant on conjugacy classes. Characters of the irreducible representations form a basis of the space of class functions on $G$. One often represents these characters in a table, called the character table. This makes the decomposition of an arbitrary representation very easy, if we know its character.

**Example 10.13.** Consider the group $S_3$ of permutations of three elements. There are three conjugacy classes: the class of the identity (with one element), the class of a 3-cycle (with two elements) and the class of any transposition (with three elements). Hence, there are three irreducible representations. The first is the trivial representation $gv = v$, the second is the sign representation $gv = (\text{sgn } g)v$, and the third is the two-dimensional representation, given by the symmetries of a regular triangle. Each column in the table below represents a function $S_3 \to \mathbb{C}$ that is constant on conjugacy classes. We present the character table for the symmetric group $S_3$:

<table>
<thead>
<tr>
<th></th>
<th>Trivial representation</th>
<th>Sign repr.</th>
<th>2-dimensional repr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>identity</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>cycles $(ijk)$</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
</tr>
<tr>
<td>transpositions $(ij)$</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
</tr>
</tbody>
</table>

The reader should check these functions are orthonormal with respect to the inner product (10.2). In fact, one builds the character table of a finite group by exploiting the orthonormality of the columns. In this manner, one obtains the $5 \times 5$ character table for $S_4$ and the $7 \times 7$ character table for $S_5$. 


10.2. Invertible Matrices and Permutations

In this section we study the representations of the groups GL($n$) and SL($n$). We cannot represent their characters by tables as there are infinitely many conjugacy classes. However, we can represent each character $\chi$ by its values on the Zariski dense subset of diagonalizable matrices. Hence, we fix the algebraic torus $T = (K^*)^n \subset GL(n)$ of diagonal matrices, and we restrict the character $\chi$ to $T$. As $\chi$ is constant on any conjugacy class and any diagonalizable matrix is conjugate to an element of $T$, the function $\chi|_T$ characterizes $\chi$. Therefore, given any representation $W$ of $GL(n)$, we restrict the group and regard $W$ as a representation of $T$. By Exercise 1 and Corollary 10.11 we know that, as a representation of $T$, the space $W$ decomposes into irreducible one-dimensional representation with multiplicities. The character of each such one-dimensional representation is a morphism $t \mapsto t^b$ from $T$ to $K^*$. Hence, by Exercise 1 from Chapter 8 it may be identified with $b \in \mathbb{Z}^n$.

In conclusion, we obtain the decomposition into one-dimensional spaces

$$(10.3) \quad W = \bigoplus_{b \in \mathbb{Z}^n} W^{ab},$$

where the torus element $t = (t_1, \ldots, t_n)$ takes any vector $w \in W_b$ to its multiple $t^bw$. The isotypic components $W^{ab}$ for the $T$-action are called weight spaces. The characters $b$ of $T$ for which $a^b \neq 0$ are called weights.

**Remark 10.14.** Let $T$ be the torus of diagonal matrices $t = \text{diag}(t_1, \ldots, t_n)$ in $GL(n)$. If $\chi$ is a character of $GL(n)$ then its restriction to $T$ is the function $\chi|_T : T \to K$, $t \mapsto \text{Tr}(\rho(t))$. Here Tr denotes the trace of a (large) square matrix. The restricted character $\chi|_T$ equals

$$\chi|_T(t) = \sum_{b \in \mathbb{Z}^n} a^b t^b.$$  

This Laurent polynomial in $t_1, \ldots, t_n$ is invariant under permuting its $n$ unknowns. This means that it is a symmetric function.

**Example 10.15.** Following Example 10.3 we consider the action of $GL(n)$ on homogeneous polynomials of degree $k$. Let $\chi$ be its character. Then $\chi|_T$ is the complete symmetric polynomial of degree $k$, i.e., $\chi|_T(t)$ is the sum of all monomials $t^a$ where $a \in \mathbb{N}^n$ and $|a| = k$.

**Example 10.16.** The group $GL(n)$ acts naturally on the $k$th exterior power $V = \bigwedge^k K^n$. Write $\rho$ for this representation and $\chi$ for its character. We fix the standard basis for $V$ given by $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$. The image $\rho(g)$ of an $n \times n$-matrix $g = (g_{ij})$ is the $k$th compound matrix, or $k$th exterior power, whose entries are the (suitably signed) $k \times k$ minors of $g$. We note that the determinant of $\rho(g)$ equals $\det(g)^{\binom{n}{n-k}}$. The restricted character $\chi|_T(t)$ is the $k$th elementary symmetric polynomial in $t_1, \ldots, t_n$. 
For a concrete example, let $k = 2$. Then $\rho(g)$ is a $\binom{n}{2} \times \binom{n}{2}$ matrix whose rows and columns are labeled by ordered pairs from $\{1, 2, \ldots, n\}$. Its entry in row $(i < j)$ and column $(k < l)$ equals $g_{ik}g_{jl} - g_{il}g_{jk}$. We have $\det(\rho(g)) = \det(g)^{n-1}$ and $\chi|_T(t) = \sum_{i<j} t_i t_j$. For $k = 1$ we have $\rho(g) = g$, so $\chi|_T(t) = t_1 + t_2 + \cdots + t_n$. For $k = n$, we get the one-dimensional representation where $\rho(g)$ is the $1 \times 1$-matrix with entry $\det(g)$, so we have $\chi|_T(t) = t_1 t_2 \cdots t_n$. This representation is trivial when restricted to $\text{SL}(n)$.

Let $\rho$ be any representation of $\text{GL}(n)$. We fix the lexicographic order on the set of weights $b$ that occur in $\rho$. Of particular importance is the highest weight. The corresponding eigenvectors $w \in W_b$ in (10.3) are called highest weight vectors. They span the highest weight space. In Example 10.15, the highest weight is $(k, 0, \ldots, 0) \in \mathbb{Z}^n$, and a highest weight vector is the monomial $x^k$. In Example 10.16, the highest weight is $(1, \ldots, 1, 0, \ldots, 0)$, and a highest weight vector is $e_1 \wedge \cdots \wedge e_k$. In both cases, the highest weight space is one-dimensional. We note that the highest weight vector does not depend on $n$, provided it exists (e.g. $n \geq k$ in the exterior power case).

Example 10.17 (Adjoint representation). The space $V = K^{n \times n}$ of $n \times n$ matrices $M$ forms a representation of $\text{GL}(n)$ under the action by conjugation: $\rho(g)(M) := g M g^{-1}$. This is the adjoint representation of $\text{GL}(n)$. For $\text{SL}(n)$ one considers the same action but on the space of traceless matrices. In that case one obtains the adjoint representation of $\text{SL}(n)$. Intersecting the torus of diagonal matrices in $\text{GL}(n)$ with $\text{SL}(n)$ we obtain an $(n-1)$-dimensional torus, with $t_1 \cdots t_n = 1$. We use the convention $t_n = t_{n-1}^{-1} \cdots t_1^{-1}$. The weights of the adjoint representation, known as roots are $t_i/t_j$ with the highest weight $t_1/t_n$ represented by $(2, 1, \ldots, 1) \in \mathbb{Z}^{n-1}$. Again, the highest weight space is one-dimensional.

Proposition 10.18. Every irreducible representation of $\text{SL}(V)$ is determined (up to isomorphism) by its highest weight, and the highest weight space is one-dimensional. A weight $(a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1}$ is the highest weight for some irreducible representation if and only if $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq 0$.

Proof. For the proof we refer to [23, Chapter 15].

Here is a tool for building irreducible representations from highest weights:

Definition 10.19. A Young diagram with $k$-rows is a nonincreasing sequence of $k$ positive integers. This is also known as a partition, and it is usually presented in the following graphical form, e.g. for a sequence $(2, 1, 1)$:
10.2. Invertible Matrices and Permutations

This Young diagram encodes the weight \((2, 1, 1)\). For \(\SL(4)\) it represents the adjoint representation. For \(\SL(5)\), this Young diagram does not represent the adjoint representation, but the highest weight vector is the same.

Proposition 10.18 says that the irreducible representations of \(\SL(n)\) are in bijection with Young diagrams having at most \(n-1\) rows. Representations of \(\GL(n)\) are similar: first, every irreducible representation \(V\) of \(\GL(n)\) is also an irreducible representation of \(\SL(n)\), so it has a Young diagram \(\lambda\). However, different representations of \(\GL(n)\) give the same representation of \(\SL(n)\) if they differ by a power of the determinant. Precisely, consider a representation \(\rho : \SL(n) \to \GL(V)\) with associated Young diagram \(\lambda\). Then we have the following representations of \(\GL(n)\) for any \(a \in \mathbb{Z}\):

\[
\rho_a(g) := (\det g)^a \cdot \sqrt[\lambda]{\det g} \cdot \rho\left(\frac{1}{\sqrt[\lambda]{\det g}} \cdot g\right).
\]

Here, the argument of \(\rho\) is in \(\SL(n)\). The irreducible representations of \(\GL(n)\) are in bijection with pairs of a Young diagram with at most \(n-1\) rows and an integer \(a \in \mathbb{Z}\). The one-dimensional representation \(g \mapsto \det(g)\) of \(\GL(n)\) corresponding to \(a = 1\) may also be represented by a Young diagram with one column and \(n\) rows. Thus for \(a \geq 0\) the representation \(\rho_a\) may be represented by a Young diagram \(\lambda\) extended by \(a\) columns of height \(n\). For a vector space \(U\), the representation of \(\GL(U)\) corresponding to a Young diagram \(\lambda\) is denoted by \(S^\lambda(U)\).

Given a Young diagram \(\lambda\), we write \(\chi_\lambda\) for the character of the irreducible representation \(S^\lambda(U)\). This is a symmetric polynomial in \(t = (t_1, \ldots, t_n)\), known as the Schur polynomial of \(\lambda\). Schur polynomials include the complete symmetric polynomials in Example 10.15, for \(\lambda = (n)\), and the elementary symmetric polynomials in Example 10.16, for \(\lambda = (1,1,\ldots,1)\).

The next result gives an explicit formula for the Schur polynomials.

**Proposition 10.20.** The Schur polynomial for \(\lambda\) is the following ratio of \(n \times n\) determinants. If \(\lambda\) has less than \(n\) rows, we extend it by zeros:

\[
\chi_\lambda(t) = \frac{\det(t_i^\lambda + n-j)_{1 \leq i,j \leq n}}{\det(t_i^{n-j})_{1 \leq i,j \leq n}}.
\]

We can find the decomposition (10.1) of a representation \(V\) into irreducibles by writing the character \(\chi_V\) as linear combination of Schur polynomials \(\chi_\lambda\) with nonnegative integer coefficients \(a_j\). These coefficients are the multiplicities. This expression is unique because the Schur polynomials form a \(\mathbb{Z}\)-linear basis for the ring of symmetric polynomials in \(n\) variables.
Example 10.21. Let $n = 3$. The Schur polynomial for $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is

$$\chi_{\lambda}(t) = \frac{1}{(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)} \det \begin{pmatrix} t_1^{\lambda_1 + 2} & t_1^{\lambda_2 + 1} & t_1^{\lambda_3} \\ t_2^{\lambda_1 + 2} & t_2^{\lambda_2 + 1} & t_2^{\lambda_3} \\ t_3^{\lambda_1 + 2} & t_3^{\lambda_2 + 1} & t_3^{\lambda_3} \end{pmatrix}.$$  

From this, we compute the three Schur polynomials of degree $|\lambda| = 3$:

- $\chi_{(3,0,0)} = t_1^3 + t_1^2 t_2 + t_1 t_2^2 + t_2^3 + t_1^2 t_3 + t_1 t_2 t_3 + t_2^2 t_3 + t_2 t_3^2 + t_3^3$
- $\chi_{(2,1,0)} = (t_1 + t_2)(t_1 + t_3)(t_2 + t_3)$
- $\chi_{(1,1,1)} = t_1 t_2 t_3$

The action of $GL(3)$ on $U = K^3$ induces an action on the space $U^{\otimes 3} \simeq K^{27}$ of $3 \times 3 \times 3$-tensors. As characters are multiplicative under tensor product,

$$\chi_{U^{\otimes 3}} = (t_1 + t_2 + t_3)^3 = \chi_{(3,0,0)} + 2 \cdot \chi_{(2,1,0)} + \chi_{(1,1,1)}.$$  

From this decomposition into Schur polynomials, we conclude the following decomposition of the triple tensor product into irreducible representations:

$$U^{\otimes 3} = S^{(3)}(U) \oplus (S^{(21)}(U) \oplus S^{(21)}(U)) \oplus S^{(111)}(U).$$  

The first summand is the space of symmetric tensors, the last summand is the space of antisymmetric tensors, and the middle summand consists of two copies of the adjoint representation of $SL(U)$ (Example 10.17).

The irreducible representations $S^\lambda(U)$ of $SL(U)$ give rise to nice projective varieties. The group $SL(U)$ acts also on the projective space $\mathbb{P}(S^\lambda(U))$. The latter action has unique closed orbit, namely the orbit of the highest weight vector. Particular examples of such highest weight orbits are:

1. Consider the highest weight vector $x^k_1 = [e_1 \cdots e_1]$ in $S^k(U)$. Its orbit in $\mathbb{P}(S^k(U))$ is the $k$-th Veronese embedding of $\mathbb{P}(U)$.

2. The orbit of $[e_1 \wedge \cdots \wedge e_k] \in \mathbb{P}(\wedge^k(U))$ is the Grassmannian $G(k,U)$ in its Plücker embedding. Here $\lambda = (1, \ldots, 1)$ as in Example 10.16.

This result provides us with a unified approach to homogeneous varieties. It could be also used to build some of the representations. Fix a Young diagram $\lambda$ and let $k\lambda$ be a Young diagram where each row is scaled by $k$. Given the homogeneous variety $X$ in $\mathbb{P}(S^\lambda(U))$ we can take the $k$-th Veronese map $v_k$ of this projective space and the linear span of $v_k(X)$ is $S^{k\lambda}(U)$. A special case of this construction is point (1) above where $X = \mathbb{P}(U)$.

Next, we present a beautiful connection between the finite groups $S_n$ and the Lie groups $SL(n)$ or $GL(n)$. This is the Schur-Weyl duality. Our description follows [23, Chapter 4]. Let us begin by going back to irreducible representations of $S_n$. Their characters form a basis of class functions. Hence the number of irreducible representations equals the number of conjugacy
10.2. Invertible Matrices and Permutations

classes. Each conjugacy class can be encoded by lengths of cycles in a decomposition of a permutation into cycles. These can be further represented by a Young diagram with \( n \) boxes: the first row represents the length of the longest cycle, the last of the shortest. Thus, the number of irreducible representations of \( S_n \) equals the number of Young diagrams with \( n \) boxes.

**Example 10.22.** The symmetric group \( S_3 \) has three conjugacy classes:

- The identity \((1)(2)(3)\) with the Young diagram \( \square \);
- Transpositions, e.g. \((12)(3)\), with the Young diagram \( \begin{array}{c} \square \\ \square \end{array} \);
- 3-cycles, e.g. \((123)\), with the Young diagram \( \begin{array}{c} \square & \square & \square \end{array} \).

We shall exhibit a natural bijection between Young diagrams \( \lambda \) with \( n \) boxes and irreducible representations \( S_\lambda \) of \( S_n \). Fix a vector space \( U \) and consider the \( n \)-fold tensor power \( U^\otimes n \). There are two groups acting on it: \( \text{GL}(U) \) - on each factor - and \( S_n \) - by permuting factors. Schur-Weyl duality provides a simultaneous decomposition of \( U^\otimes n \) with respect to both groups.

**Theorem 10.23** (Schur-Weyl duality). Suppose \( \dim(U) \geq n \). Then

\[
U^\otimes n = \sum_{|\lambda|=n} S_\lambda \otimes S^\lambda(U),
\]

where the sum is over all Young diagrams \( \lambda \) with precisely \( n \) boxes.

When \( n = 2 \) and \( \dim U \geq 2 \) we obtain \( U^\otimes 2 = S^2(U) \oplus \wedge^2 U \), as there are only two irreducible representations of \( S_2 \), both one-dimensional. This recovers the fact every square matrix is uniquely the sum of a symmetric matrix and a skew-symmetric matrix. The \( S_2 \) action on matrix space \( U^\otimes 2 \) is transposition, which acts trivially on \( S^2(U) \) and changes the sign on \( \wedge^2 U \).

The case \( n = 3 \) is the first interesting one. The three irreducible representations \( S_\lambda \) of \( S_3 \) in Example 10.13 correspond to the three outer summands in \((10.5)\). Note that \( \dim(S_\lambda) = 2 \) for \( \lambda = (2,1) \). The middle summand in \((10.5)\) is the 16-dimensional space \( S_{(21)} \oplus S^{(21)}(U) \).

In general, one can use \((10.6)\) to define and construct the irreducible representations \( S_\lambda \) of the symmetric group \( S_n \). These representation are known as Specht modules. By Schur-Weyl duality, the dimension of \( S_\lambda \) equals the multiplicity \( a_\lambda \) of \( S^\lambda(U) \) in \( U^\otimes n \). These is seen from the decomposition of \( U^\otimes n \) as a \( \text{GL}(U) \) representation into isotypic components:

\[
U^\otimes n = \bigoplus a_\lambda (S^\lambda(U))^{a_\lambda}.
\]

The multiplicities \( a_\lambda \) can be found using Schur polynomials as in \((10.4)\). For each isotypic component \((S^\lambda(U))^{a_\lambda}\) consider the highest weight space.
This is the space of eigenvectors of the torus action with weight $\lambda$. The permutation group $S_n$ acts on the highest weight space. This representation of $S_n$ is irreducible. The Specht module $S_{\lambda}$ is what we need for (10.6).

Let us return to the familiar example $n = 2$ of decomposing matrices into symmetric and skew-symmetric. The highest weight vector $e_{1}e_{1} = e_{1} \otimes e_{1}$ of $S^2(U)$ is invariant with respect to transposition, i.e. it provides the trivial representation of the two-element group $S_2$. The highest weight vector $e_{1} \wedge e_{2} = \frac{1}{2}(e_{1} \otimes e_{2} - e_{2} \otimes e_{1})$ of $\wedge^2(U)$ changes sign when transposed, i.e. it provides the sign representation of the two-element group $S_2$.

**Example 10.24** ($n = 3$). We consider $\lambda = (2, 1)$ for $U^{\otimes 3}$. The isotypic component $(S^{(21)}(U))^2$ in the middle of (10.5) has a 2-dimensional subspace $S_{\lambda}$ of highest weight vectors. One possible basis of this space consists of the tensors $e_{112} + e_{211} - 2e_{121}$ and $e_{121} + e_{211} - 2e_{112}$, where $e_{ijk} := e_{i} \otimes e_{j} \otimes e_{k}$.

### 10.3. Exploiting Symmetry

Representation theory is useful for many questions concerning tensors. Properties and varieties that occur naturally tend to be invariant under group actions on tensors, and this can be exploited for algorithms and applications.

We begin by showing how to obtain lower bounds for the border rank of a tensor. This has many applications, notably in complexity theory. One well-known instance is the matrix multiplication tensor $M_n$, which we saw in Section 9.3. Bounds on rank and border rank of $M_n$ translate into complexity bounds for optimal algorithms for multiplying two $n \times n$ matrices.

Let $A, B, C$ be three vector spaces and let $T \in A \otimes B \otimes C$ be a tensor. As described in Section 9.2, we can flatten it to obtain the linear map

$$\hat{T} : A^* \rightarrow B \otimes C.$$  

(10.7)

Given bases, we can think of $\hat{T}$ as a matrix. If the tensor $T$ has rank one, i.e. $T = a \otimes b \otimes c$, then the matrix $\hat{T}$ has rank one. If $T$ is a sum of $r$ tensors of rank one then $\hat{T}$ is a sum of $r$ matrices of rank one. Thus, rank and border rank of the tensor $T$ are bounded below by the rank of the matrix $\hat{T}$.

The best possible lower bound one can possibly obtain by flattening is the maximum of $\dim(A), \dim(B), \dim(C)$. Our aim is to give larger lower bounds on the border rank of some tensors $T$. This requires new ideas. One of the tools—Strassen’s invariant—was presented in Section 9.2. We now show how representation theory can lead us to a more general result.

The groups $\text{GL}(A), \text{GL}(B)$ and $\text{GL}(C)$ act on our tensors, and this action preserves the rank. In what follows, all spaces are viewed as representations of these groups, and linear maps are morphisms of representations. We first
tensor the map $[10.7]$ with the identity on $\bigwedge^k C$. This gives the linear map

$$\tilde{T} \otimes \text{Id}_{\bigwedge^k C} : A^* \otimes \bigwedge^k C \to B \otimes C \otimes \bigwedge^k C,$$

$$f \otimes (c_1 \wedge \cdots \wedge c_k) \mapsto \tilde{T}(f) \otimes (c_1 \wedge \cdots \wedge c_k).$$

We have $\text{brk} T \geq \text{rk}(\tilde{T}) = \text{rk}(\tilde{T} \otimes \text{Id}_{\bigwedge^k C})/(\text{dim} \bigwedge^k C)$. The tensor product $C \otimes \bigwedge^k C$ is a reducible representation of $\text{GL}(C)$. For instance, the map

$$C \otimes \bigwedge^k C \to \bigwedge^{k+1} C, \quad c_0 \otimes (c_1 \wedge \cdots \wedge c_k) \mapsto c_0 \wedge c_1 \wedge \cdots \wedge c_k$$

has a non-trivial kernel. By tensoring this with $\text{Id}_B$ we obtain a linear map

$$B \otimes C \otimes \bigwedge^k C \to B \otimes \bigwedge^{k+1} C.$$ 

We now compose the last map with $\tilde{T} \otimes \text{Id}_{\bigwedge^k C}$ to obtain our final map

$$T_{k,C} : A^* \otimes \bigwedge^k C \to B \otimes \bigwedge^{k+1} C.$$

It is instructive to fix bases for $A, B, C$ and to explicitly write the matrix that represents the linear map $T_{k,C}$. Each entry in that matrix is either zero or it equals an entry of the given tensor $T = (t_{ijk})$, up to sign.

**Example 10.25.** Suppose that $A = K^2, B = K^3, C = K^4$ and $k = 2$. Then $T = (t_{ijk})$ is a $2 \times 3 \times 4$ tensor. The domain and the range of our linear map $T_{2,C}$ are the 12-dimensional vector spaces $K^2 \otimes \bigwedge^2 K^4$ and $K^3 \otimes \bigwedge^3 K^4$. With respect to their standard bases, $T_{C,2}$ is given by the $12 \times 12$-matrix

$$
\begin{pmatrix}
1|12 & 1|13 & 1|14 & 1|23 & 1|24 & 1|34 & 2|12 & 2|13 & 2|14 & 2|23 & 2|24 & 2|34 \\
1|123 & t_{113} & -t_{112} & 0 & t_{111} & 0 & 0 & t_{213} & -t_{212} & 0 & t_{211} & 0 & 0 \\
1|124 & t_{114} & 0 & -t_{112} & 0 & t_{111} & 0 & t_{214} & 0 & -t_{212} & 0 & t_{211} & 0 \\
1|134 & 0 & t_{114} & -t_{113} & 0 & 0 & t_{111} & 0 & t_{214} & -t_{213} & 0 & t_{211} & 0 \\
1|234 & 0 & 0 & 0 & t_{114} & -t_{113} & t_{112} & 0 & 0 & 0 & t_{214} & -t_{213} & t_{212} \\
2|123 & t_{123} & -t_{122} & 0 & t_{121} & 0 & 0 & t_{223} & -t_{222} & 0 & t_{221} & 0 & 0 \\
2|124 & t_{124} & 0 & -t_{122} & 0 & t_{121} & 0 & t_{224} & 0 & -t_{222} & 0 & t_{221} & 0 \\
2|134 & 0 & t_{124} & -t_{123} & 0 & 0 & t_{121} & 0 & t_{224} & -t_{223} & 0 & t_{221} & 0 \\
2|234 & 0 & 0 & 0 & t_{124} & -t_{123} & t_{122} & 0 & 0 & 0 & t_{224} & -t_{223} & t_{222} \\
3|123 & t_{133} & -t_{132} & 0 & t_{131} & 0 & 0 & t_{233} & -t_{232} & 0 & t_{231} & 0 & 0 \\
3|124 & t_{134} & 0 & -t_{132} & 0 & t_{131} & 0 & t_{234} & 0 & -t_{232} & 0 & t_{231} & 0 \\
3|134 & 0 & t_{134} & -t_{133} & 0 & 0 & t_{131} & 0 & t_{234} & -t_{233} & 0 & t_{231} & 0 \\
3|234 & 0 & 0 & 0 & t_{134} & -t_{133} & t_{132} & 0 & 0 & 0 & t_{234} & -t_{233} & t_{232} \\
\end{pmatrix}
$$

If the tensor $T$ has rank 1, so its entries factor as $t_{ijk} = x_i y_j z_k$, then the matrix $T_{2,C}$ has rank 3. Also, if $T$ has rank 2 then $T_{2,C}$ has rank 6, if $T$ has rank 3 then $T_{2,C}$ has rank 9, and if $T$ has rank $\geq 4$ then $T_{2,C}$ is invertible.
Returning to the general case, suppose that \( T = a \otimes b \otimes c \) is a rank 1 tensor in \( A \otimes B \otimes C \). Then the image the linear map \( T_{k,C} \) equals the subspace
\[
Kb \otimes \left( Kc \wedge \bigwedge^k (C/Kc) \right) \cong K^1 \otimes \bigwedge^k K^{\dim(C)-1}.
\]
In particular, the rank of \( T_{k,C} \) is at most \( \binom{\dim C - 1}{k} \) when \( T \) has rank 1. Further, all our constructions are linear in the entries \( t_{ijk} \) of the tensor \( T \), i.e. \( (T_1 + \cdots + T_r)_{k,C} = (T_1)_{k,C} + \cdots + (T_r)_{k,C} \). Thus, if \( T \) has border rank at most \( r \) then the rank of the matrix \( T_{k,C} \) is at most \( r \cdot \left( \binom{\dim C - 1}{k} \right) \). This reasoning implies the following lower bound on the border rank of a tensor.

**Proposition 10.26.** The border rank of any tensor \( T \) is bounded below by
\[
brk(T) \geq \text{rk}(T_{k,C})/\binom{\dim C - 1}{k}.
\]

This implies that the \( s \times s \) minors of the matrix \( T_{k,C} \) vanish on all tensors whose border rank is less than \( s/\binom{\dim C - 1}{k} \). This gives a method for identifying explicit polynomials that vanish on tensors of border rank up to
\[
\text{(10.8)} \quad \min\left( (\dim B)\binom{\dim C}{k}, (\dim A)\binom{\dim C}{k+1}/\binom{\dim C - 1}{k} \right).
\]
These explicit polynomials are the appropriate minors of the matrix \( T_{k,C} \).

Proposition 10.26 can now be used to give lower bounds on the border-rank of a specific tensor \( T \) we are interested in. This is done by showing that the matrix \( T_{k,C} \) has high rank for some \( k \). Here is a concrete numerical example. Fix an even integer \( m > 0 \), set \( k = m/2 \), and assume \( \dim(A) = \dim(B) = \dim(C) = m \). Thus we consider \( m \times m \times m \) tensors. The matrix \( T_{C,k} \) has format \( m(k) \times m(k+1) \). Therefore, its rank can be as large as \( m \cdot \frac{m}{(k+1)} \). If this happens then we get the following lower bound:
\[
brk(T) \geq m \cdot \frac{m}{(k+1)}/\binom{m-1}{k} = \frac{m^2}{k+1}.
\]
This should be compared with what one can hope to get from the usual flattening of \( T \). The \( m \times m^2 \) matrix \( \tilde{T} \) can at best give the bound \( \text{brk}(T) \geq \text{rk}(\tilde{T}) = m \). Thus the bound from Proposition 10.26 is almost twice as good. This technique can be generalized by replacing the exterior powers in our construction by other Schur modules. The projection into such modules are the Young symmetrizers, and the resulting generalizations of our matrices \( T_{k,C} \) are known as the Young flattenings of the tensor \( T \); see [39, Section 8.2].

Young flattenings have been a key tool recently for proving lower bounds on the complexity of matrix multiplication. This is obtained by finding a lower bound on the border rank of the matrix multiplication tensor \( T = M_n \), which has format \( m \times m \times m \) for \( m = n^2 \). To estimate the rank of the matrices \( T_{k,C} \), one also uses representation theory. This is beyond the scope of the
book. Just as an example we state an inequality due to Landsberg and Ottaviani \[40\]. They proved that the border rank of the tensor $M_n$ satisfies
\[
brk(M_n) \geq n^2(\frac{2n-1}{n-1}) / (\frac{2n-2}{n-1}) = 2n^2 - n.
\]
Further improvements on the lower bounds can be found in \[37, 38\].

Representation theory can be exploited to reduce the complexity of high-dimensional objects with symmetry. For instance, it is very useful for describing the ideals of varieties with a group action. Suppose a group $G$ acts on a projective space $\mathbb{P}^n$ and that $X \subset \mathbb{P}^n$ is a variety that is mapped to itself by $G$. We will illustrate our approach for two particular examples:

1. $X = G(2,5) \subset \mathbb{P}(\wedge^2 \mathbb{C}^5)$ is the Grassmannian and $G = \text{GL}(5)$;
2. $X = \sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3) \subset \mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$ is the secant variety of all tensors of border rank at most four and $G = \text{GL}(4) \times \text{GL}(4) \times \text{GL}(4)$.

As $G$ acts on $\mathbb{P}^n$, it also acts on the polynomial ring $K[x] = K[x_0, \ldots, x_n]$, and this restricts to an action on the radical ideal $I(X)$. The vector space $I(X)_k$ of polynomials of degree $k$ in this ideal is a representation of the group $G$. This has several advantages. Given just one polynomial $f \in I(X)$, we immediately obtain many, as $\langle Gf \rangle \subseteq I(X)$. Further, we can decompose $I(X)_k$ into irreducibles. Given just one polynomial from each irreducible piece, usually the highest weight vector, we can reconstruct the whole $I(X)_k$. This approach is extremely useful when combined with numerical methods. Indeed, $I(X)_k$ is often of huge dimension and it would be impossible to write down a basis. Yet it could happen that it has just a few irreducible representations and we just need to remember $G$ and a few polynomials.

To make this explicit, we first note that there are no linear functions vanishing on either the Grassmannian or the secant variety $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$. In both cases $I_1 = 0$ is the trivial representation. The space of linear functions on the ambient space of $G(2,5)$ equals $\wedge^2((\mathbb{C}^5)^*)$. This is the irreducible representation of $\text{GL}(5)$ given by the Young diagram $\tiny \begin{array}{c} \square \\
\end{array}$ with highest weight vector $e_1^* \wedge e_2^*$. The irreducible representations for a product of groups are the tensor products of irreducible representations of each factor. Hence, irreducible representations of $\text{GL}(4) \times \text{GL}(4) \times \text{GL}(4)$ will be represented by triples of Young diagrams, each one with at most four rows. The highest weight vector is just the tensor product of highest weight vectors.

The linear functions on the ambient space of $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ are given by $\tiny \begin{array}{c} \square \otimes \square \otimes \square \end{array}$. We see that a coordinate in the ambient space is a product of three coordinates: one for each $\mathbb{P}^3$. The space of quadrics is already more complicated, however the ideal is still trivial. Indeed, for $X = \sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ we have $I(X)_2 = I(X)_3 = I(X)_4 = 0$. This is either a challenging exercise or a less challenging literature check \[35\]. Hence, the
first interesting case is $I(X)_5$. This is a space of degree five polynomials in 64 variables. The dimension of this space equals 1728. This sounds scary, but everything becomes easy when we apply representation theory.

Let us begin by describing the quadrics in the ideal of $G(2, 5)$. The space of all quadrics equals $S^2(\Lambda^2(\mathbb{C}^5)^*)$, so its dimension is \( \binom{\binom{5}{2} + 1}{2} = 55 \). The composition of symmetric and wedge products is an example of plethysm. No general formulas are known for plethysm of irreducible representations. However, each special case can be dealt with. The above representation is a sum of two irreducibles, namely \( \begin{array}{c} \hline
\end{array} \) and \( \begin{array}{c}
\hline
\end{array} \) of dimensions 50 and 5.

From Section 10.2, we know that the second Veronese embedding of $G(2, 5)$ spans the representation \( \begin{array}{c}
\hline
\end{array} \). As the linear functions on the Veronese embedding are the quadratic functions before the embedding, we deduce that

$$\mathbb{C}[x]/I(G(2, 5))_2 \simeq \begin{array}{c}
\hline
\end{array} \quad \text{and hence} \quad I(G(2, 5))_2 \simeq \begin{array}{c}
\hline
\end{array} .$$

Writing $p_{ij}$ for $e^*_i \wedge e^*_j$ the highest weight vector of the 5-dimensional space $I(G(2, 5))_2$ is the Plücker quadric $p_{23}p_{14} - p_{13}p_{24} + p_{12}p_{34}$, which is familiar from \([5, 3]\). Indeed, it has the correct weight and belongs to the ideal.

We now pass to the secant variety $X = \sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$. We seek quintics that vanish on $4 \times 4 \times 4$ tensors of border rank four. For this we apply the method described at the beginning of this section. We first focus on tensors $T \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$. Such a tensor may be represented as three $4 \times 4$ matrices $A, B, C$. The map $T_{1,\mathbb{C}^3}$ for $e_i \in \mathbb{C}^3$ and $f \in \mathbb{C}^4$ is defined by:

$$T_{1,\mathbb{C}^3}(e_i \otimes f) = (Af) \otimes e_1 \wedge e_i + (Bf) \otimes e_2 \wedge e_i + (Cf) \otimes e_3 \wedge e_i.$$  

Note that one of the three terms on the right is zero. In matrix form,

$$T_{1,\mathbb{C}^3} = \begin{pmatrix}
0 & -C & B \\
-C & 0 & A \\
-B & A & 0
\end{pmatrix}.$$  

If the matrices $A, B, C$ are symmetric, then negating the second column block gives a skew-symmetric matrix. The principal minors of a skew-symmetric matrix with unknown entries are squares of polynomials known as Pfaffians. The vanishing of Pfaffians of size $2s$, hence of degree $s$, ensures that a skew-symmetric matrix is of rank at most $2s - 2$. In our case, by Proposition 10.26, the $10 \times 10$ Pfaffians are the degree five polynomials vanishing on $X$, as they provide the correct rank condition for $T_{1,\mathbb{C}^3}$. 
10.3. Exploiting Symmetry

For the general case, let us first assume that $B$ is invertible. Using row and column operations we can transform the above matrix to the form:

$$
\begin{pmatrix}
0 & 0 & B \\
0 & AB^{-1}C - CB^{-1}A & A \\
-B & A & 0
\end{pmatrix}.
$$

The rank of this matrix equals 8 plus the rank of $AB^{-1}C - CB^{-1}A$. In particular, the rank is 8 if and only if $AB^{-1}C - CB^{-1}A = 0$. This is not a polynomial equation, but we can transform it by multiplying with $\det(B)$:

$$(10.9) \quad A \cdot \text{adj}(B) \cdot C - C \cdot \text{adj}(B) \cdot A = 0.$$

Here $\text{adj}(B)$ is the adjoint matrix, whose entries are the $3 \times 3$ minors of $B$. Hence the entries of $(10.9)$ have degree $5 = 1 + 3 + 1$ in the entries of $A, B, C$. They are known as Strassen equations. If we specialize $(10.9)$ to symmetric matrices $A, B, C$, then we obtain the Aronhold invariant in (9.15).

The upper left entry of $(10.9)$ is a quintic with 180 terms, but it uses only 30 out of all 64 variables. It is the highest weight vector of the representation

This is irreducible of dimension 576. By permuting the three factors, we get two further irreducibles. Together they span $I(X)_5$, which has dimension $1728 = 3 \cdot 576$. We construct all of these quintics by applying the symmetric group $S_3$ and $\text{GL}(4) \times \text{GL}(4) \times \text{GL}(4)$ to one small quintic with 180 terms.

There is one big difference between our Grassmannian and our secant variety. A theorem of Kostant states that the ideal of a homogeneous variety, such as a Grassmannian, is generated by quadrics. By contrast, ideals of higher secant varieties are only known in very few cases. The above ideal $I(X)$ is not generated in degree five. Describing the generators of $I(X)$ is known as the Salmon Problem. Salmon is not a mathematician, but a fish. The problem was posed by Elizabeth Allman, to highlight the importance of secant variety $X$ for phylogenetics. It represents the general Markov model for a tree with three leaves attached to the root, each with four states.

We stress the fact that there are many practical applications of representation theory. For instance, the book of Serre [50], to which we referred several times, grew out of lectures for quantum chemists and physicists. The book by Diaconis [17] highlights applications in probability and statistics.
Exercises

(1) (a) Prove that, over an algebraically closed field, every irreducible representation of an abelian group is one-dimensional.

(b) Explain the correspondence between characters of a torus \( T = (\mathbb{C}^*)^n \), as defined in Chapter 8, and irreducible representations of \( T \).

(2) Derive the character table of the symmetric group \( S_4 \). Hint:

\[
1^2+1^2+2^2+3^2+3^2 = 24.
\]

What is the geometric meaning of the 3-dimensional representations?

(3) Let \( f : V_1 \rightarrow V_2 \) be a morphism between representations of a group \( G \). Show that the kernel, image and cokernel of \( f \) are also representations of \( G \). Prove also that morphisms between two representations are closed under taking scalar multiples and sums, i.e. they form a vector space.

(4) Derive the character table of the symmetric group \( S_5 \). Hint:

\[
1^2+1^2+2^2+4^2+4^2+5^2+5^2+6^2 = 120.
\]

Write matrices \( \rho(g) \) for the 6-dimensional irreducible representation.

(5) Let \( V_1 \) and \( V_2 \) be two representations of a group \( G \).

(a) Prove that \( \text{Hom}(V_1, V_2) \) is also a representation of \( G \). How can you characterize morphisms of representations inside \( \text{Hom}(V_1, V_2) \)?

(b) In terms of multiplicities of isotypic components, what is the dimension of the space of morphisms of representations \( V_1 \rightarrow V_2 \)?

(c) Conclude that the multiplicity of an irreducible representation \( W \) in \( V_1 \) equals the dimension of morphisms of representations \( W \rightarrow V_1 \).

(6) Let \( V \) be a representation of \( \text{GL}(n) \). Its character \( \chi_V \) is a Laurent polynomial in \( t_1, \ldots, t_n \). Show that \( S^2(V) \) and \( \wedge^2 V \) are also representations of \( \text{GL}(V) \), and compute the characters \( \chi_{S^2(V)} \) and \( \chi_{\wedge^2 V} \) in terms of \( \chi_V \).

(7) Describe the 2-dimensional irreducible representation from Example 10.13 explicitly. Map each of the six permutations of \( \{1, 2, 3\} \) to a \( 2 \times 2 \) matrix.

(8) Let \( \rho \) be the action of \( \text{GL}(6) \) on \( \wedge^3 \mathbb{K}^6 \). What is the highest weight? What is the Young diagram? Find the entries of the \( 20 \times 20 \)-matrix \( \rho(g) \).

(9) Is every \( 2 \times 2 \times 2 \) tensor a symmetric tensor plus a skew-symmetric tensor?

(10) If \( U = \mathbb{K}^n \), what is the dimension of \( S^3(\mathbb{K}) \)? Give a polynomial in \( n \).

(11) What is the dimensions of the vector space \( S^3(S^3(\mathbb{K})) \)? Find a weight basis. Write down the character of this representation of \( \text{GL}(3) \). Can you decompose it into Schur polynomials? What does plethysm mean?
(12) What are the orbits of points in the adjoint representation? Are they closed? What is the dimension of a general orbit? What is the vanishing ideal of such an orbit, e.g. for \( n = 3 \)?

(13) Show that the representation \( K^2 \) of the additive group \((K, +)\) in Example 10.9 is not a (nontrivial) sum of irreducible representations.

(14) What is the determinant of the “large matrix” from Example 10.3?

(15) What is the determinant of the \( 12 \times 12 \)-matrix \( T_{C;2} \) in Example 10.25?

(16) Let \( A = K^3, B = C = K^4 \) and \( k = 2 \). What ranks are possible for the matrix \( T_{C;2} \)? Give an explicit tensor \( T \) such that \( T_{C;2} \) has maximal rank.

(17) Write the space \( I(G(3, 6)_2) \) in Example 5.9 as a representation of \( GL(6) \).
Chapter 11

Invariant Theory

“Like the Arabian phoenix rising out of its ashes, the theory of invariants ... is once again at the forefront of mathematics.”
Gian-Carlo Rota

What is geometry? An answer to this question was proposed by Felix Klein’s *Erlanger Programm*. According to Klein, a quantity is geometric if it is invariant under the action of an underlying group of transformations. Thus, in short, geometry is invariant theory. For example, Euclidean geometry is the study of quantities, expressed in the coordinates of points, that are invariant under the Euclidean group. From the modern point of view, invariant theory can be seen as a branch of representation theory. However, that view does not do justice to the tremendous utility of invariant theory for dealing with geometric objects. In particular, in algebraic geometry, invariants are used to construct quotients of algebraic varieties modulo groups that act on them. This results in a concise description of orbit spaces. The study of such spaces is *Geometric Invariant Theory*. Our aim in this chapter is to give a first introduction to this theory, starting with actions by finite groups.

11.1. Finite Groups

We fix the polynomial ring $K[x] = K[x_1, \ldots, x_n]$ over a field $K$ of characteristic zero. The group $\text{GL}(n, K)$ of invertible $n \times n$ matrices acts on $K^n$. This induces an action on the ring of polynomial functions on $K^n$. Namely, if $\sigma = (\sigma_{ij})$ is a matrix in $\text{GL}(n, K)$ and $f$ is a polynomial in $K[x]$ then $\sigma f$ is the polynomial that is obtained from $f$ by replacing the variable $x_i$ by the linear form $\sum_{j=1}^n \sigma_{ij} x_j$ for $i = 1, \ldots, n$. 
11. Invariant Theory

Let $G$ be a subgroup of $\text{GL}(n, K)$. A polynomial $f \in K[x]$ is an invariant of the group $G$ if $\sigma f = f$ for all $\sigma \in G$. We write $K[x]^G$ for the set of all such invariants. This set is a subring of the polynomial ring $K[x]$ because the sum of two invariants is again an invariant, and same for the product.

In this chapter we discuss two scenarios. In this section we consider finite groups $G$, and in the next one we consider representations of nice, i.e. reductive, infinite groups like $\text{SL}(d, K)$ and $\text{SO}(d, K)$. A celebrated theorem of Hilbert shows that the invariant ring is finitely generated in this case. After two initial examples, we begin by proving this for finite groups $G$.

**Example 11.1.** Let $G$ be the group of $n \times n$ permutation matrices. The invariant ring $K[x]^G$ consists of all polynomials $f$ that are invariant under permuting the coordinates. This means that $f$ satisfies

$$f(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n}) = f(x_1, x_2, \ldots, x_n)$$

for all permutations $\pi$ of $\{1, 2, \ldots, n\}$.

Such polynomials are called symmetric. The invariant ring $K[x]^G$ is generated by the $n$ elementary symmetric polynomials $E_1, \ldots, E_n$ defined below. See [56, Theorem 1.1.1] for a proof. The elementary symmetric polynomials $E_i$ are the coefficients of the following auxiliary polynomial in one variable $z$:

$$(11.1) \quad (z + x_1)(z + x_2) \cdots (z + x_n) = z^n + \sum_{i=1}^{n} E_i(x)z^{n-i}.$$ 

We also set $E_0 = 1$. Alternatively, since $\text{char}(K) = 0$, the invariant ring $K[x]^G$ can also be generated by the power sums

$$P_j(x) = x_1^j + x_2^j + \cdots + x_n^j \quad \text{for } j = 1, 2, \ldots, n.$$ 

The formulas that write the elementary symmetric polynomials $E_i$ in terms of the power sums $P_j$, and vice versa, are known as Newton’s Identities:

$$(11.2) \quad kE_k = \sum_{i=1}^{k} (-1)^{i-1}E_{k-i}P_i \quad \text{and} \quad P_k = (-1)^{k-1}kE_k + \sum_{i=1}^{k-1} (-1)^{k-1-i}E_{k-i}P_i \quad \text{for } 1 \leq k \leq n.$$ 

Next, we provide geometric motivation for studying the ring of invariants. Suppose a group $G$ acts on an $n$-dimensional vector space $K^n$. Our aim is to describe the space of orbits $Q$, i.e. a geometric object, whose points correspond to orbits. We are not claiming that such a space has always the structure of a variety, but let us assume this for a moment. Following the approach presented in Chapters 1 and 2 we try to describe $Q$ through polynomial functions on it. By mapping each point to the orbit it belongs to, we expect to get a quotient map $K^n \to Q$. By dualizing, we should get an inclusion from the ring of polynomial functions on $Q$ into $K[x]$. Note that a function on $Q$ gives rise to a polynomial that is constant on the orbits of $G$. 


As invariants are exactly the polynomial functions that are constant along \( G \)-orbits, we see that \( K[Q] \) maps to \( K[x]^G \). Thus invariants offer an algebraic view on the space of orbits. We may define \( Q \) as the spectrum of \( K[x]^G \). In the quotient space \( Q = \text{Spec } K[x]^G \), also denoted by \( K^n//G \), the (closed) points should correspond to the orbits. This interpretation is only informal, as the details are actually quite subtle. Note, in particular, that we have not proved that indeed there is a bijection between (closed) points of \( Q \) and orbits. This is the case when e.g. \( G \) is finite, but not in general. Making it all precise is the aim of Geometric Invariant Theory.

**Example 11.2.** For \( n = 2 \), consider the following representation of the cyclic group of order 4. These are the rotational symmetries of the square:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
-1 & 0 \\
0 & -1 \\
0 & -1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
-1 & 0 \\
-1 & 0 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
0 & -1 \\
-1 & 1 \\
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

As we will prove in Example 11.7, its invariant ring is generated by

\[
I_1 = x_1^2 + x_2^2, \quad I_2 = x_1^2 x_2^2, \quad I_3 = x_1^3 x_2 - x_1 x_2^3.
\]

These three invariants are algebraically dependent. Using implicitization, we compute the cubic relation they satisfy. With this, we can write

\[
K[x_1, x_2]^G = K[I_1, I_2, I_3] \cong K[y_1, y_2, y_3]/\langle y_1^2 y_2 - 4y_2^2 - y_3^2 \rangle.
\]

The spectrum of the ring (11.4) corresponds to the cubic surface in \( K^3 \) defined by the equation \( y_1^2 y_2 = 4y_2^2 + y_3^2 \). The points on this surface are in one-to-one correspondence with the \( G \)-orbits on \( K^2 \). This bijection is given by the map \( K^2 \to K^3, (x_1, x_2) \mapsto (I_1, I_2, I_3) \) which is constant on orbits.

In what follows, let \( G \) be a finite subgroup of \( \text{GL}(n, K) \). One can create invariants by averaging polynomials. The Reynolds operator, denoted by a star, is the following map from arbitrary polynomials to invariants:

\[
* : K[x] \to K[x]^G, \quad f \mapsto f^* := \frac{1}{|G|} \sum_{\sigma \in G} \sigma f.
\]

Each of the following properties of the Reynolds operator is easily verified:

**Lemma 11.3.** The Reynolds operators * has the following three properties:

(a) The map * is \( K \)-linear, i.e. \((\lambda f + \nu g)^* = \lambda f^* + \nu g^* \) for all \( f, g \in K[x] \) and \( \lambda, \nu \in K \).

(b) The map * restricts to the identity on \( K[x]^G \), i.e. \( I^* = I \) for all invariant polynomials \( I \).

(c) The map * is a \( K[x]^G \)-module homomorphism, i.e. \((fI)^* = f^* I \) for all \( f \in K[x] \) and \( I \in K[x]^G \).

The next result marks the beginning of Commutative Algebra in 1890.
Theorem 11.4 (Hilbert’s Finiteness Theorem). The invariant ring $K[x]^G$ of any finite matrix group $G \subset \text{GL}(n, K)$ is finitely generated as a $K$-algebra.

We present the proof under the hypothesis that $K$ has characteristic zero. However, the result holds for every field $K$. For a proof in characteristic $p$ see [16]. That setting is known as modular invariant theory.

Proof. Let $\mathcal{I}_G = \langle K[x]^G \rangle$ be the ideal in $K[x]$ that is generated by all homogeneous invariants of positive degree. By Lemma 11.3 (a), every invariant is a $K$-linear combination of symmetrized monomials $(x^a)^*$. Thus $\mathcal{I}_G$ is generated by the infinite set $\{ (x^a)^* : a \in \mathbb{N}^n \setminus \{0\} \}$. By Hilbert’s Basis Theorem 1.14, the ideal $\mathcal{I}_G$ is finitely generated, so that a finite subset of integer vectors $a$ in $\mathbb{N}^n$ suffices for generation. In conclusion, there exist homogeneous invariants $I_1, I_2, \ldots, I_m$ such that $\mathcal{I}_G = \langle I_1, I_2, \ldots, I_m \rangle$.

We claim that these $m$ invariants generate the invariant ring $K[x]^G$ as a $K$-algebra. Suppose the contrary, and let $I$ be a homogeneous element of minimal degree in $K[x]^G \setminus K[I_1, I_2, \ldots, I_m]$. Since $I \in \mathcal{I}_G$, we have $I = \sum_{j=1}^m f_j I_j$ for some polynomials $f_j \in K[x]$. As the invariants $I_j$ are homogeneous, so are the polynomials $f_j$ and their degrees are all strictly less than the degree of the invariant $I$ that is not in the subring.

We now apply the Reynolds operator on both sides of the equation $I = \sum_{j=1}^m f_j I_j$. This yields

$$I = I^* = \left( \sum_{j=1}^m f_j I_j \right)^* = \sum_{j=1}^m f_j^* I_j.$$

Here we are using the properties (b) and (c) in Lemma 11.3. The new coefficients $f_j^*$ are homogeneous invariants whose degrees are less than $\deg(I)$. From the minimality assumption on the degree of $I$, we get $f_j^* \in K[I_1, \ldots, I_m]$ for $j = 1, \ldots, m$. This implies $I \in K[I_1, \ldots, I_m]$, which is a contradiction to our assumption. This completes the proof of Theorem 11.4 $\square$

Theorem 11.5 (Noether’s Degree Bound). If $G$ is finite and $\text{char}(K) = 0$ then the invariant ring $K[x]^G$ is generated by homogeneous invariants whose degree is at most the group order $|G|$.

Proof. Let $u = (u_1, \ldots, u_n)$ be new variables. We extend the action of $G$ to $K[u, x]$ by $gu_i = u_i$ for all $i$. For any $d \in \mathbb{N}$, we consider the expression

$$S_d(u, x) = \left[ (u_1 x_1 + \cdots + u_n x_n)^d \right]^* = \frac{1}{|G|} \sum_{\sigma \in G} \left[ u_1 (\sigma x_1) + \cdots + u_n (\sigma x_n) \right]^d.$$

This is a polynomial in $u$ whose coefficients are polynomials in $x$. Up to a multiplicative constant, these coefficients are the invariants $(x^a)^*$ where $|a| = d$. By definition, all polynomials in $u$ are fixed under the action of $\ast$.  


Consider the $|G|$ expressions $u_1(\sigma x_1) + \cdots + u_n(\sigma x_n)$, one for each group element $\sigma \in G$. The polynomial $S_d(u, x)$, up to a multiplicative constant, is the $d$th power sum of these expressions. The power sums for $d > |G|$ are polynomials in the first $|G|$ power sums. Such a representation is derived from Newton’s Identities (11.2). It implies that all $u$-coefficients of $S_d(u, x)$ for $d > |G|$ are polynomial functions in the $u$-coefficients of $S_d(u, x)$ for $d \leq |G|$. Hence all invariants $(x^a)^*$ with $|a| > |G|$ are polynomial functions (over $K$) in the invariants $(x^b)^*$ with $|b| \leq |G|$. This proves the claim.

We note that Example 11.2 attains Noether’s degree bound. Here, the group has order 4, and the invariant ring requires a generator of degree 4.

Our next theorem is a useful tool for constructing the invariant ring. It says that we can count invariants by averaging the reciprocal characteristic polynomials of the group elements.

**Theorem 11.6 (Molien).** The Hilbert series of the invariant ring $K[x]^G$ is the rational generating function

$$
\sum_{d=0}^{\infty} \dim_K(K[x]^G_d)z^d = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{\det(\text{id} - z\sigma)}.
$$

The coefficient of $z^d$ in this formal generating function is the number of linearly independent invariants of degree $d$.

**Proof.** See [56, Theorem 2.2.1].

**Example 11.7.** Consider the cyclic group $G = \mathbb{Z}_4$ in Example 11.2. For the four matrices $\sigma$ in Example 11.2, the quadratic polynomials $\det(\text{id} - z\sigma)$ are $(1-z)^2$, $(1+z)^2$ and twice $1+z^2$. Adding up their reciprocals and dividing by $|G| = 4$, we see that the Hilbert series of $K[x]^G$ is

$$
\frac{1 + z^4}{(1-z^2)(1-z^4)} = 1 + z^2 + 3z^4 + 3z^6 + 5z^8 + \cdots.
$$

This agrees with the Hilbert series of the ring on the right in (11.4), where $\deg(y_1) = 2$ and $\deg(y_2) = \deg(y_3) = 4$. Indeed, we see that every element of that ring can be uniquely represented as a sum of a polynomial in $K[y_1, y_2]$ and $y_3$ times a polynomial in $K[y_1, y_2]$. One says that the ring is a free module with basis $\{1, y_3\}$ over $K[y_1, y_2]$. This explains the numerator and denominator on the left of (11.7), and it proves that the invariants $I_1, I_2, I_3$ do indeed generate $K[x]^G$.

### 11.2. Classical Invariant Theory

Hilbert’s Finiteness Theorem also holds for an infinite group $G \subset \text{GL}(n, K)$ that has a Reynolds operator $*$ which satisfies the properties (a), (b) and (c)
in Lemma 11.3. This is the case when $G$ is reductive. Indeed, homogeneous polynomials of degree $d$ in $n$ variables form a representation $K[x]^G_d$ and $K[x]^G_d$ is a subrepresentation. If we know there exists a complementary subrepresentation $H$ such that $H \oplus K[x]^G_d = K[x]_d$, then we may define $\ast$ as the projection with kernel $H$. Summing over all degrees $d \in \mathbb{N}$ and extending the projection $\ast$ linearly to $K[x]$, we obtain the Reynolds operator for $G$.

**Corollary 11.8.** Fix a reductive group $G$ of $n \times n$-matrices. If $\{g_1, g_2, \ldots, g_m\}$ is a set of homogeneous polynomials that generates the ideal $\mathcal{I}_G$ then its image $\{g_1^\ast, g_2^\ast, \ldots, g_m^\ast\}$ under the Reynolds operator generates the invariant ring $K[x]^G$ as a $K$-algebra.

**Proof.** Let $M = \langle x_1, \ldots, x_n \rangle$ be the homogeneous maximal ideal in $K[x]$, and consider the finite-dimensional vector space $\mathcal{I}_G/M \mathcal{I}_G$. It has a basis of invariants since $\mathcal{I}_G$ is generated by invariants. This means that the Reynolds operator acts as the identity on $\mathcal{I}_G/M \mathcal{I}_G$. The images of $g_1, g_2, \ldots, g_m$ also span $\mathcal{I}_G/M \mathcal{I}_G$ as a vector space, and hence so do the invariants $g_1^\ast, g_2^\ast, \ldots, g_m^\ast$. By Nakayama’s Lemma, we find that $g_1^\ast, g_2^\ast, \ldots, g_m^\ast$ generate the ideal $\mathcal{I}_G$. As in the proof of Theorem 11.4 we conclude that $g_1^\ast, g_2^\ast, \ldots, g_m^\ast$ generate the $K$-algebra $K[x]^G$. □

Classical invariant theory was primarily concerned with the case when $G$ is a representation of the group $\text{SL}(d, K)$ of $d \times d$-matrices with determinant 1. Here $d$ is an integer that is usually much smaller than $n$. We continue to assume that $K$ is a field of characteristic zero. We have assumed that $G$ is the image of a group homomorphism $\text{SL}(d, K) \rightarrow \text{GL}(n, K)$. It is known that $\text{SL}(d, K)$ is a reductive group, i.e. there also exists an averaging operator $\ast : K[x] \rightarrow K[x]^G$ which has the same formal properties as the averaging operator of a finite group, stated in Lemma 11.3. See Remark 10.8.

That Reynolds operator $\ast$ can be realized either by integration or by differentiation. In the first realization, one replaces the sum in (11.5) by an integral. Namely, one takes $K = \mathbb{C}$ and one integrates over the compact subgroup $\text{SU}(d, \mathbb{C})$ with respect to Haar measure. The same kind of integral also works in Theorem 11.6. If $G = \text{SL}(d, \mathbb{C})$ then one can compute the Hilbert series of the invariant ring by averaging reciprocal characteristic polynomials. In other words, Molien’s Theorem generalizes to $G = \text{SL}(d, \mathbb{C})$.

An alternative to integrating with respect to Haar measure on $\text{SU}(d, \mathbb{C})$ is a certain differential operator known as Cayley’s $\Omega$-process. This process, which is explained in [56, Section 4.3], can also be used to transform arbitrary polynomials into invariants.

A third method for computing invariants is plain old linear algebra. Indeed, suppose we fix an integer $d \in \mathbb{N}$ and we seek a basis for the space $K[x]^G_d$
of homogeneous invariants of degree $d$. We then pick a general polynomial $f$ of degree $d$ with unknown coefficients, and we examine the equations $\sigma f = f$ for $\sigma \in G$. Each of these translates into a linear system of equations in the unknown coefficients of $f$. By taking enough matrices $\sigma$, we obtain a linear system of equations whose solutions are precisely the invariants of degree $d$. In the case when $G$ is a connected Lie group, like $\text{SL}(d, \mathbb{C})$, one can replace the condition $\sigma f = f$ by requiring that $f$ is annihilated by the associated Lie algebra. Setting up these linear equations and solving them is usually quite efficient on small examples. See [56], Section 4.5.

In what follows we take the matrix group to be an $n$-dimensional polynomial representation of $G = \text{SL}(d, K)$ for some $d, n \in \mathbb{N}$. This is a direct sum of irreducible representations, indexed by partitions, as in Chapter 10.

Example 11.9. Let $U = (K^d)^m$ be the space of $d \times m$-matrices. The group $G = \text{SL}(d, K)$ acts naturally on column vectors of length $d$, so we view $U$ as the direct sum of $m$ copies of the defining representation, one copy for each column in our matrix. Equivalently, $G$ acts on $U$ by matrix multiplication on the left. This induces an action on the ring $K[U]$ of polynomials in the entries of a $d \times m$ matrix of variables. If $m < d$ then this action has no non-constant invariants. If $m \geq d$ then the $\binom{m}{d}$ maximal minors of the $d \times m$ matrix are invariants. This invariance holds because the determinant of the product of two $d \times d$-matrices is the product of the determinants. It is known that the invariant ring $K[U]^G$ is generated by these $\binom{m}{d}$ determinants. This is the First Fundamental Theorem of Invariant Theory; cf [56], Section 3.2.

Note that we already encountered the invariant ring $K[U]^G$ in Chapter 5. This ring is the coordinate ring of the Grassmannian of $d$-dimensional subspaces in $K^m$. Thus, $K[U]^G$ is isomorphic to a polynomial ring in $\binom{m}{d}$ variables, modulo the ideal of quadratic Plücker relations.

Arguably, the most important irreducible representations of the group $G = \text{SL}(d, K)$ are the $p$-th symmetric powers of the defining representation $K^d$. Here $p$ can be any positive integer. We denote such a symmetric power by $V = K[u_1, \ldots, u_d]_p = \text{Sym}_p(K^d)$. Its elements are homogeneous polynomials of degree $p$ in $d$ variables. The $G$-module $V$ has dimension $n = \binom{p+d-1}{p}$. A basis consists of the monomials of degree $p$ in $d$ variables. The action of $G$ on $V$ is simply by linear change of coordinates.

Example 11.10 ($d=2, p=3$). Fix the space $V = \text{Sym}_3(K^2)$ of binary cubics

$$f(u_1, u_2) = x_1u_1^3 + x_2u_1^2u_2 + x_3u_1u_2^2 + x_4u_2^3.$$  

The coefficients $x_i$ are the coordinates on $V \simeq K^4$. The way we set things up, the group $\text{SL}(2, K)$ acts on this space by left multiplication, in its guise
as the group $G$ of $4 \times 4$-matrices of the form

$$\phi(\sigma) = \begin{bmatrix} \sigma_1^2 & \sigma_1^2 \sigma_{12} & \sigma_{11} \sigma_{12}^2 & \sigma_{12}^3 \\ 3 \sigma_1^2 \sigma_{21} & \sigma_2^2 + 2 \sigma_{11} \sigma_{12} \sigma_{21} & \sigma_{12}^2 + 2 \sigma_{11} \sigma_{12} \sigma_{22} & 3 \sigma_{12} \sigma_{22}^2 \\ 3 \sigma_{11} \sigma_{21}^2 & \sigma_{12} \sigma_{21} + 2 \sigma_{11} \sigma_{12} \sigma_{22} & \sigma_{12} \sigma_{22}^2 + 2 \sigma_{11} \sigma_{12} \sigma_{22} & 3 \sigma_{12} \sigma_{22}^2 \\ \sigma_{21}^2 \sigma_{22} & \sigma_{21} \sigma_{22}^2 & \sigma_{22}^3 & \end{bmatrix}. $$

For $\sigma \in G = \text{SL}(2, K)$, the determinant of this $4 \times 4$-matrix $\phi(\sigma)$ is equal to $(\sigma_{11} \sigma_{22} - \sigma_{12} \sigma_{21})^6 = 1$. The $G$-action on $V$ is given by $x \mapsto \phi(\sigma)x$ where $x$ is the column vector $(x_1, x_2, x_3, x_4)^T$. One invariant under this action is the discriminant of the binary cubic $f(u_1, u_2)$, which is

$$\Delta = 27x_1^2 x_4^2 - 18x_1 x_2 x_3 x_4 + 4x_1 x_3^3 + 4x_2^3 x_4 - x_2^2 x_3^2.$$ 

It turns out that the discriminant generates the invariant ring:

$$K[x]^G = K[\Delta].$$

Every $G$-invariant can be written uniquely as a polynomial in $\Delta$.

Invariants of binary forms ($d = 2$) are a well-studied subject in invariant theory. Complete lists of generators for the invariant ring are known up to degree $p = 10$. For $p = 2$, there is also only the discriminant $\Delta = x_2^2 - 4x_1 x_3$. For $p = 4$, we have two generating invariants of degree 2 and 3 respectively. However, for $p = 10$, the invariant ring has 106 minimal generators.

### 11.3. Geometric Invariant Theory

According to Felix Klein, invariant theory plays a fundamental role for geometry. A polynomial in the coordinates of a space is invariant under the group of interest if and only if that polynomial expresses a geometric property. For instance, consider the space $V$ of binary cubics $f$ in Example 11.10. The hypersurface defined by $f$ in $\mathbb{P}^1$ consists of three points. The vanishing of the invariant $\Delta$ means that these three points are not all distinct.

In geometric invariant theory, one considers the variety $V(I_G)$ defined by all homogeneous invariants of positive degree. This variety is known as the nullcone. Its points are known as unstable points. For a finite group $G$, the nullcone consists just of the origin. In symbols, $V(I_G) = \{0\}$.

For $G = \text{SL}(d, K)$ the situation is more interesting, and the geometry of the nullcone is very important for understanding the invariant ring $K[x]^G$. Corollary [11.8] says, more or less, that computing $K[x]^G$ is equivalent to finding polynomial equations that define the nullcone.

**Example 11.11** ($d = p = 3$). We consider the 10-dimensional vector space $V = \text{Sym}_3(K^3)$ whose elements are the ternary cubics

$$f(u) = x_1 u_1^3 + x_2 u_2^3 + x_3 u_3^3 + x_4 u_1^2 u_2 + x_5 u_2^2 u_3 + x_6 u_1^2 u_3 + x_7 u_2^2 u_3 + x_8 u_3^2 u_1 + x_9 u_3^2 u_2 + x_{10} u_1 u_2 u_3.$$
11.3. Geometric Invariant Theory

The group $G = \text{SL}(3, K)$ acts on $V$ by linear change of coordinates. The corresponding invariant ring is generated by two invariants $I_4$ and $I_6$ of degrees 4 and 6 respectively. In symbols, $K[x]^G = K[I_4, I_6]$. The degree 4 invariant unique up to scaling. It is the following sum of 25 monomials:

$$I_4 = x_0^4 - 8x_0^2x_4x_9 - 8x_0^2x_5x_7 - 8x_0^2x_6x_8 - 216x_0x_1x_2x_3 + 24x_0x_1x_7x_9 + 24x_0x_2x_5x_8 + 24x_0x_3x_4x_6 + 24x_0x_4x_7x_8 + 144x_1x_2x_8x_9 + 24x_0x_5x_6x_9 + 144x_1x_3x_6x_7 - 48x_1x_6x_9^2 - 48x_1x_7x_8 - 48x_2x_4x_8^2 + 144x_2x_3x_4x_5 - 48x_2x_6x_9^2 - 48x_3x_2x_7^2 - 48x_3x_5x_6^2 + 16x_1^2x_9^2 - 16x_4x_5x_7x_9 - 16x_4x_6x_8x_9 + 16x_5^2x_7^2 - 16x_5x_6x_7x_8 + 16x_6^2x_8^2.$$

The degree 6 invariant is similarly unique. It is a sum of 103 monomials:

$$I_6 = x_0^6 - 12x_0^4x_4x_9 - 12x_0^4x_5x_7 - 12x_0^4x_6x_8 + 540x_0^3x_1x_2x_3 + \cdots + 96x_3^2x_6^2x_7^2 - 64x_6^3x_8^3.$$

The invariant $I_4$ is the Aronhold invariant. This plays an important role in the theory of tensor decomposition. Indeed, we can regard $f$ as a symmetric $3 \times 3 \times 3$-tensor. A random tensor $f$ has rank 4. The Aronhold invariant $I_4$ vanishes for those tensors of border rank $\leq 3$. In other words, $I_4 = 0$ holds if and only if $f$ is a sum of three cubes of linear forms, or can be approximated by a sequence of such. See the discussion of ranks of tensors in Chapter 9.

On the geometric side, we can identify $f$ with the cubic curve $V(f)$ it defines in the projective plane $\mathbb{P}^2$. To a number theorist, this is an elliptic curve. An important invariant of this curve is the discriminant $\Delta$.

This discriminant of a ternary cubic has degree 12. An explicit formula is

$$\Delta = I_4^3 - I_6^2. \quad (11.10)$$

This expression vanishes if and only if the cubic curve $V(f)$ has a singular point. Typically, this singularity is a node. In the special case when both $I_4$ and $I_6$ vanish, that singular point is a cusp. Thus, for ternary cubics, the nullcone $V(I_G)$ is given by plane cubics that have a cusp. The moduli space of elliptic curves is parametrized by the $j$-invariant, which equals $I_4^3/\Delta$.

We now present a general purpose algorithm, due to Harm Derksen, for computing the invariant ring of a reductive algebraic group $G$ that acts polynomially on a vector space $V = K^n$. The group $G$ can be represented as an algebraic variety inside $\text{GL}(n, K)$, that is, by polynomial equations in the entries of an unknown $n \times n$-matrix. This works for both finite groups and for polynomial representations of $\text{SL}(d, K)$, such as the ones discussed above. As before, we use the notation $\sigma \mapsto \phi(\sigma)$ to write the representation of $G$ on $V = K^n$ explicitly. The ring $K[\sigma]$ denotes the coordinate ring of $G$, which is typically represented as a polynomial ring modulo an ideal.

The product $G \times V \times V$ is an algebraic variety, with coordinates $(\sigma, x, y)$. Inside this coordinate ring $K[\sigma, x, y]$, let $J_G$ be the ideal generated by the
n entries of the vector \( y - \phi(\sigma)x \). This ideal is radical, and it is prime when \( G \) is a connected group like SL(\( d, K \)). Its variety describes the action of the group. The elimination ideal \( J_G \cap K[x, y] \) is also radical (resp. prime). Its variety contains pairs of points in \( V \) that lie in the same \( G \)-orbit.

**Theorem 11.12 (Derksen’s Algorithm).** The ideal \( I_G \) of the nullcone is the image in \( K[x] \) of the elimination ideal \( J_G \cap K[x, y] \) under the substitution \( y = 0 \). From any finite list of ideal generators of \( I_G \), algebra generators for the invariant ring \( K[x]^G \) are found via Corollary 11.18.

**Proof.** Let \( I \) be any homogeneous invariant of positive degree. Then \( I(x) \equiv I(\phi(\sigma)x) \equiv I(y) \) modulo the ideal \( J_G \) that defines the group action. Therefore, \( I(x) - I(y) \) lies in the elimination ideal \( J_G \cap K[x, y] \). We find \( I(x) \) in the ideal that is obtained by substituting \( y = 0 \). This proves that \( I_G \) is contained in the ideal output by Derksen’s Algorithm. For the converse direction, we refer to the argument given in the proof of [15, Theorem 3.1]. □

**Example 11.13 (p=2).** Consider the 3-dimensional space \( V = \text{Sym}_2(K^2) \) of binary quadrics

\[
f(u_1, u_2) = x_1u_1^2 + x_2u_1u_2 + x_3u_2^2.
\]

The coordinate ring of the variety \( \text{SL}(2, K) \times V \times V \) is the polynomial ring

\[
K[\sigma, x, y] = K[\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, x_1, x_2, x_3, y_1, y_2, y_3]
\]

modulo the principal ideal \( \langle \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21} - 1 \rangle \). Note that this ring has 10 generators. The ideal that encodes our action equals

\[
J_G = \langle \sigma_{11}^2x_1 + \sigma_{11}\sigma_{21}x_2 + \sigma_{21}^2x_3 - y_1, \sigma_{12}^2x_1 + \sigma_{12}\sigma_{22}x_2\sigma_{22}x_3 - y_3, 2\sigma_{11}\sigma_{21}x_2 + (\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21})x_2 + 2\sigma_{21}\sigma_{22}x_3 - y_2 \rangle.
\]

We eliminate the four variables \( \sigma_{ij} \) that represent for the group elements. This yields the principal ideal

\[
J_G \cap K[x, y] = \langle 4x_1x_3 - x_2^2 - 4y_1y_3 + y_2^2 \rangle.
\]

We now set \( y_1 = y_2 = y_3 = 0 \). The result is the familiar discriminant \( \Delta = 4x_1x_3 - x_2^2 \). In this manner, Derksen’s Algorithm finds the invariant ring for binary quadrics. It confirms that \( K[x]^G = K[\Delta] \).

In Example 11.10, we determined the invariant ring for \( \text{SL}(2, K) \) acting on \( 2 \times 2 \times 2 \) tensors that are symmetric. In what follows, we extend this computation to non-symmetric tensors. Thus, we present a case study in invariant theory for \( d = 2 \) and \( n = 8 \). We identify \( K^8 \) with the space \( (K^2)^{\otimes 3} \) of \( 2 \times 2 \times 2 \)-tensors. The corresponding polynomial ring is denoted by

\[
K[x] = K[x_{111}, x_{112}, x_{121}, x_{122}, x_{211}, x_{212}, x_{221}, x_{222}]
\]

The group \( G = \text{SL}(2, K) \) acts on \( K^2 \) by matrix-vector multiplication. This action extends naturally to the triple tensor product of \( K^2 \). Explicitly, if
\[ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \] is a \( 2 \times 2 \)-matrix in \( G \) then \( \sigma \) acts by performing the following substitution in each polynomial on \( K[x] \):

\[ x_{ijk} \mapsto \sum_{r=1}^{2} \sum_{s=1}^{2} \sum_{t=1}^{2} x_{rst} \sigma_{ri} \sigma_{sj} \sigma_{tk}. \]  

Here are two nice polynomials that are invariant under this action:

**Example 11.14.** Up to scaling, there is a unique polynomial of degree two that is invariant under \( G = \text{SL}(2,K) \). That invariant is the following quadric, which we call the hexagon invariant:

\[ \text{Hex}(x) = x_{112}x_{122} - x_{122}x_{121} + x_{121}x_{221} - x_{221}x_{211} + x_{211}x_{212} - x_{212}x_{112}. \]

Another nice invariant is the hyperdeterminant, which has degree four:

\[ \text{Det}(x) = x_{221}x_{112}^2 + x_{211}x_{122}^2 + x_{121}x_{212}^2 + x_{111}x_{222}^2 + 4x_{111}x_{221}x_{122}x_{212} + 4x_{121}x_{211}x_{112}x_{222} - 2x_{211}x_{221}x_{112}x_{212} - 2x_{121}x_{211}x_{112}x_{221} - 2x_{111}x_{221}x_{112}x_{222} - 2x_{111}x_{211}x_{122}x_{222}. \]

One checks by computation that the substitution (11.11) maps the hexagon invariant \( \text{Hex}(x) \) to itself times the third power of \( \det(\sigma) = \sigma_{11} \sigma_{22} - \sigma_{12} \sigma_{21} \). Similarly, the hyperdeterminant \( \text{Det}(x) \) transforms to itself times \( \det(\sigma)^6 \). Hence both are left invariant when \( \det(\sigma) = 1 \).

Invariants can be used to test whether two tensors lie in the same orbit. Here is a concrete example. We write our \( 2 \times 2 \times 2 \) tensors as vectors in \( \mathbb{R}^8 \) as follows: \( c = (c_{111}, c_{112}, c_{121}, c_{122}, c_{211}, c_{212}, c_{221}, c_{222}) \). The following two tensors appear in the theory of signatures of paths, seen briefly in Section 8.3. It is of interest to know whether their \( G \)-orbits agree up to scaling:

\[ c_{\text{axis}} = \left( \frac{1}{6}, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0, \frac{1}{6} \right) \quad \text{and} \quad c_{\text{mono}} = \left( \frac{1}{6}, \frac{1}{2}, \frac{1}{6}, \frac{4}{15}, 0, \frac{2}{15}, 0, \frac{4}{15} \right). \]

The two polynomials in Example 11.14 are relative invariants of the GL(2) action on the tensor space \( \mathbb{R}^8 \). The following rational function is an absolute invariant. It is homogeneous of degree zero, so it represents an invariant rational function on the projective space \( \mathbb{P}^7 \):

\[ \frac{\text{Hex}(x)^2}{\text{Det}(x)}. \]

We find that the rational invariant (11.12) evaluates to 81 on \( c_{\text{axis}} \), and it evaluates to 45 on \( c_{\text{mono}} \). Hence the orbit closures of our two special core tensors of format \( 2 \times 2 \times 2 \) are disjoint in \( \mathbb{P}^7 \).

We now come to determination of the ring of invariants for the \( G \)-action on the space \( K^8 \) of \( 2 \times 2 \times 2 \) tensors. Using Derksen’s Algorithm, we derive:
Theorem 11.15. The invariant ring $K[x]^{\operatorname{SL}(2)}$ of $2 \times 2 \times 2$ tensors has Krull dimension five. It is minimally generated by 13 invariants, namely the hexagon invariant of degree two, eight invariants of degree four (including the hyperdeterminant), and four invariants of degree six.

In addition to the hyperdeterminant, there are three additional invariants of degree four that deserve special attention. Each has 17 terms when expanded. One of these invariants is

\[
(11.13) \quad x_{111}x_{222} - x_{212}x_{121}^2 + x_{121}x_{222}x_{112}^2 + x_{111}x_{212}x_{122}^2 + x_{121}x_{222}x_{211}^2
\]

The other two invariants in this family are obtained by permuting indices.

Corollary 11.16. The three quartics in (11.13) together with Hex and Det form an algebraically independent system of five primary invariants. All other invariants in $K[x]^{\operatorname{SL}(2)}$ are integral over the polynomial subring generated by these five. The five primary invariants cut out the null cone $\mathcal{V}(K[x]^{\operatorname{SL}(2)_+})$, which is a variety of dimension four and degree 12 in $\mathbb{P}^7$.

It is instructive to restrict the 13 generating invariants in Theorem 11.15 to the 4-dimensional subspace $\operatorname{Sym}_3(K^2)$ of symmetric $2 \times 2 \times 2$ tensors. We discussed this in Example 11.10. The restriction is carried out by setting

\[
x_{111} = x_1, \quad x_{112} = x_{121} = x_{211} = \frac{1}{3} x_2, \quad x_{122} = x_{212} = x_{221} = \frac{1}{3} x_3, \quad x_{222} = x_4.
\]

The resulting symmetric tensors correspond to binary cubics (11.8). The hyperdeterminant and the five other generators of degree four all specialize to the discriminant $\Delta$ of the binary cubic. The other eight generators of $K[x]^{\operatorname{SL}(2)}$, including the hexagon invariant, specialize to zero. In this manner, the invariant ring in Theorem 11.15 maps onto the invariant ring of binary cubics. What about the other irreducible $\operatorname{SL}(2)$ modules in $(K^2)^{\otimes 3}$?

Exercises

1. Let $G$ be the symmetry group of the square $[-1, 1]^2$ in the plane $\mathbb{R}^2$. This is an order 8 subgroup in $\operatorname{GL}(2, \mathbb{R})$. List all eight matrices. Determine the invariant ring $\mathbb{R}[x_1, x_2]^G$.

2. Let $G$ be the symmetry group of the regular 3-cube, as a subgroup of $\operatorname{GL}(3, \mathbb{R})$. How many matrices are in $G$, and what are their characteristic polynomials? Determine the Molien series (11.7) of this group. What does it tell you about the invariant ring?
(3) Fix $n = 5$. Let $\psi(j)$ denote the number of monomials in the expansion of the power sum $P_j$ in terms of the elementary symmetric functions $E_1, E_2, E_3, E_4, E_5$. Compute $\psi(j)$ for some small values, say $j \leq 20$. Guess a formula for $\psi(j)$. Can you prove it?

(4) Show that Noether’s Degree Bound is always tight for finite cyclic groups.

(5) Find a subgroup of $\text{GL}(4, K)$ that has order 15. Compute the invariant ring of your matrix group.

(6) Let $T$ be the group of $3 \times 3$ diagonal matrices with determinant 1, acting on the space $V = \text{Sym}_3(K^3)$ of ternary cubics. This group is the torus $T \cong (K^\ast)^2$. Determine the invariant ring $K[V]^T$. Do you see any relationship to the invariants in Example 11.11?

(7) Let $G = A_n$ be the alternating group of order $n!/2$. Its elements are the even permutation matrices. Determine the invariant ring $K[x]^G$.

(8) List all 103 monomials of the invariant $I_6$ of ternary cubics in Example 11.11. Give an explicit formula, in terms of the ten coefficients $x_1, x_2, \ldots, x_9, x_0$, for the discriminant and for the j-invariant.

(9) Consider the action of $\text{SL}(3, K)$ on the space $\text{Sym}_2(K^3) \cong K^6$ of symmetric $3 \times 3$-matrices. The entries of the $6 \times 6$ matrix $\phi(\sigma)$ are quadratic forms in $\sigma_{11}, \sigma_{12}, \ldots, \sigma_{33}$. Write this matrix explicitly, similarly to the $4 \times 4$ matrix $\phi(\sigma)$ shown in Example 11.10. What is the invariant ring?

(10) Using Derksen’s Algorithm, determine the invariant ring for binary quartics ($d = 2, p = 4$). How many minimal generators does this ring have?

(11) Using an internet source, find the generators of the binary forms of degree $p = 5$. Write them explicitly as polynomials in $x_1, x_2, x_3, x_4, x_5, x_6$.

(12) The rotation group $\text{SO}(2, \mathbb{R})$ acts by left multiplication on the space of $2 \times 2$-matrices. Determine the invariant ring.

(13) Hilbert’s 14th problem asked: Is the invariant ring of every matrix group $G \subset \text{GL}(n, K)$ finitely generated? Find the answer and explain it.
Semidefinite Programming

“Premature optimization is the root of all evil”, Donald Knuth

The transition from linear algebra to nonlinear algebra has a natural
counterpart in convex optimization, namely the passage from linear pro-
gramming to semidefinite programming. This transition is the topic of this
chapter. The term “program” or “programming”, as it is used in this chap-
ter, is simply an old-fashioned way of saying “optimization problem”.

Linear programming concerns the solution of linear systems of inequal-
ities, and the optimization of linear functions subject to linear constraints.
The feasible region is a convex polyhedron, and the optimal solutions form a
face of that polyhedron. In semidefinite programming we work in the space
of symmetric $n \times n$-matrices. The inequality constraints now stipulate that
some linear combination of matrices be positive semidefinite. The feasible
region given by such constraints is a closed convex set, known as a spectrahedron.
We again seek to optimize a linear function, but over a spectrahedron
instead of a polyhedron. The condition for a polynomial to be a sum of
squares may be regarded as a semidefinite program. This gives a connection
to the real Nullstellensatz (Chapter 6), thereby establishing semidefinite
programming as a key tool for computing in real algebraic geometry.

12.1. Spectrahedra

In this chapter we work over the field $\mathbb{R}$ of real numbers. This field comes
with an order. The Spectral Theorem in Linear Algebra states that all
The eigenvalues of a symmetric matrix \( A \in \text{Sym}_2(\mathbb{R}^n) \) are real. Moreover, there is an orthonormal basis of \( \mathbb{R}^n \) consisting of eigenvectors of \( A \). We say that the matrix \( A \) is \textit{positive definite} if it satisfies the following conditions. It is a basic fact about quadratic forms that these three conditions are equivalent:

1. All \( n \) eigenvalues of \( A \) are positive real numbers.
2. All \( 2^n \) principal minors of \( A \) are positive real numbers.
3. Every non-zero column vector \( u \in \mathbb{R}^n \) satisfies \( u^T A u > 0 \).

Here, by a \textit{principal minor} we mean the determinant of any square submatrix of \( A \) whose set of column indices agree with its set of row indices. For the empty set, we get the \( 0 \times 0 \) minor of \( A \), which equals 1. Next there are the \( n \) diagonal entries of \( A \), which are the \( 1 \times 1 \) principal minors, and finally we have the determinant of \( A \), which is the unique \( n \times n \) principal minor.

Each of the three conditions (1), (2) and (3) behaves as expected when we pass to the closure. This is not obvious because the closure of an open semialgebraic set \( \{ f > 0 \} \), where \( f \in \mathbb{R}[x] \), is generally smaller than the closed semialgebraic set \( \{ f \geq 0 \} \) that is defined by the same polynomial \( f \).

**Example 12.1.** Let \( f = x^3 + x^2 y + xy^2 + y^3 - x^2 - y^2 \). The set \( \{ f > 0 \} \) is the open halfplane above the line \( x + y = 1 \) in \( \mathbb{R}^2 \). The closure of the set \( \{ f > 0 \} \) is the corresponding closed halfplane. It is properly contained in \( \{ f \geq 0 \} \) which also contains the origin \((0,0)\).

Luckily, no such thing happens with condition (2) for positive definite matrices. In fact, the closure of the set of positive-definite matrices is obtained by allowing the case of equality in each of the three conditions.

**Theorem 12.2.** For a symmetric \( n \times n \) matrix \( A \), the following three conditions are equivalent:

1. All \( n \) eigenvalues of \( A \) are nonnegative real numbers.
2. All \( 2^n \) principal minors of \( A \) are nonnegative real numbers.
3. Every non-zero column vector \( u \in \mathbb{R}^n \) satisfies \( u^T A u \geq 0 \).

If this holds then \( A \) is called \textit{positive semidefinite}. The semialgebraic set \( \text{PSD}_n \) of positive semidefinite \( n \times n \) matrices is a full-dimensional closed convex cone in \( \text{Sym}_2(\mathbb{R}^n) \).

We use the notation \( X \succeq 0 \) to express that a symmetric matrix \( X \) is positive semidefinite. A \textit{spectrahedron} \( S \) is the intersection of the cone \( \text{PSD}_n \) with an affine-linear subspace \( \mathcal{L} \) of the ambient space \( \text{Sym}_2(\mathbb{R}^n) \). Hence, spectrahedra are closed convex semialgebraic sets.

A space \( \mathcal{L} \) of symmetric matrices is either given parametrically, or as the solution set to a system of, possibly inhomogeneous, linear equations.
In the latter representation,

\[(12.1) \quad \mathcal{L} = \{ X \in \text{Sym}_2(\mathbb{R}^n) : \langle A_1, X \rangle = b_1, \ldots, \langle A_s, X \rangle = b_s \}. \]

Here \( A_1, A_2, \ldots, A_s \in \text{Sym}_2(\mathbb{R}^n) \) and \( b_1, b_2, \ldots, b_s \in \mathbb{R} \) are fixed. We employ the standard inner product in the space of square matrices, which is given by the trace of the matrix product:

\[(12.2) \quad \langle A, X \rangle := \text{trace}(AX) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{ij}. \]

The associated spectrahedron \( S = \mathcal{L} \cap \text{PSD}_n \) consists of all positive semi-definite matrices that lie in the subspace \( \mathcal{L} \). The same matrices can be used to describe a different subspace in its parametric representation, namely

\[(12.3) \quad \mathcal{L} = \{ A_0 + x_1 A_1 + \cdots + x_s A_s : (x_1, \ldots, x_s) \in \mathbb{R}^s \}. \]

If this formulation is used, then it is customary to identify the spectrahedron with its preimage in \( \mathbb{R}^s \). In symbols,

\[(12.4) \quad S = \{ (x_1, \ldots, x_s) \in \mathbb{R}^s : A_0 + x_1 A_1 + \cdots + x_s A_s \succeq 0 \}. \]

**Proposition 12.3.** Every convex polyhedron is a spectrahedron. Convex polyhedra are precisely the spectrahedra that arise when the subspace \( \mathcal{L} \) consists only of diagonal \( n \times n \) matrices.

**Proof.** Suppose that the matrices \( A_0, A_1, \ldots, A_s \) are diagonal matrices. Then (12.4) is the solution set in \( \mathbb{R}^s \) of a system of \( n \) inhomogeneous linear inequalities. Such a set is a convex polyhedron. Every convex polyhedron in \( \mathbb{R}^s \) has such a representation. We simply write its defining linear inequalities as the diagonal entries of the matrix \( A_0 + x_1 A_1 + \cdots + x_s A_s \). \( \square \)

In the previous proposition, the subspace \( \mathcal{L} \) is an affine-linear subspace of \( \text{Sym}_2(\mathbb{R}^n) \) that consists only of diagonal matrices. The formula \( S = \mathcal{L} \cap \text{PSD}_n \) with \( \mathcal{L} \) as in (12.1) corresponds to the standard representation of a convex polyhedron, as the set of non-negative points in an affine-linear space. Here the equations for \( X \) in (12.1) include those that require the off-diagonal entries of each \( X \) to be zero:

\[ \langle X, E_{ij} \rangle = x_{ij} = 0 \quad \text{for} \ i \neq j. \]

The other matrices \( A_i \) are diagonal and their \( b_i \) are typically nonzero.

**Example 12.4.** Let \( \mathcal{L} \) be the space of symmetric 3 \times 3 matrices whose three diagonal entries are all equal to 1. This is an affine-linear subspace of dimension \( s = 3 \) in \( \text{Sym}_3(\mathbb{R}^3) \cong \mathbb{R}^6 \). The spectrahedron \( S = \mathcal{L} \cap \text{SDP}_3 \) is the
yellow convex body seen in Chapter 1, Figure 1. To draw this spectrahedron in $\mathbb{R}^3$, one uses the representation (12.4), namely

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}.$$ 

The boundary of $S$ consists of all points $(x, y, z)$ where the matrix has determinant zero and its nonzero eigenvalues are positive. The determinant is a polynomial of degree three in $x, y, z$, so the boundary lies in a cubic surface in $\mathbb{R}^3$. This cubic surface also contains points where the three eigenvalues are positive, zero and negative. Such points are drawn in red in Figure 1 of Chapter 1. They lie in the Zariski closure of the yellow boundary points.

We next slice our 3-dimensional spectrahedron to get a planar picture. 

**Example 12.5.** Suppose that $L \subset \text{Sym}_2(\mathbb{R}^3)$ is a general 2-dimensional plane that intersects the 6-dimensional cone $\text{PSD}_3$. The spectrahedron $S$ is a convex body in the plane $\mathbb{R}^2$ whose boundary is a smooth cubic curve, drawn in red in Figure 1. On that boundary, the $3 \times 3$ determinant vanishes and the other two eigenvalues are positive. For points $(x, y) \in \mathbb{R}^2 \setminus S$, the matrix has at least one negative eigenvalue. The black curve lies in the Zariski closure of the red curve. It separates points in $\mathbb{R}^2 \setminus S$ whose remaining two eigenvalues are positive from those with two negative eigenvalues.

![Figure 1](image)

**Figure 1.** A plane curve of degree three (left) and its dual curve of degree six (right). The red part on the left bounds a spectrahedron while that on the right bounds its convex dual. The points on the dual curve are lines that are tangent to the primal curve. Biduality holds.

To be explicit, suppose that our planar cubic spectrahedron is written as

$$S = \left\{ (x, y) \in \mathbb{R}^2 : \begin{pmatrix} 1 & x & x + y \\ x & 1 & y \\ x + y & y & 1 \end{pmatrix} \succeq 0 \right\}.$$
The cubic curve is the locus where the $3 \times 3$ matrix in (12.5) is singular. Its determinant equals

$$f = 2x^2y + 2xy^2 - 2x^2 - 2xy - 2y^2 + 1.$$  

The curve $\{f = 0\}$ has four connected components in $\mathbb{R}^2$, one in red and three in black, as shown in Figure 1 (left). The boundary of the cubic spectrahedron $S$ is the convex part of the curve that is shown in red.

The picture on the right in Figure 1 shows the dual curve. This lives in the dual plane whose points $(u, v)$ represent the lines $\ell = \{(x, y) : ux + vy = 1\}$ in the given plane $\mathbb{R}^2$. The points in the dual curve correspond to lines $\ell$ that are tangent to the original curve. For an introduction to duality in algebraic geometry we refer to [5, Section 5.2.4]. Chapter 5 in [5] argues that this algebraic duality agrees with duality in convexity or optimization.

The curve that is dual to (12.6) has degree six, and its equation can found by the following computation of an elimination ideal in $\mathbb{R}[x, y, u, v]$:

$$\langle f(x, y), u \cdot x + v \cdot y - 1, \partial f/\partial x \cdot v - \partial f/\partial y \cdot u \rangle \cap \mathbb{R}[u, v] = \langle 8u^6 - 24u^5v + 21u^4v^2 - 2u^3v^3 + 21u^2v^4 - 24uv^5 \\
+ 8u^6 - 24u^5 + 60u^4v - 24u^3v^2 - 24u^2v^3 + 60uv^4 \\
- 24v^5 + 12u^4 - 24u^3v + 36u^2v^2 - 24uv^3 + 12v^4 + 24u^3 \\
- 36u^2v - 36uv^2 + 24v^3 - 24u^2 + 24uv - 24v^2 + 4 \rangle.$$  

Do try this out! The black points on the sextic in Figure 1 correspond to lines that are tangent at black points of the cubic, and similarly for the red points. Moreover, the convex set enclosed by the red sextic on the right in Figure 1 is dual, in the sense of convexity, to the spectahedron on the left.

The polynomials in (12.6) and (12.7) have degree three and six respectively, confirming what was asserted in the caption to Figure 1. A random line $L$ will meet the curve in three (left) or six (right) complex points. Consider the point $p$ on the other side that is dual to $L$, that is points of $L$ correspond to lines through $p$. There are three (right) or six (left) complex lines through $p$ that are tangent to the curve.

### 12.2. Optimization and Duality

We now finally come to *Semidefinite Programming* (SDP). This refers to the problem of maximizing or minimizing a linear function over a spectrahedron. Linear programming is the special case when the spectrahedron consists of diagonal matrices. If the spectrahedron is given in its standard form representation (12.1), then we get the SDP in its primal form:

$$\text{Minimize } \langle C, X \rangle \text{ subject to } \langle A_1, X \rangle = b_1, \langle A_2, X \rangle = b_2, \ldots, \langle A_s, X \rangle = b_s \text{ and } X \succeq 0.$$  

(12.8)
Here \( C = (c_{ij}) \) is a \( n \times n \) matrix that represents the cost function. Every convex optimization problem has a dual problem. On first glance, it is not so easy to relate SDP duality to duality in algebraic geometry, as seen for plane curves in Figure 1. Making that connection is the point of [5 Chapter 5].

We abbreviate \( b^T x = \sum_{i=1}^s b_i x_i \). The semidefinite programming problem dual to (12.8) takes the following form

\[
\text{(12.9) Maximize } b^T x \text{ subject to } C - x_1 A_1 - x_2 A_2 - \cdots - x_s A_s \succeq 0.
\]

In this formulation, the spectrahedron of feasible points lives in \( \mathbb{R}^s \), similarly to (12.4). We refer to either formulation (12.8) or (12.9) as a semidefinite program, also abbreviated SDP. The relationship between the primal SDP and the dual SDP is given by the following theorem:

**Theorem 12.6 (Weak Duality).** If \( x \) is any feasible solution to (12.9) and \( X \) is any feasible solution to (12.8) then

\[
b^T x \leq \langle C, X \rangle.
\]

If the equality \( b^T x = \langle C, X \rangle \) holds then both \( x \) and \( X \) are optimal.

The term feasible means only that the point \( x \) resp. \( X \) satisfies the equations and inequalities that are required in (12.9) resp. (12.8). The point is optimal if it is feasible and it solves the program, i.e. it attains the minimum resp. maximum value for that optimization problem.

**Proof.** The inner product of two positive semidefinite matrices is a non-negative real number:

\[
0 \leq \langle C - \sum_{i=1}^s x_i A_i, X \rangle = \langle C, X \rangle - \sum_{i=1}^s x_i \cdot \langle A_i, X \rangle = \langle C, X \rangle - b^T x.
\]

This shows that the optimal value of the minimization problem (12.8) is an upper bound for the optimal value of the maximization problem (12.9). If the equality is attained by a pair \((X, x)\) of feasible solutions then \( X \) must be optimal for (12.8) and \( x \) must be optimal for (12.9). \( \Box \)

There is also a Strong Duality Theorem which states that, under suitable hypotheses, the duality gap \( \langle C, X \rangle - b^T x \) must attain the value zero for some feasible pair \((X, x)\). These hypotheses are always satisfied for diagonal matrices, and we recover the Duality Theorem for Linear Programming as a special case. Note that this is equivalent to Farkas’ Lemma in Theorem 6.13.

Interior point methods for Linear Programming are numerical algorithms that start at an interior point of the feasible polyhedron and create a path from that point towards an optimal vertex. The same class of algorithms works for Semidefinite Programming. These run in polynomial time and are well-behaved in practice. Semidefinite Programming has a much larger expressive power than Linear Programming. Many more problems can be phrased as an SDP. We illustrate this with an example from linear algebra.
Example 12.7 (The largest eigenvalue). Let $A$ be a positive definite symmetric $n \times n$ matrix. Consider the problem of computing its largest eigenvalue $\lambda_{\text{max}}(A)$. We can solve this without writing down the characteristic polynomial and extracting its roots. Let $C = \text{Id}$ be the identity matrix and consider the SDP problems (12.8) and (12.9) with $s = 1$ and $b = 1$. They are

12.8 Minimize $\text{trace}(X)$ subject to $\langle A, X \rangle = 1$.
12.9 Maximize $x$ subject to $\text{Id} - xA \succeq 0$.

If $x^*$ is the common optimal value of (12.8'), (12.9') then $\lambda_{\text{max}}(A) = 1/x^*$.

The inner product $\langle A, X \rangle = \text{trace}(A \cdot X)$ of two positive semidefinite matrices $A$ and $X$ can only be zero when their matrix product $A \cdot X$ is zero.

We record this for our situation:

Lemma 12.8. If the duality gap $\langle C, X \rangle - b^T x$ is zero then the product of symmetric matrices $(C - \sum_{i=1}^{s} x_i A_i) \cdot X$ is the zero matrix.

This lemma implies the following algebraic reformulation of duality:

Corollary 12.9. Consider the following system of $s$ linear equations and $\left(\begin{array}{c} n+1 \\ 2 \end{array}\right)$ bilinear equations in the $\left(\begin{array}{c} n+1 \\ 2 \end{array}\right) + s$ coordinates of the pair $(X, x)$:

(12.10) $\langle A_1, X \rangle = b_1, \ldots, \langle A_s, X \rangle = b_s$ and $(C - \sum_{i=1}^{s} x_i A_i) \cdot X = 0$.

If $X \succeq 0$ and $C - \sum_{i=1}^{s} x_i A_i \succeq 0$ then $X$ is optimal for (12.8) and $x$ is optimal for (12.9).

The equations (12.10) are the Karush-Kuhn-Tucker (KKT) equations. These play a major role when one explores semidefinite programming from an algebraic perspective. In particular, they allow us to study the nature of the optimal solution as function of the data. A key feature of the KKT system is that the two optimal matrices have complementary ranks. This follows from the complementary slackness condition on the right of (12.10):

$$\text{rank}(C - \sum_{i=1}^{s} x_i A_i) + \text{rank}(X) \leq n.$$ 

In particular, if $X$ is known to be nonzero then the determinant of $C - \sum_{i=1}^{s} x_i A_i$ vanishes. For instance, for the eigenvalue problem in Example 12.7 we have $(\text{Id} - xA) \cdot X = 0$ and $\langle A, X \rangle = 1$. This implies $\det(\text{Id} - xA) = 0$, so $1/x$ is a root of the characteristic polynomial.

Example 12.10. Consider the problem of maximizing a linear function $\ell(x, y) = ux + vy$ over the spectrahedron $S$ in (12.5). This is the primal
SDP \((12.8)\) with \(s = 2\) and \(b = (u, v)\) and

\[
A_1 = -\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}
\quad \text{and} \quad
A_2 = -\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{pmatrix}.
\]

The KKT system \((12.10)\) consists of eight equations in eight unknowns:

\[
2x_{12} + 2x_{13} + u = 2x_{13} + 2x_{23} + v = 0 \quad \text{and} \quad
\begin{pmatrix}
1 & x & x+y \\
x & 1 & y \\
x+y & y & 1
\end{pmatrix} \cdot
\begin{pmatrix}
x_11 & x_{12} & x_{13} \\
x_{12} & x_{22} & x_{23} \\
x_{13} & x_{23} & x_{33}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Here \(u\) and \(v\) are parameters. By eliminating the variables \(x_{ij}\) from the eight equations, we obtain an ideal \(I\) in \(\mathbb{Q}[u, v, x, y]\) that characterizes the optimal solution \((x^*, y^*)\) to our SDP as an algebraic function of \((u, v)\). Let \(\ell^*\) now be a new unknown, and consider the elimination ideal \((I + \langle ux + vy - \ell^* \rangle) \cap \mathbb{Q}[u, v, \ell^*]\). Its generator is a ternary sextic in \(u, v, \ell^*\). This is precisely the homogenization of the dual sextic in \((12.7)\). It expresses the optimal value \(\ell^*\) as an algebraic function of degree six in the cost \((u, v)\).

This relationship between the dual hypersurface and the optimal value function generalizes to arbitrary polynomial optimization problems, including semidefinite programs. This is the content of [5, Theorem 5.23]. We refer to the book [5], and especially Chapter 5, for further reading on spectrahedra, semidefinite programming, and the relevant duality theory.

A fundamental task in Convex Algebraic Geometry [5] is computing the convex hull of a given algebraic variety or semialgebraic set. Recall that the convex hull of a set is the smallest convex set containing the given set. Spectrahedra or their linear projections, known as spectrahedral shadows, can be used for this task. This matters for optimization since minimizing a linear function over a set is equivalent to minimizing over its convex hull.

**Example 12.11** (Toeplitz Spectrahedron). Consider the convex body

\[
(12.11) \quad K = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{bmatrix} 1 & x & y & z \\ x & 1 & x & y \\ y & x & 1 & x \\ z & y & x & 1 \end{bmatrix} \succeq 0 \right\}.
\]

The determinant of the given Toeplitz matrix of size \(4 \times 4\) factors as

\[
(x^2 + 2xy + y^2 - xz - x - z - 1)(x^2 - 2xy + y^2 - xz + x + z - 1).
\]

The spectrahedron \((12.11)\) is the convex hull of the cosine moment curve

\[
(12.12) \quad \left\{ (\cos(\theta), \cos(2\theta), \cos(3\theta)) : \theta \in [0, \pi] \right\}.
\]
12.2. Optimization and Duality

The curve and its convex hull are shown on the left in Figure 2. The two endpoints, \((x, y, z) = (1, 1, 1)\) and \((x, y, z) = (-1, 1, -1)\), correspond to matrices of rank 1. All other points on the curve have rank 2.

We call (12.11) the **Toeplitz spectrahedron**. To construct this convex body geometrically, we form the cone from each endpoint over the cosine curve, and we intersect these two quadratic cones. The two cones intersect along this curve and the line through the endpoints of the cosine curve.

Shown on the right in Figure 2 is the convex body \(K^*\) dual to the Toeplitz spectrahedron \(K\). Its points \((a_1, a_2, a_3)\) correspond to the trigonometric polynomials \(1 + a_1 \cos(\theta) + a_2 \cos(2\theta) + a_3 \cos(3\theta)\) that are nonnegative on \([0, \pi]\). This convex body \(K^*\) is not a spectrahedron because it has a non-exposed edge, that is an edge which is not an intersection of \(K^*\) with a kernel of a linear form that is nonnegative on \(K^*\) (cf. [5] Exercise 6.13).

We close this section with a reformulation of SDP that will appeal to algebra students. It expresses the primal (12.8) and the dual (12.9) in a form that looks symmetric. After replacing the constraint matrices \(A_1, A_2, \ldots, A_s\) by linear combinations, we may assume that \(b_1 = 1\) and \(b_2 = \cdots = b_s = 0\).

Let \(V\) be the subspace of \(\text{Sym}_2(\mathbb{R}^n)\) spanned by \(\{A_2, \ldots, A_s\}\), and let \(U\) be the subspace of \(\text{Sym}_2(\mathbb{R}^n)\) spanned by \(\{C, A_1\}\) and \(V\). We assume \(\dim(V) = s - 1\) and \(\dim(U) = s + 1\). Our instance defines a pair of flags

\[
\begin{align*}
(12.13) & \quad V \subset U \subset \text{Sym}_2(\mathbb{R}^n) \\
& \quad U^\perp \subset V^\perp \subset \text{Sym}_2(\mathbb{R}^n)^*.
\end{align*}
\]

The KKT equations (12.10) can now be written as

\[
(12.14) \quad X \in V^\perp \quad \text{and} \quad Y \in U \quad \text{and} \quad X \cdot Y = 0.
\]
Here the matrices $X$ and $Y$ are regarded up to scaling, that is, they are unknown points in the projective space $\mathbb{P}(\text{Sym}^2(\mathbb{R}^n)) = \mathbb{P}^{n+1}_2$. The flag $V \subset U$ is our data. They specify a variety in $\mathbb{P}^{n+1}_2 \times \mathbb{P}^{n+1}_2$ via (12.14).

Corollary 12.12. The semidefinite programming problem (12.8) is equivalent to solving the bilinear system of equations (12.14) subject to $X, Y \succeq 0$.

Proof. This is a reformulation of Corollary 12.9.$\square$

### 12.3. Sums of Squares

Semidefinite programming can be used to model and solve arbitrary polynomial optimization problems. The key to this is the representation of nonnegative polynomials in terms of sums of squares, or, more generally, the Real Nullstellensatz (cf. Chapter 6). We explain this for the simplest scenario, namely the problem of unconstrained polynomial optimization.

Let $f(x_1, \ldots, x_n)$ be a polynomial of even degree $2p$, and suppose that $f$ attains a minimal real value $f^*$ on $\mathbb{R}^n$. Our goal is to compute $f^*$ and a point $u^* \in \mathbb{R}^n$ such that $f(u^*) = f^*$. Minimizing a function is equivalent to finding the best possible lower bound $\lambda$ for that function. Our goal is therefore equivalent to solving the following optimization problem:

(12.15)\[ \text{Maximize } \lambda \text{ such that } f(x) - \lambda \geq 0 \text{ for all } x \in \mathbb{R}^n. \]

This is a difficult problem. Instead, we consider the following relaxation:

(12.16)\[ \text{Maximize } \lambda \text{ such that } f(x) - \lambda \text{ is a sum of squares in } \mathbb{R}[x]. \]

Here *relaxation* means that we modified the set of feasible solutions. Indeed, every sum of squares is nonnegative, but not every nonnegative polynomial is a sum of squares of polynomials. For instance, the Motzkin polynomial $x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is nonnegative but it is not a sum of squares of polynomials, by Example 6.11. For that reason, the optimal value of (12.16) is always a lower bound for the optimal value of (12.15), but the two values can be different in some cases. However, here is the good news:

Proposition 12.13. The optimization problem (12.16) is a semidefinite program.

Proof. Let $x^{[p]}$ be the column vector whose entries are all monomials in $x_1, \ldots, x_n$ of degree $\leq p$. Thus $x^{[p]}$ has length $\binom{n+p}{n}$. Let $G = (g_{ij})$ be a symmetric $\binom{n+p}{n} \times \binom{n+p}{n}$ matrix with unknown entries. Then $(x^{[p]})^T \cdot G \cdot x^{[p]}$ is a polynomial of degree $d = 2p$ in $x_1, \ldots, x_n$. We set

(12.17)\[ f(x) - \lambda = (x^{[p]})^T \cdot G \cdot x^{[p]}. \]
By collecting coefficients of the $x$-monomials, we get a system of $(2^{p+n} + n)$ linear equations in $1 + \binom{n+p}{2}$ unknowns, namely $\lambda$ and the matrix entries $g_{ij}$. Suppose the linear system \((12.17)\) has a solution $(G, \lambda)$ such that $G$ is positive semidefinite. Then we can write $G = H^T H$ where $H$ is a real matrix with $r$ rows and $(p+n)$ columns. (This is known as a Cholesky factorization of $G$.) The polynomial in \((12.17)\) then equals
\[ f(x) - \lambda = (Hx)^T \cdot (Hx). \]
This is the scalar product of a vector of length $r$ with itself. Hence $f(x) - \lambda$ is a sum of squares. Conversely, every representation of $f(x) - \lambda$ as a sum of squares of polynomials uses polynomials of degree $\leq p$, and it can hence be written in the form as in \((12.18)\).

Our argument shows that the problem \((12.16)\) is equivalent to
\[ \text{Maximize } \lambda \text{ subject to } (G, \lambda) \text{ satisfying the linear equations } (12.17) \text{ and } G \succeq 0. \]
This is a semidefinite programming problem, so the proof is complete. \qed

If $n = 1$ or $d = 2$ or $(n = 2$ and $d = 4$) then every nonnegative polynomial is a sum of squares. It is precisely in those special cases when the two problems \((12.15)\) and \((12.19)\) are equivalent. See [5, Section 3.1.2].

**Example 12.14** $(n = 1, p = 2, d = 4)$. Suppose we seek to find the minimum of the degree 4 polynomial $f(x) = 3x^4 + 4x^3 - 12x^2$. Of course, we know how to do this using Calculus. Here we explain the SDP approach.

The linear equations \((12.17)\) have a one-dimensional space of solutions. Introducing a parameter $\mu$ for that line, the solutions can be written as
\[ (12.20) \quad f(x) - \lambda = \begin{pmatrix} x^2 & x & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & \mu - 6 \\ 2 & -2\mu & 0 \\ \mu - 6 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}. \]
Consider the set of all pairs $(\lambda, \mu)$ such that the $3 \times 3$ matrix in \((12.20)\) is positive semidefinite. This set is a cubic spectrahedron in the plane $\mathbb{R}^2$, just like that shown on the left in [1]. We seek to maximize $\lambda$ over all points in that cubic spectrahedron. The optimal point equals $(\lambda^*, \mu^*) = (-32, -2)$. Substituting this into the matrix in \((12.20)\) we obtain a positive definite matrix of rank 2. This can be factored as $G = H^T H$, where $H$ has format $2 \times 3$. The resulting representation \((12.18)\) as a sum of two squares equals
\[ f(x) - \lambda^* = f(x) + 32 = (\sqrt{3}x - \frac{4}{\sqrt{3}}) \cdot (x + 2)^2 + \frac{8}{3}(x + 2)^2. \]
The right hand side is nonnegative for all $x$. It takes the value 0 at $t = -2$. \[ \square \]
Any polynomial optimization problem can be translated into a relaxation that is a semidefinite programming problem. For unconstrained optimization we saw this in Proposition 12.13. Remarkably, the same approach works for arbitrary optimization problems where both the objective function and the constraints are given by polynomials. Indeed, suppose we wish to minimize \( f(x) \) subject to some polynomial constraints. We seek a certificate for \( f(x) - \lambda < 0 \) to have no solution. This certificate is promised by the Real Nullstellensatz or the Positivstellensatz. We discussed these in Chapter 6.

If we fix a degree bound then the existence of a certificate translates into a semidefinite program, and so does the additional requirement for \( \lambda \) to be minimal. The translation is explained in [5] Section 3.4.3. This relaxation may or may not give the correct solution for some fixed degree bound.

If one increases the degree bound then the SDP formulation is more likely to succeed, albeit at the expense of having to solve a much larger problem. This is a powerful and widely used approach to polynomial optimization, known as \textit{SOS programming}. The term \textit{Lasserre hierarchy} refers to varying the degree bounds. It is often found in the computer science literature.

Every spectrahedron \( S = \mathcal{L} \cap \text{PSD}_n \) has a special point in its relative interior. This point, defined as the unique matrix in \( S \) whose determinant is maximal, is known as \textit{analytic center}. Finding the analytic center of \( S \) is a convex optimization problem, since the function \( X \mapsto \log \det(X) \) is strictly convex on the cone of positive definite matrices \( X \). The analytic center is important for semidefinite programming because it serves as the starting point for interior point methods. Indeed, the \textit{central path} of an SDP starts at the analytic center and runs to the optimal face. It is computed by a sequence of numerical approximations.

\textbf{Example 12.15.} The determinant function takes on all values between 0 and 1 on the spectrahedron \( S \) in [12.5]. The value 1 is attained only by the identity matrix, for \( (x,y) = (0,0) \). This point is the analytic center of \( S \).

We close by relating spectrahedra and their analytic centers to statistics. For further reading on this connection we refer to the article [60]. Every positive definite \( n \times n \) matrix \( \Sigma = (\sigma_{ij}) \) is the \textit{covariance matrix} of a multivariate normal distribution. Its inverse \( \Sigma^{-1} \) is also symmetric and positive definite. The matrix \( \Sigma^{-1} \) is known as the \textit{concentration matrix}.

A \textit{Gaussian graphical model} is specified by requiring that some off-diagonal entries of \( \Sigma^{-1} \) are zero. These entries correspond to the non-edges of the graph. Maximum likelihood estimation (MLE) for this graphical model translates into a matrix completion problem. Suppose that \( S \) is the sample covariance matrix of a given data set. We regard \( S \) as a partial matrix, with visible entries only on the diagonal and on the edges of the graph.
12.3. Sums of Squares

One considers the set of all completions of the non-edge entries that make the matrix $S$ positive definite. The set of all these completions is a spectrahedron. Maximum likelihood estimation for the data $S$ in the graphical model amounts to maximizing the logarithm of the determinant. We hence seek to compute the analytic center of the spectrahedron of all completions.

**Example 12.16** (Positive definite matrix completion). Suppose that the eight entries $\sigma_{ij}$ in the following symmetric $4 \times 4$-matrix are visible, but the entries $x$ and $y$ are unknown:

$$
\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & x & \sigma_{14} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} & y \\
x & \sigma_{23} & \sigma_{33} & \sigma_{34} \\
\sigma_{14} & y & \sigma_{34} & \sigma_{44}
\end{pmatrix}.
$$

(12.21)

This corresponds to the graphical model of the four-cycle 12, 23, 34, 41. Given visible entries $\sigma_{ij}$, we consider the set of pairs $(x, y)$ that make $\Sigma$ positive definite. This is the interior of a planar spectrahedron $S_\sigma$ bounded by a quartic curve. The MLE is the analytic center of $S_\sigma$.

One is also interested in conditions on the $\sigma_{ij}$ such that $\text{int}(S_\sigma)$ is non-empty. When can we find $(x, y)$ that make $\Sigma$ positive definite? One necessary condition is that the diagonal entries $\sigma_{ii}$ are positive. A further condition is that the four visible principal $2 \times 2$-minors are positive:

$$
\sigma_{11}\sigma_{22} > \sigma_{12}^2, \quad \sigma_{22}\sigma_{33} > \sigma_{23}^2, \quad \sigma_{33}\sigma_{44} > \sigma_{34}^2, \quad \sigma_{11}\sigma_{44} > \sigma_{14}^2.
$$

(12.22)

But these necessary conditions are not sufficient. The answer is as follows.

The region of all matrices $\sigma$ with $\text{int}(S_\sigma) \neq \emptyset$ is a semialgebraic convex cone. Its boundary is a hypersurface of degree eight. The polynomial defining that hypersurface equals

$$
\sigma_{11}^2 \sigma_{22}^2 \sigma_{33}^2 - 2\sigma_{22} \sigma_{23}^2 \sigma_{34} \sigma_{12}^2 \sigma_{14} - \sigma_{11} \sigma_{33} \sigma_{44} \sigma_{14} \sigma_{12}^2 \sigma_{23}^2 - 2\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{44} \sigma_{14} \sigma_{12}^2 \sigma_{23}^2 + 4\sigma_{33} \sigma_{44} \sigma_{12}^2 \sigma_{14}^2 \sigma_{23}^2 + 8\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{44} \sigma_{14} \sigma_{12} \sigma_{23} \sigma_{34} - 4\sigma_{33} \sigma_{44} \sigma_{12}^2 \sigma_{14} \sigma_{23} \sigma_{34} - 4\sigma_{22} \sigma_{33} \sigma_{12}^2 \sigma_{14} \sigma_{23} \sigma_{34} + \sigma_{22}^2 \sigma_{33} \sigma_{14}^2 \sigma_{12} \sigma_{23} - 4\sigma_{11} \sigma_{44} \sigma_{12} \sigma_{14} \sigma_{23} \sigma_{34} - 2\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{44} \sigma_{12}^2 \sigma_{23}^2 - 2\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{14}^2 \sigma_{12} \sigma_{23} + 4\sigma_{22} \sigma_{33} \sigma_{12}^2 \sigma_{14}^2 \sigma_{23}^2 - 2\sigma_{11} \sigma_{22} \sigma_{44} \sigma_{12} \sigma_{14} \sigma_{23} \sigma_{34} + 4\sigma_{11} \sigma_{44} \sigma_{12} \sigma_{14} \sigma_{23} \sigma_{34} + 4\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{14}^2 \sigma_{12} \sigma_{23} - 4\sigma_{11} \sigma_{22} \sigma_{12} \sigma_{14} \sigma_{23} \sigma_{34} + 4\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{14} \sigma_{12} \sigma_{23} + \sigma_{11}^2 \sigma_{22}^2 \sigma_{14}^2 \sigma_{23}^2 + \sigma_{33}^2 \sigma_{14}^2 \sigma_{12}^2.
$$

This polynomial is found by eliminating $x$ and $y$ from the determinant and its partial derivatives with respect to $x$ and $y$, after saturating by the ideal of $3 \times 3$-minors. For more details on this example see [60, Theorem 4.8].
Exercises

(1) Prove Theorem 12.2.

(2) Show that a real symmetric matrix $G$ is positive semidefinite if and only if it admits a Cholesky factorization $G = H^T H$ over the real numbers, with $H$ upper triangular.

(3) What is the largest eigenvalue of any of the $3 \times 3$ matrices in the set $S$ in (12.5)? Express this problem as a semidefinite program and solve it.

(4) Maximize and minimize the linear function $13x + 17y + 23z$ over the spectrahedron $S$ in Example 12.4. Use SDP software if you can.

(5) Maximize and minimize the linear function $13x + 17y + 23z$ over the Toeplitz spectrahedron in Example 12.11. Use SDP software if you can.

(6) Write the dual SDP and solve the KKT system for the previous two problems. Also write down the symmetric formulation in Corollary 12.12.

(7) Compute the convex body dual to the spectrahedron $S$ in Example 12.4.

(8) Consider the problem of minimizing the univariate polynomial $x^6 + 5x^3 + 7x^2 + x$. Express this problem as a semidefinite program and solve it.

(9) True or false: A spectrahedron $S = \mathcal{L} \cap \text{PSD}_n$ is a polyhedron if and only if the linear space $\mathcal{L}$ consists only of diagonal matrices.

(10) In the partial matrix (12.21) set $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{44} = 5$, $\sigma_{12} = \sigma_{23} = \sigma_{34} = 1$ and $\sigma_{14} = 2$. Compute the spectrahedron $S_\sigma$, draw a picture, and find the analytic center of $S_\sigma$. What is the statistical interpretation?

(11) Find numerical values for the eight entries $\sigma_{ij}$ of the partial $4 \times 4$ matrix in (12.21) such that (12.22) holds but $\text{int}(S_\sigma) = \emptyset$.

(12) Let $s = 3$ and $n = 4$, pick a general flag $V \subset U$ as in (12.13), and consider the subvariety of $\mathbb{P}^9 \times \mathbb{P}^9$ defined by the KKT equations (12.14). Working over $\mathbb{C}$, show that this variety is finite and determine its cardinality.

(13) Study the convex hull of following curve which seems similar to (12.12):

$$\{ (\cos(\theta), \cos(2\theta), \sin(3\theta)) : \theta \in [0, \pi] \}.$$

Can you draw a picture? Is this convex body a spectrahedron?
Combining interacts in many fruitful ways with algebra and geometry, for instance in the interplay between polytopes and toric varieties we saw in Chapter 8. We introduce topics that are important for nonlinear algebra. The first such topic is matroid theory. Matroids encode independence, just like groups encode symmetry. The theory of matroids has many connections to toric geometry and we will present a few of them. One of our main aims is to highlight connections between toric varieties, matroids, and Grassmannians (Chapter 5). In these directions, lattice polytopes play a prominent role. We conclude by presenting a snapshot of generating functions. Their role as Hilbert series brings us back to the two key invariants of a variety: dimension and degree. The emphasis of this chapter is not on combinatorics itself, but rather on the role it plays within nonlinear algebra. In short, what we offer here is a pinch of combinatorics in a vast sea of algebra.

13.1. Matroids

In this section we give an introduction to the theory of matroids. The name matroids reveals that these can be seen as generalizations of matrices. Indeed, as we will soon see, every matrix over a field defines a matroid.

We fix a finite set $E$, which we refer to as the ground set of a matroid. We shall distinguish a family of subsets of $E$ that are called independent. Thus a matroid $M$ is a family $\mathcal{I} \subset 2^E$ of subsets $E$ that we refer to as independent sets. These are of course assumed to satisfy certain axioms.
These axioms reflect the familiar scenario when $E$ is a set of vectors in a vector space, and independence simply means linear independence.

A first observation is that whenever we have an independent set $I \subset E$, it is reasonable to assume that every subset of $I$ is also independent. We obtain the first axiom of a matroid for the family $\mathcal{J}$:

1. If $I \in \mathcal{J}$ and $J \subset I$, then $J \in \mathcal{J}$.

What we defined so far is a simplicial complex. Another requirement is for $\mathcal{J}$ to be nonempty. Equivalently, we want the empty set to be independent:

2. We have $\emptyset \in \mathcal{J}$.

To obtain a matroid, we need one more axiom. To motivate it we make the following observation. Consider finite linearly independent subsets $I, J \subset V$ of a vector space $V$, if $|I| < |J|$. Then we can extend $I$ by an element of $j \in J$, in such a way that $I \cup \{j\}$ is still linearly independent. This simple observation is precisely what we need to get the last axiom for the family $\mathcal{J}$:

3. If $I, J \in \mathcal{J}$ and $|I| < |J|$ then there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{J}$.

**Definition 13.1.** A *matroid* is a family of subsets $\mathcal{J}$ satisfying Axioms 1,2,3.

Exercise 1 asks to prove the assertion in the next sentence.

**Example 13.2.** The following families $\mathcal{J}$ are matroids:

- (Representable matroid) Let $V$ be a vector space over an arbitrary field $F$. Let $E \subset V$ be a nonempty, finite subset. We define $\mathcal{J}$ to be the family of subsets of $E$ that are linearly independent. We say that the matroid is *representable* over $F$. In coordinates, $V \simeq F^n$ and we may write the set $E$ as the rows of an $|E| \times n$ matrix.

- (Graphic matroid) Let $G$ be a graph with edge set $E$. Let $\mathcal{J}$ be the family of those subsets of $E$ that do not contain a cycle. Equivalently $\mathcal{J}$ is the family of forests in $G$.

- (Algebraic matroid) Let $F \subset K$ be an arbitrary field extension. Let $E$ be a finite subset of $K$. Let $\mathcal{J}$ be the family of subsets of $E$ that are algebraically independent over $F$.

- (Uniform matroid) Let $E$ be a finite set and $k \leq |E|$. Let $\mathcal{J}$ be the family of subsets of cardinality at most $k$. This matroid is denoted by $U_{k,E}$ or $U_{k,|E|}$.

Matroids are known for having many equivalent definitions, depending on one’s point of view. For example, thanks to the first axiom, to determine a matroid we do not have to know all independent sets, just those that are inclusion maximal. By analogy to linear algebra, the inclusion maximal
independent sets are called bases. It turns out - as the reader is asked to prove in Exercise 2 - that a nonempty family $\mathcal{B} \subset 2^E$ of subsets of $E$ is the set of bases of some matroid if and only if the following axiom is satisfied:

- For all bases $B_1, B_2 \in \mathcal{B}$ and each element $b_2 \in B_2 \setminus B_1$ there exists $b_1 \in B_1 \setminus B_2$ such that $(B_1 \setminus \{b_1\}) \cup \{b_2\} \in \mathcal{B}$.

The seemingly weak axiom on $\mathcal{B}$ in fact implies the following two statements:

- For all $B_1, B_2 \in \mathcal{B}$ and $b_2 \in B_2 \setminus B_1$ there exists $b_1 \in B_1 \setminus B_2$ such that both $(B_1 \setminus \{b_1\}) \cup \{b_2\}$ and $(B_2 \setminus \{b_2\}) \cup \{b_1\}$ are bases in $\mathcal{B}$.

- For all $B_1, B_2 \in \mathcal{B}$ and any subset $A_2 \subset B_2 \setminus B_1$ there exists a subset $A_1 \subset B_1 \setminus B_2$ such that both $(B_1 \setminus A_1) \cup A_2$, $(B_2 \setminus A_2) \cup A_1 \in \mathcal{B}$.

The first point is known as the symmetric exchange property and the second one as the multiple symmetric exchange property. The facts that both exchange properties hold is nontrivial - we refer the reader to the proofs in \cite{8,63}. We will soon see the algebraic meaning of the exchange properties.

Exercise 3 states that all bases of a matroid have the same cardinality. The cardinality of a basis is known as the rank of a matroid. More generally for a matroid on a ground set $E$ we may define the rank of any subset $A \subset E$.

**Definition 13.3.** For a matroid on the ground set $E$, specified by its collection of independent sets $\mathcal{I} \subset 2^E$, we define the rank function:

$$r : 2^E \to \mathbb{Z}, \ A \mapsto \max_{I \in \mathcal{I}} \{|I \cap A|\}.$$ 

Equivalently, the rank of a set $A$ is the cardinality of a largest independent set contained in $A$.

We note that, for a representable matroid, the rank is simply the dimension of the vector subspace spanned by the given vectors. For any matroid, representable or not, the rank function $r$ satisfies the following:

- $0 \leq r(A)$ for all $A \subset E$ and $r(\emptyset) = 0$.
- $r(A) \leq r(A \cup \{b\}) \leq r(A) + 1$ for all $A \subset E$, $b \in E$.

Further, the rank function has one more property, known as submodularity:

- for all $A, B \subset E$ we have $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

In Exercise 7 the reader is asked to prove that any function $r : 2^E \to \mathbb{Z}$ satisfying the three axioms above is a rank function of a matroid. The independent sets can be reconstructed as those $I \subset E$ for which $r(I) = |I|$. Thus, the above rank axioms, give another possible definition of a matroid.

Matroid theory has numerous connections to topics seen earlier in this book. Two of them will be explored in the two sections that follow. We
close this section by touching upon two other connections, namely to solving polynomial equations (Chapter 3) and to tropical algebra (Chapter 7).

Suppose we are given a matroid $M$ and a field $K$. Two basic questions arise: Is $M$ representable over $K$? What is the set of all matrices that represent $M$? This leads to a system of polynomial equations. Namely, let $E = \{1, 2, \ldots, m\}$ and $n = r(E)$ the rank of $M$. Let $R(M)$ denote the set of all matrices $X \in K^{m \times n}$ whose rows represent $M$. This is known as the realization space of the matroid $M$. Thus a subset $I$ of the rows of $X$ is linearly independent in $K^n$ if and only if it is independent in $M$. For any subset $B \subset E$ with $|B| = n$, we write $\det_B(X)$ for the $n \times n$ determinant given by the rows of $X$ that are indexed by $B$. By definition, $R(M)$ equals

$$\{ X \in K^{m \times n} : \det_B(X) \neq 0 \text{ for } B \in \mathcal{B} \text{ and } \det_B(X) = 0 \text{ for } B \not\in \mathcal{B} \}.$$

This is an affine variety. The inequations can be written as one equation

$$z \cdot \prod_{B \in \mathcal{B}} \det_B(X) = 1,$$

where $z$ is a new variable.

With this, the set $R(M)$ is a closed subvariety in the affine space $K^{mn+1}$.

**Example 13.4** (Fano Plane). The Fano plane is the matroid $M$ of rank $n = 3$ on $m = 7$ elements. The nonbases are 124, 235, 346, 457, 561, 672, 713. These are the triples in the ground set that are not bases. Note that scaling rows and columns by a nonzero scalar, does not change the matroid. Every element $X \in R(M)$ satisfies $\det_{123}(X) \neq 0$, so we may assume that $X$ has an identity matrix in rows 1, 2, 3. After further scaling the rows and columns,

$$X^T = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & z \\ 0 & 1 & 0 & x & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & y & 1 & 1 \end{pmatrix}.$$

The zero entries in columns 4,5,7 come from the nonbases 124, 235, 713. Similarly the nonzero entries are specified by the bases and, if possible, scaled to one. The constraints $\det_B(X) = 0$ from the four nonbases: 346, 457, 561, 672, and the fact that $x, y, z \neq 0$ span the ideal

$$\langle x - 1, xyz + 1, y - 1, z - 1 \rangle \subset K[x, y, z].$$

If the characteristic of our field $K$ is different from 2 then this is the unit ideal and hence $M$ is not representable. However, if char$(K) = 2$ then the point $(x, y, z) = (1, 1, 1)$ is a solution and gives a representation of $M$.

We have seen that representability of matroids leads to varieties. Remarkably, also the converse holds. Given any affine variety $V$ that is defined over $\mathbb{Z}$, one can construct a matroid $M$ such that $R(M)$ is essentially equal to $V$. This is the content of the celebrated Mnëv’s Universality Theorem.
Matroids play a fundamental role also in tropical geometry. Following Chapter 7 for every ideal $I$ in a Laurent polynomial ring $K[x^\pm]$ we have an associated tropical variety $\text{trop}(\mathcal{V}(I))$. This tropical variety is a subset of $\mathbb{R}^n$. It is called a tropical linear space if $I$ is generated by linear polynomials.

**Example 13.5.** Let $n = 3$ and $I = \langle x_1 + x_2 + x_3 + 1, x_1 + 2x_2 + 3x_3 + 4 \rangle$. The classical variety $\mathcal{V}(I)$ is a line in the affine 3-space $K^3$. Its tropicalization $\text{trop}(\mathcal{V}(I))$ is a balanced graph in $\mathbb{R}^3$ with four unbounded rays in the coordinate directions $e_1, e_2, e_3$ and $-e_1 - e_2 - e_3$. This tropical line captures the structure of the associated matroid, i.e. the uniform matroid $U_{2, 4}$. It is represented by the matrix seen in the ideal generators: $X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}^T$.

In general one defines a tropical linear space $\text{trop}(M)$ for any matroid $M$. This construction is explained in [41, Section 4.2]. These are the linear spaces in tropical geometry. They are fundamental for the concept of a tropical manifold. In classical mathematics, a manifold is a space that looks locally like a linear space. Hence, a tropical manifold is a space that looks locally like a matroid $M$. Each of its charts is a tropical linear space $\text{trop}(M)$.

### 13.2. Lattice Polytopes

In this section we study the interplay between toric geometry and matroids. Starting from the basics, we begin by recalling the definition of a lattice polytope that we know from Chapter 8. We work in the real vector space $\mathbb{R}^n$. A polytope $P$ is the convex hull of a finite set of points $p_1, \ldots, p_k \in \mathbb{R}^n$:

$$P := \{ x \in \mathbb{R}^n : x = \sum_{i=1}^k \lambda_i p_i \text{ for some real } \lambda_1, \ldots, \lambda_k \geq 0, \sum_{i=1}^k \lambda_i = 1 \}.$$ 

We call $P$ a lattice polytope if we can find $p_1, \ldots, p_k \in \mathbb{Z}^n$. If none of the $p_i$ is redundant in the representation of $P$ then $p_1, \ldots, p_k$ are the vertices of $P$.

To pass from a combinatorial object, like a matroid, to its polytope, we proceed as follows. Elements $e \in E$ correspond to standard basis vectors $b_e$ of $\mathbb{R}^{|E|}$. Any subset $A \subset E$ can be identified with a point $p_A := \sum_{e \in A} b_e \in \mathbb{R}^{|E|}$. In this way a family of subsets may be identified with a set of points.

**Definition 13.6** (Matroid basis polytope). Let $M$ be a matroid on the ground set $E$, with set of bases $\mathcal{B}$. We use the notation introduced above. We define the matroid basis polytope $P_M \subset \mathbb{R}^{|E|}$ as the convex hull of the points $p_B := \sum_{e \in B} b_e \in \mathbb{R}^{|E|}$, where $B$ ranges over the set $\mathcal{B}$ of all bases.

By construction, $P_M$ is a lattice polytope. We will study its toric variety $X_{P_M}$, as defined in Section 8.2. From now on we write $X_M$ since $P_M$ is determined by the matroid $M$. By definition, $X_M$ is the toric variety in
the projective space $\mathbb{P}^{n-1}$ that is paramatrized by the monomials $\prod_{i \in B} t_i$ where $B$ runs over $\mathcal{B}$. We have $\dim(X_M) \leq n-1$ and equality usually holds.

**Example 13.7.** Fix the uniform matroid $M = U_{2,4}$. Then $P_M$ is the regular octahedron. The toric variety $X_M \subset \mathbb{P}^5$ has the parametric representation

$$ (t_1, t_2, t_3, t_4) \mapsto (t_1 t_2, t_1 t_3, t_1 t_4, t_2 t_3, t_2 t_4, t_3 t_4). $$

The variety $X_M$ has dimension 3 and degree 4. Using implicitization as in Section 4.2, we find that its defining prime ideal is $\langle y_1 y_6 - y_2 y_5, y_2 y_5 - y_3 y_4 \rangle$.

One important property a lattice polytope and its associated toric variety can have is normality. This was introduced in Definitions 8.7 and 8.19. A result due to Neil White states that normality always holds for toric varieties arising from matroids [62].

**Theorem 13.8.** A matroid basis polytope is normal in the lattice it spans.

We shall present a proof of this theorem. Our approach rests on the following important combinatorial result. For a proof we refer to [45, 12.3.1].

**Theorem 13.9 (Matroid Union Theorem).** Let $M_1, \ldots, M_k$ be matroids on the same ground set $E$ with respective families of independent sets $\mathcal{I}_1, \ldots, \mathcal{I}_k$ and rank functions $r_1, \ldots, r_k$. Let

$$ \mathcal{I} := \{I \subset E : I = \bigcup_{i=1}^{k} I_i \text{ for } I_i \in \mathcal{I}_i\}. $$

Then $\mathcal{I}$ is also the family of independent sets for a matroid, known as the union of $M_1, \ldots, M_k$. The rank function for the union matroid satisfies:

$$ r(A) = \min_{S \subset A} \{|A \setminus S| + \sum_{i=1}^{k} r_i(S)\}. $$

As a corollary, we obtain the following theorem due to Jack Edmonds.

**Theorem 13.10.** Let $M$ be a matroid with rank function $r_M$. Then its ground set $E$ can be partitioned into $k$ independent sets if and only if $|A| \leq k \cdot r_M(A)$ for all subsets $A \subset E$.

**Proof.** The implication $\Rightarrow$ is straightforward.

For the other implication consider the union $U$ of $M$ with itself $k$ times. We apply the matroid union theorem to compute the rank of $E$:

$$ r_U(E) = \min\{|E| - |B| + k \cdot r_M(B)\}. $$

Clearly by assumption $|E| - |B| + k \cdot r_M(B) \geq |E|$ and equality holds for $B = \emptyset$. Hence, $r_U(E) = |E|$. This means that $E$ is an independent set in $U$, and hence by definition it is a union of $k$ independent sets of $M$. \qed
13.2. Lattice Polytopes

Let $M$ be a matroid with the family of independent sets $\mathcal{I}$. We take $E = \{1, 2, \ldots, n\}$ and consider any subset $E' \subset E$. The restriction of $M$ to $E'$ is the matroid where $A \subset E'$ is independent if and only if $A \in \mathcal{I}$.

Proof of Theorem 13.8. Let $p \in kP_M$. We know that $p = \sum_{B \in \mathcal{B}} \lambda_B p_B$ with $\sum \lambda_B = k$ and $0 \leq \lambda_B \in \mathbb{Q}$. After clearing denominators, we have

$$dp = \sum \lambda_B p_B,$$

where $\sum \lambda_B' = dk$ and $0 \leq \lambda_B' \in \mathbb{Z}$. By restricting the matroid $M$, we may assume that all coordinates of $p = (p_1, \ldots, p_n)$ are nonzero.

We define two matroids. The first matroid $N$ is on the ground set $E_N := \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq p_i\}$. In other words, we replace a point $i$ in the original matroid by $p_i$ new points. A subset $\{(i_1, j_1), \ldots, (i_s, j_s)\} \subset E_N$ is independent in the matroid $N$ if and only if:

- all $i_q$'s are distinct,
- $\{i_1, \ldots, i_s\}$ is an independent set in $M$.

Bases of $N$ map naturally to bases of $M$. Also the rank function for $N$ is the same as the one for $M$ if we forget the second coordinates. The point $p$ has a decomposition as a sum of $k$ points corresponding to bases of $M$ if and only if the matroid $N$ is covered by $k$ bases (i.e. the ground set is a union of $k$ bases). Hence, by Theorem 13.10 our aim is to prove the following statement: For any $A \subset E_N$ we have $|A| \leq kr_N(A)$.

For the second matroid $N'$, we replace any point of $N$ by $d$ new points. The ground set of $N'$ equals $E_{N'} := \{(i, j, l) : 1 \leq i \leq n, 1 \leq j \leq p_i, 1 \leq l \leq d\}$. A subset $\{(i_1, j_1, l_1), \ldots, (i_s, j_s, l_s)\} \subset E_N$ is independent if and only if:

- all $i_q$’s are distinct,
- $\{i_1, \ldots, i_s\}$ is an independent set in $M$.

We have a natural projection $\pi : E_{N'} \to E_N$ given by forgetting the last coordinate. We note that $r_{N'}(\pi^{-1}(A)) = r_N(A)$. As the point $dp$ is decomposable we know that the matroid $N'$ can be covered by $kd$ bases. Hence, $|B| \leq dk \cdot r_{N'}(B)$ for any $B \subset E_{N'}$. Applying this to $\pi^{-1}(A)$ we obtain:

$$k|A| = |\pi^{-1}(A)| \leq dk \cdot r_{N'}(\pi^{-1}(A)) = dk \cdot r_N(A).$$

This is equivalent to the statement we wanted to prove.

Our next aim is to relate matroids with the geometry of special subvarieties of Grassmannians. We recall that one definition of the Grassmannian is that $G(k, n)$ is the orbit of $[e_1 \wedge \cdots \wedge e_k] \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ under the action of the group $GL(n)$ of invertible $n \times n$ matrices. While the Grassmannian is an orbit of the big group $GL(n)$, we may ask how smaller groups act on $G(k, n)$. In particular, consider the torus $T := (K^*)^n$ of diagonal matrices. This acts
on $\mathbb{P}(\bigwedge^k K^n)$ and on $G(k, n)$. However, $G(k, n)$ is generally not an orbit or orbit closure of $T$, since $\dim(G(k, n)) = k(n - k)$ exceeds $\dim(T) = n$. Let us fix a point $p \in G(k, n)$. The following questions motivate us:

- What is the $T$-orbit of $p$?
- What is the closure of this orbit?
- How can we describe this variety?

A beautiful answer was provided by Gelfand, Goresky, MacPherson and Serganova [24]. A point $p = [v_1 \wedge \cdots \wedge v_k] \in G(k, n)$ represents the subspace $V = \langle v_1, \ldots, v_k \rangle$ in $K^n$. We write the vectors $v_1, \ldots, v_k$ as a $k \times n$ matrix $N_p$. From Chapter 5, we know that the coordinates of $p \in \mathbb{P}(\bigwedge^k K^n)$ are the maximal minors of $N_p$. How does a torus element $t = (t_1, \ldots, t_n) \in T$ act on $p$? In general, $t$ acts on the coordinate of $e_{i_1} \wedge \cdots \wedge e_{i_k}$ by rescaling it with $t_{i_1} \cdots t_{i_k}$. Hence, the orbit of $p$ is the image of the monomial map

$$T \ni (t_1, \ldots, t_n) \mapsto (t_1 \cdots t_k \det_{i_1 \cdots i_k}(N_p))_{1 \leq i_1 < \cdots < i_k \leq n} \in \mathbb{P}(\bigwedge^k K^n).$$

**Example 13.11.** Fix $k = 2, n = 4$ and let $p \in G(2, 4)$ be the rowspace of

$$N_p = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

This is the same subspace as that seen in Examples 13.5 and 13.7. In Plücker coordinates on the Grassmannian, the corresponding point in $\mathbb{P}^5$ equals

$$(e_1 + e_2 + e_3 + e_4) \wedge (e_1 + 2e_2 + 3e_3 + 4e_4) = e_1 \wedge e_2 + 2e_1 \wedge e_3 + 3e_1 \wedge e_4 + e_2 \wedge e_3 + 2e_2 \wedge e_4 + e_3 \wedge e_4.$$

Hence, the $T$-orbit of $p$ in $G(2, 4) \subset \mathbb{P}^5$ is parameterized as follows:

$$(13.2) \quad (t_1, t_2, t_3, t_4) \mapsto (t_1t_2, 2t_1t_3, 3t_1t_4, t_2t_3, 2t_2t_4, t_3t_4).$$

This is almost the same as the monomial map in (13.1). The only difference is the constants given by minors of the matrix $N_p$. However, these constants do not depend on $t \in T$. We can define an automorphism of $\mathbb{P}(\bigwedge^k K^n)$ that turns the orbit to an image of a monomial map, by rescaling the coordinates. The prime ideal defining this orbit equals $\langle 4y_1y_6 - y_2y_5, 3y_2y_5 - 4y_3y_4 \rangle$.

In general, if all $k \times k$-minors of $N_p$ are nonzero, in which case the matroid of $N_p$ is uniform, then the $T$-orbit of $p$ is isomorphic to the image of the map given by all squarefree monomials of degree $k$. The associated matroid polytope is the hypersimplex. If the matroid is not uniform then we delete all zero coordinates of $p$ and we consider the torus orbit in $\mathbb{P}^{[3]-1}$.

For instance, if the lower right matrix entry “4” in Example 13.11 is replaced by “3” then (13.2) becomes $(t_1, t_2, t_3, t_4) \mapsto (t_1t_2, 2t_1t_3, 2t_1t_4, t_2t_3, t_2t_4)$ and the image of this map is the hypersurface in $\mathbb{P}^4$ defined by $\langle y_2y_5 - y_3y_4 \rangle$. 
Our discussion establishes the following general result.

**Proposition 13.12.** The closure of the $T$-orbit of any point $p = [v_1 \wedge \cdots \wedge v_k]$ in a Grassmannian $G(k, n)$ is isomorphic to the toric variety represented by the matroid basis polytope, for the representable matroid defined by columns of the $k \times n$ matrix $N_p$ with $i$-th row equal to $v_i$.

The results of Chapter 8 combined with Theorem 13.8 imply:

**Proposition 13.13.** Any torus orbit closure in any Grassmannian in the Plücker embedding is projectively normal.

We conclude that the toric varieties associated with matroids include all torus orbit closures in Grassmannians. But they are more general because there exist matroids that are not representable. We now present a toric variety that arises as torus orbit closure in $G(3, 7)$ only when char$(K) = 2$.

**Example 13.14** (Fano matroid). Fix $n = 7$ and let $M$ be the Fano matroid from Example 13.4. This has 7 nonbases and hence $|\mathcal{B}| = 28 = \binom{7}{3} - 7$ bases. Thus, the toric variety $X_M$ lives in $\mathbb{P}^{27}$. It has dimension 6 and degree 232. The prime ideal of $X_M$ is minimally generated by 140 quadrics $y_i y_j - y_k y_l$.

We now turn to the interpretation of basis exchange properties in terms of toric ideals. Consider a matroid with basis polytope $P$. We recall that:

- the ideal of the associated toric variety is generated by binomials,
- every binomial in the ideal corresponds to an integral relation among lattice points of $P$.

How do these statements specialize in the case of matroids? Vertices of $P$ are characteristic vectors of bases. A sum of vertices is a sum of characteristic vectors. This corresponds to taking a union of bases as *multisets*.

**Example 13.15.** Consider the uniform matroid $U_{2,4}$. Its matroid polytope is a regular octahedron. A typical integral relation among its vertices is

$$(1, 1, 0, 0) + (0, 0, 1, 1) = (1, 0, 1, 0) + (0, 1, 0, 1).$$

As a union of bases this corresponds to the identity

$$\{1, 2\} \cup \{3, 4\} = \{1, 3\} \cup \{2, 4\}.$$  

With appropriate indexing of the variables, this translates into the quadratic binomial $y_{12}y_{34} - y_{13}y_{24}$ in the ideal of the associated toric threefold in $\mathbb{P}^5$.

Two multisets of bases are *compatible* if their union (as multisets) is the same. Equivalently, every element of $E$ belongs to the same number of bases in the first and second multiset of bases. We conclude that the binomials in the ideal of the toric variety represented by matroid basis polytope are
in bijection with pairs of compatible multisets of bases. The quadrics in a toric ideal correspond to pairs of basis pairs \((\{B_1, B_2\}, \{B_3, B_4\})\) satisfying \(B_1 \cup B_2 = B_3 \cup B_4\). In transitioning between these basis pairs, we change

- \(B_1\) by subtracting a set \(A_1 \subset B_1 \setminus B_2\) and adding \(A_2 \subset B_2 \setminus B_1\) and
- \(B_2\) by adding to it \(A_1\) and subtracting \(A_2\).

We see that quadrics in the ideal correspond to multiple symmetric exchanges. It follows that symmetric basis exchanges form a distinguished set of quadrics in the ideal. The following four conjectures are due to White.

**Conjecture 13.16.** The following is true for arbitrary matroids \(M\):

- **Representable case:** The ideal of any \(T\)-orbit in a Grassmannian is:
  1. generated by quadrics,
  2. generated by quadrics corresponding to symmetric basis exchanges.

- **General case:** Any two finite multisets of bases \((B_i), (B_j)\) such that \(\bigcup B_i = \bigcup B_j\) can be transformed to one another in a finite number of such steps that:
  1. we replace two bases \(B, B'\) in one multiset by two bases \(\tilde{B}, \tilde{B}'\) obtained by multiple symmetric exchange (i.e. \(B \cup B' = \tilde{B} \cup \tilde{B}'\)),
  2. we replace two bases \(B, B'\) in one multiset by two bases \(\tilde{B}, \tilde{B}'\) obtained by a symmetric exchange (i.e. \(B = \tilde{B} \cup \{b_1\}\setminus\{b_2\}\) and \(B' = \tilde{B}' \cup \{b_2\}\setminus\{b_1\}\)).

In these conjectures, the general case implies the representable case.

### 13.3. Generating Functions

In this section we introduce multivariate generating functions that are given by rational functions. The key example is multigraded Hilbert series. We discuss methods for computing them, and we explore connections to regular triangulations. In particular, we discuss Ehrhart series of lattice polytopes.

We start with the familiar example of the polynomial ring \(K[x]\). In Chapter 1 we introduced its Hilbert function \(d \mapsto h(d)\). The value \(h(d) = \dim_K K[x]_d\) is the number of monomials of degree \(d\), or, equivalently, the number of lattice points \(u\) satisfying \(u \geq 0\) and \(\sum_{i=1}^n u_i = d\). Let \(\Delta\) be the standard simplex, i.e. the convex hull of the basis vectors. The Hilbert function counts the number of lattice points in dilations of \(\Delta\), precisely \(h(d) = |d\Delta \cap \mathbb{Z}^n|\). Further, by Example 1.22 the Hilbert series equals

\[
\text{HS}(z) = \sum_{q=0}^{\infty} |q\Delta \cap \mathbb{Z}^n|z^q = \frac{1}{(1 - z)^n},
\]
Our next aim is to refine our counting. So far we have treated all monomials of the same degree on an equal footing. What happens if we try to remember each monomial, not only its degree? Then we obtain the generating function

\[ MHS(x) = \prod_{i=1}^{n} \frac{1}{1 - x_i} = \sum_{u \in \mathbb{N}^n} x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}. \]

Note that the polytopes \( q\Delta \) are slices of the nonnegative orthant \( C = \mathbb{R}_{\geq 0}^n \). The sum in Equation (13.3) may be regarded as a sum over all lattice points of \( C \). We next replace \( C \) by more general cones.

**Definition 13.17.** Let \( C \subset \mathbb{R}^n \) be a rational pointed polyhedral cone. We define the associated multigraded Hilbert series as a formal power series:

\[ MHS_C(x) = \sum_{c \in C \cap \mathbb{Z}^n} x^c. \]

If \( C \subset \mathbb{R}_{\geq 0}^n \) then we reconstruct the Hilbert series of \( K[C] \) from the multigraded Hilbert series, by setting \( x_1 = \cdots = x_n = z \). However, \( MHS_C \) remembers much more information: all lattice points of the cone. Our next aim is to represent \( MHS_C \) as a rational function, just as we did for \( C = \mathbb{R}_{\geq 0}^n \).

A cone generated by linearly independent vectors is called simplicial.

**Lemma 13.18.** Let \( C \) be a simplicial cone with generators \( c_1, \ldots, c_d \in \mathbb{Z}^n \). There is a Laurent polynomial \( \kappa_C(x) \) with non-negative coefficients such that

\[ MHS_C(x) = \frac{\kappa_C(x)}{\prod_{i=1}^{d} (1 - x^{c_i})}, \]

**Proof.** Consider the following half-open parallelepiped:

\[ P := \{ x \in C : x = \sum_{i=1}^{d} \lambda_i c_i, 0 \leq \lambda_1, \ldots, \lambda_d < 1 \}. \]

Since the \( c_i \) are linearly independent and they generate \( C \) as a cone, every lattice point \( c \in C \) has a unique representation \( c = p + \sum_{i=1}^{d} s_i c_i \), where \( p \in P \) is a lattice point and \( s_i \) are non-negative integers. We obtain:

\[ MHS_C(x) = \sum_{c \in C \cap \mathbb{Z}^n} x^c = \sum_{p \in P \cap \mathbb{Z}^n} x^p \left( \prod_{i=1}^{d} \left( \sum_{s_i=0}^{\infty} x^{s_i c_i} \right) \right) \]

\[ = \sum_{p \in P \cap \mathbb{Z}^n} x^p \left( \prod_{i=1}^{d} \frac{1}{1 - x^{c_i}} \right) = \frac{\sum_{p \in P \cap \mathbb{Z}^n} x^p}{\prod_{i=1}^{d} (1 - x^{c_i})}. \]

Hence \( \kappa_C(x) \) is the sum of all monomials representing lattice points in \( P \).

**Remark 13.19.** We did freely manipulate with infinite summations in Lemma 13.18. This is justified by the hypothesis that \( C \) is pointed. This ensures that the series is absolutely convergent in some neighborhood of 0.
Proposition 13.20. Let $C$ be a pointed, rational polyhedral cone with generators $c_1, \ldots, c_d \in \mathbb{Z}^n$. Here $C$ is not necessarily simplicial. Then we have

$$MHS_C(x) = \frac{\kappa_C(x)}{\prod_{i=1}^d (1 - x_i^e)},$$

where $\kappa_C(x)$ is a Laurent polynomial with integral coefficients.

Proof. We triangulate the cone $C$, i.e. we write it as a union of simplicial cones with rays $c_i$ that intersect in lower-dimensional simplicial cones. This may be done e.g. by induction on the number $d$. Using inclusion-exclusion over all cones, we write $MHS_C(x)$ as an alternating sum of multivariate Hilbert series for simplicial cones. Lemma 13.18 now implies the claim. □

Remark 13.21. We note that the proofs of Proposition 13.20 and Lemma 13.18 give us an algorithm to compute the multigraded Hilbert series. It also gives a combinatorial interpretation of the numerator $\kappa_C(x)$.

The case when $C$ is a cone over a lattice polytope $P \subset \mathbb{R}^n$ is particularly nice. We identify $P$ with the polytope $P \times \{1\} \subset \mathbb{R}^n \times \mathbb{R}$. Let $C \subset \mathbb{R}^{n+1}$ be the cone over $P$, i.e. the smallest cone that contains $P \times \{1\}$.

Proposition 13.22. Let $C$ and $P$ be as defined above. The Hilbert function for $C$ with respect to the grading induced by the last variable is given by:

$$h(q) = |qP \cap \mathbb{Z}^{n+1}|.$$

It counts the number of lattice points in dilations of $P$. It is known as the Ehrhart polynomial of $P$. It coincides with a polynomial for $q \in \mathbb{Z}_{\geq 0}$.

Proof. The only nontrivial statement is that $h(q)$ is equal to a polynomial for all positive integers $q$. By induction on $d := \dim P$ and by triangulating $P$, it is enough to prove the statement when $P$ is a simplex with vertices $v_1, \ldots, v_d$. Let $f(q) := \binom{d+q-1}{q} = \binom{d+q-1}{d-1}$ for $q \geq 0$ and $f(q) = 0$ for $q < 0$. Similarly to the proof of Lemma 13.18 we have:

$$h(q) = \sum_{i=0}^{d-1} a_i f(q - i),$$

where $a_i$ is the number of lattice points with the last coordinate $i$ in the set

$$\{x : x = \sum_{i=1}^d \lambda_i (v_i, 1), 0 \leq \lambda_1, \ldots, \lambda_d < 1\}.$$

We only have to consider $a_i$ for $i < d$, as there are no lattice points in this parallelepiped with last coordinate greater or equal to $d$. 

We note that $f$ is not a polynomial if we consider the negative arguments. The punchline is that the polynomial $g(q) := (d+q-1)(d+q-2) \cdots q/(d-1)!$ equals $f$ also for negative, integral $q$, as long as $q \geq -d+1$. Hence, we have:

$$h(q) = \sum_{i=0}^{d-1} a_i g(q - i) \text{ for } q \in \mathbb{Z}_{\geq 0}.$$ 

This sum is a polynomial in $q$. This is the Ehrhart polynomial of $P$. □

Given a lattice polytope $P$ with $N$ lattice points, we associate to it a projective toric variety $X_P$ in $\mathbb{P}^{N-1}$ as in Chapter 8. Hence, we obtain a binomial ideal $I_P \subset K[x] = K[x_1, \ldots, x_N]$. Fix a term order $\prec$. The initial ideal in $\prec(I_P)$ satisfies:

- if a product $m$ of distinct variables does not belong to $\mathrm{rad} \in \prec(I_P)$, then neither does any monomial that divides $m$.

This may be restated as:

- subsets $S$ of variables such that $\prod_{x \in S} x \notin \mathrm{rad} \in \prec(I_P)$ form a simplicial complex.

Our aim is to obtain a nice, geometric description of this simplicial complex. The main idea is that the variables $x_i$ are in bijection with lattice points of $P$. Let $\Delta$ be a subdivision of $P$ into polytopes $P_i$ that are convex hulls of sets of points $S$, such that the product of variables in $S$ is not in $\mathrm{rad} \in \prec(I_P)$.

**Example 13.23.** Let $P$ be the square $\overline{conv}((0,0,1), (0,1,1), (1,0,1), (1,1,1))$. The associated projective variety is a surface in $\mathbb{P}^3$ defined by $x_1x_4 - x_2x_3$. We fix a term order for which $x_1x_4$ is the leading term. The subdivision $\Delta$ of $P$ contains two triangles: $\overline{conv}((0,0,1), (0,1,1), (1,0,1), (1,1,1))$ and $\overline{conv}((0,1,1), (1,0,1), (1,1,1))$. The minimal nonface is the pair of vertices $(0,0,1)$ and $(1,1,1)$. This pair corresponds to the unique generator $x_1x_4$ of the initial ideal. If we change the term order so that $x_2x_3$ becomes the leading term, then we obtain a different triangulation of $P$, given by the other diagonal.

Our next result relates Gröbner bases to triangulations.

**Theorem 13.24.** Using the notation introduced above, $\Delta$ is a triangulation of $P$. The minimal non-faces of $\Delta$ correspond to generators of $\mathrm{rad} \in \prec(I_P)$.

**Proof.** The proof can be found in [57, Chapter 8]. □

**Definition 13.25.** The triangulations of the form $\Delta$, induced by any term order, are called regular. There exist triangulations that are not regular.

The story that we are telling has in fact three sides: combinatorial, algebraic and geometric. From the combinatorial point of view, we are
triangulating a lattice polytope \( P \) into simplices. The algebraic part is the finest: we degenerate a toric ideal \( I_P \) to a monomial ideal that shares with \( I_P \) all the most important invariants, like dimension and degree. Let us now describe the geometry in this picture. Here we degenerate the variety \( \mathcal{V}(I_P) \) to \( \mathcal{V}(\text{in}_{\mathcal{L}}(I_P)) \). One of the problems is that \( \text{in}_{\mathcal{L}}(I_P) \) maybe not radical, thus we may loose some information, however let us ignore this for a moment.

What is \( \mathcal{V}(\text{in}_{\mathcal{L}}(I_P)) \)? This variety equals \( \mathcal{V}(\text{rad}(\text{in}_{\mathcal{L}}(I_P))) \), so the question is: what is the set of solutions of a squarefree monomial ideal? The answer is given by the solution to Exercise 12 in Chapter 2. We recall this:

- the variety \( \mathcal{V}(\text{rad}(\text{in}_{\mathcal{L}}(I_P))) \) is a union of coordinate subspaces. Each is spanned by basis vectors \((e_i)_{i \in S}\) with \( \prod_{i \in S} x_i \notin \text{rad}(\text{in}_{\mathcal{L}}(I_P)) \).

In particular: the simplices in the induced triangulation of \( P \) correspond naturally to components of \( \mathcal{V}(\text{in}_{\mathcal{L}}(I_P)) \) - as the triangulation breaks the polytope into simple pieces, our variety breaks into simple components.

We note that the idea of computing the dimension and degree of an ideal, by passing to the initial ideal is equivalent to the idea of computing the Hilbert series of a cone, by subdividing it into simplicial cones.

We know that the dimension of the (projective) toric variety associated to a polytope \( P \) equals the dimension of \( P \). What about the degree?

**Proposition 13.26.** Let \( P \subset \mathbb{R}^d \) be a lattice polytope, where \( P \cap \mathbb{Z}^d \) spans the lattice \( \mathbb{Z}^d \). The degree of the ideal \( I_P \) equals the volume of \( P \) times \( d! \).

**Sketch of the proof.** The degree times the factorial of \( \dim(X_P) \) is the leading coefficient of the the Ehrhart polynomial \( h \). Thus it is enough to show the following claim: for any \( \epsilon > 0 \) there exists a constant \( C \) with

\[
(13.4) \quad (\vol P - \epsilon)q^d - C \leq h_P(q) \leq (\vol P + \epsilon)q^d + C \quad \text{for all } q \in \mathbb{Z}_{>0},
\]

where \( h_P(q) \) is the number of lattice points in \( qP \). It is easy to prove the inequality \([13.4]\) for rational polytopes that are products of intervals. This fact implies the claim, by covering \( P \) with small products of intervals, according to the definition of the Lebesgue measure.

Thus, \( h_P \) is of degree \( d \) and has the leading coefficient equal to \( \vol P \). The proposition follows by the definition of the degree. \( \square \)

**Example 13.27.** Let \( P \) be a \( d \)-dimensional simplex, given as the convex hull of 0 and \( d \) basis vectors. The Ehrhart polynomial is given by \( h(q) = \binom{d+q}{d} = \frac{1}{d!}q^d + \text{lower order terms} \). Indeed \( \vol P = \frac{1}{d!} \) and \( \dim P = d \).

The usual Euclidean volume multiplied by \( d! \) is known as the *normalized volume*. The simplex \( \text{conv}(0, e_1, e_2, \ldots, e_d) \) has normalized volume equal to one. Every lattice polytope has normalized volume that is a positive integer, equal to the degree of the toric variety, if one works in the correct lattice.
How is the triangulation compatible with the degree computation? The sum of volumes of its (maximal) simplices is equal to the volume of $P$.

**Theorem 13.28.** Let $P$ be a $d$ dimensional lattice polytope whose lattice points generate the lattice $\mathbb{Z}^d$. Let $I_P$ be the associated toric ideal. Let $<$ be a term order and $\Delta$ the associated triangulation of $P$.

1. The minimal primes of $\text{in}_<(I_P)$ are in bijection with the maximal simplices in the triangulation $\Delta$.

2. The primary ideal corresponding to a minimal prime of $\text{in}_<(I_P)$ has as degree the normalized volume of the associated simplex in $\Delta$.

---

**Exercises**

1. Show that Example 13.2 presents matroids.

2. (a) Fix the family of independent sets $\mathcal{I}$ for a matroid $M$. Prove that the inclusion maximal elements in $\mathcal{I}$ satisfy the axioms for bases.

   (b) Fix a nonempty set $\mathcal{B} \subset 2^E$ satisfying the axiom for basis of a matroid. Prove that $\mathcal{I} := \{I \subset E : \exists B \in \mathcal{B} : I \subset B\}$ satisfies the axioms for the independent sets.

3. Prove that all bases in a matroid have the same cardinality.

4. Prove that the points $p_B$ in Definition 13.6 are vertices of the polytope $P_M$. Prove that these are the only lattice points in $P_M$.

5. In this exercise we examine matroid duality.

   (a) Let $\mathcal{B} \subset 2^E$ be a set of bases of a matroid $M$. Let $\mathcal{B}^* := \{B \subset E : E \setminus B \in \mathcal{B}\}$. Prove that $\mathcal{B}^*$ is a set of bases of a matroid $M^*$. The matroid $M^*$ is known as the dual matroid (of $M$).

   (b) Prove that a dual of a representable matroid is representable.

6. Prove that for any matroid the rank function is submodular.

7. Prove that any function $2^E \to \mathbb{Z}$ satisfying the three axioms of the rank function is indeed a rank function of some matroid.

8. How many distinct torus orbit closures are there in $G(2,4)$? How many are there up to isomorphism (of algebraic varieties)?


10. Let $M$ be the rank 3 matroid on $E = \{1, 2, \ldots, 8\}$ with nonbases $124, 235, 346, 457, 568, 671, 782, 823$. Is $M$ representable? Compute $\mathcal{R}(M)$.

11. Let $I$ be an ideal generated by three linear polynomials in five variables, so that $V(I)$ is a plane in $K^5$. Determine the tropical plane trop($V(I)$).

12. Use Theorem 13.28 to triangulate the matroid polytopes of $U_{3,5}$ and $U_{3,6}$.
(13) Carry out the computation reported in Example 13.14. Identify an initial monomial ideal of this toric ideal. Compute its Hilbert series MHS.

(14) Find a matroid $M$ of rank 4 that has precisely 12 bases. Determine its rank function $r$ and its polytope $P_M$. Verify White’s conjecture for $M$.

(15) Let $C$ be the cone over the regular 3-cube $[0,1]^3$. Compute $\text{MHS}_C$. 

Bibliography


Index

adjacency matrix, 102
adjoint matrix, 163
adjoint representation, 154
affine cone, 27
affine variety, 19, 114
algebraic statistics, 22
algebraic torus, 108, 113, 153
analytic center, 192
Aronhold invariant, 141, 175
Artin’s Theorem, 90
assignment problem, 103
associated prime, 46

Bézout’s Theorem, 26, 77
Bertini’s Theorem, 27
binomial, 116
Birch point, 123
border rank, 138

Catalan number, 81
character, 114, 151–153
Chevalley’s Theorem, 29
Chinese remainder theorem, 23
Cholesky factorization, 191
class function, 152
closed image, 58, 61, 114
complementary slackness, 187
complete intersection, 26
complete symmetric polynomial, 153
cone, 117
conic curve, 31
constructible set, 25, 67
coordinate ring, 22
cosine moment curve, 188
covariance matrix, 130
cubic curve, 32
cusp, 35
cuspidal curve, 35, 115
degree, 2, 14, 15, 25
do a Grassmannian, 81
Derksen’s Algorithm, 176
Descartes’ Rule of Signs, 40
Dickson’s Lemma, 7
dimension, 14, 15, 25
do a face, 117
discriminant, 174, 176, 178
dual curve, 185

Ehrhart
polynomial, 206, 207
series, 204
eigenspace
of a tropical matrix, 106
eigenvalue, 105, 187
eigenvector
of a symmetric matrix, 130
of a tensor, 134
elementary symmetric polynomial, 153, 168
elimination ideal, 58, 60, 61, 63
elliptic curve, 32, 175
embedded prime, 47
Erlanger Program, 167
Euclidean Algorithm, 4
Euclidean algorithm, 6
Index

exponent of matrix multiplication, 144
Extended Buchberger Algorithm, 87
Extended Euclidean Algorithm, 6

face, 117
facet, 118
Fano plane, 198
Farkas' Lemma, 92
flag of linear subspaceses, 78
flattening of a tensor, 137, 158
Fundamental Theorem of Algebra, 24
Fundamental Theorem of tropical geometry, 109

Gaussian graphical model, 192
Gröbner basis, 9, 60, 76, 86
Grassmannian, 62, 71, 73, 156, 173, 203

hexagon invariant, 177
highest weight, 154
Hilbert function, 12, 15, 204, 206
Hilbert polynomial, 13–15
Hilbert series, 12, 15, 171, 204
multigraded, 205
Hilbert's Basis Theorem, 9, 42
Hilbert's Finiteness Theorem, 170
Hilbert's Nullstellensatz, 76, 88
homogeneous ideal, 27
homogeneous polynomial, 27
homogeneous variety, 73, 156
hyperdeterminant, 62, 63, 69, 138, 177

ideal, 3
ideal quotient, 4, 48
implicitization, 61, 116
independence model, 21, 122
initial
form, 108
ideal, 8, 207
monomial, 8

integer decomposition property, 120
integral domain, 5
invariant ring, 168, 171
irreducible
ideal, 44
representation, 149, 154
variety, 20, 22
isotypic component, 151

j-invariant, 175
Jacobian matrix, 26

Kalman variety, 30
KKT equations, 187, 189, 194
Kleene plus, 105
Kostant's theorem, 163

Lagrangian Grassmannian, 83
Lasserre hierarchy, 192
lattice, 114
lattice polytope, 119, 199, 204
Laurent monomial, 114
Laurent polynomial, 108
lexicographic order, 8, 11
Linear Programming Duality, 92
Littlewood-Richardson rule, 80
log-linear model, 123

Möbius strip, 31
Macaulay2, 76
Maschke's theorem, 149
matrix completion, 59, 192
matrix multiplication tensor, 141, 145
matroid, 196
algebraic, 196
basis of, 197
basis polytope, 199, 203
graphic, 196
rank, 197
representable, 196
restriction, 201
uniform, 196
union, 200
maximal ideal, 5
maximum likelihood, 122, 123, 125, 192
minimal prime, 47, 209
modular invariant theory, 170
Molien's Theorem, 171, 172
moment map, 121, 128
monomial ideal, 8, 49, 207
monomial order, 7
morphism
of representations, 149
Motzkin polynomial, 90, 93, 190
multigraded Hilbert series, 205

Newton's identities, 168
nilpotent
matrix, 89
ring element, 4
Noether's Degree Bound, 170
Noetherian ring, 44
nonnegative rank, 141
normal
polytope, 127
Index
variety, 118, 120
normal polytope, 200
normal variety, 115
normalized volume, 208
nullcone, 174
Nullstellensatz, 85
certificate, 86
odeco, 134
orthogonal decomposition
of a matrix, 132
orthogonally decomposable tensor, 134
oval, 31
PDE, 49
Pfaffian, 62
pfaffian, 76, 162
PID, 4
Pieri rule, 81
Plücker quadric, 75, 162
Plücker relations, 62, 74, 76, 173
plethysm, 162
polytope, 119, 199
positive definite, 182
positive semidefinite, 182
positive toric variety, 121
Positivstellensatz, 92
power iteration, 134
power sum polynomial, 168
primary decomposition, 45
primary ideal, 5, 43, 45, 209
prime ideal, 5, 20, 22, 43
projective closure, 28
projective space, 27, 30
projective variety, 27, 73, 119, 207
pseudo-line, 31
Puiseux series, 40, 99, 107
quadratic module, 92
quotient space, 169
radical ideal, 5, 89
radical of an ideal, 17, 88
rank
of a tensor, 133
radical map, 130
rational normal curve, 29
read radical, 93
real Nullstellensatz, 90, 93, 181
realization space, 198
reduced Gröbner basis, 9
reductive group, 150
representation, 148
completely reducible, 150
semi-simple, 150
resultant, 63
Reynolds operator, 169
Riemann sphere, 30
Robinson polynomial, 94
roots
in representation theory, 154
Salmon problem, 163
saturated monoid, 115
saturation, 48
scheme, 54
Schubert calculus, 77, 79
Schubert variety, 78
Schur polynomial, 155
Schur's Lemma, 149
Schur-Weyl duality, 157
SDP, 185
secant variety, 139, 161
Segre variety, 119, 136
semi-algebraic set, 25, 66
semi-simple
representation, 150
semiring, 98
simplicial complex, 196
singular locus, 26
singular point, 26
singular value decomposition, 132
singular vector, 131, 136
smooth point, 26
SOS programming, 192
Specht module, 157
spectrahedral shadow, 188
spectrahedron, 182
spectral decomposition, 132
Spectral Theorem, 181
spectrum of a ring, 22
standard monomial, 10
standard Young tableau, 81
Strassen equation, 163
Strassen invariant, 140
Strassen's Algorithm, 143
submodularity, 197
sub-representation, 149
sum of squares, 90
supporting hyperplane
of a cone, 118
Sylvester matrix, 64
symmetric exchange property, 197
symmetric polynomials, 168
symmetric rank, 139
tangent space, 26
tangential variety, 139
Tarski-Seidenberg Theorem, 66
tensor decomposition, 133
Toeplitz spectrahedron, 188
toric ideal, 116
toric model, 123
toric variety, 114, 119, 125, 203, 207
  normal, 115
torus orbit, 120
transportation polytopes, 122
triangular prism, 121
triangulation, 204, 207, 209
tropical
  convexity, 106
determinant, 103
eigenvalue and eigenvector, 104
polynomial, 107
polytope, 106
projective torus, 106
rank, 104, 109
semiring, 97
triangle, 107
variety, 109
zero, 107
tropical linear space, 199
tropical polynomial, 100
tropical rank one, 102
tropical root, 100
tropicalization
  of a polynomial, 100, 107
  of a variety, 109
  of determinant, 103
ultrametric, 99
unstable points, 174
valuation, 99
value group, 99
variety, 19, 22
Veronese variety, 69, 119, 156
W-state, 137, 145
Weak Duality Theorem, 186
Weierstrass form, 32
weight spaces, 153
weights, 153
Young diagram, 82, 154, 155
Young flattening, 160
Zariski topology, 20, 22
zero divisor, 5