Problem Sheet 6

1. Asymptotic fraction of values Let $\nu \in Y^p(\Omega, \mathbb{R}^n)$ be a Young measure with generating sequence $(V_j) \subset \mathrm{L}^p(\Omega, \mathbb{R}^n)$. Show that for every closed set $E \subset \mathbb{R}^n$ it holds that

$$\nu_x(E) = \lim_{r \to 0} \lim_{j \to \infty} \frac{|\{x \in B_r(x) \colon V_j(x) \in E\}|}{|B_1(0)| r^d}$$

Also show that this formula fails in general when E is not closed.

2. An existence result Let $f \colon \mathbb{R}^{m \times n} \to \mathbb{R}$ be Borel-measureable and strongly quasiconvex, i.e. there exists $\lambda > 0$ such that $z \to f(z) - \gamma |z|^2$ is quasiconvex. Assume further that for some $\Lambda > 0$ and all $z \in \mathbb{R}^{m \times n}$, $|f(z)| \leq \Lambda(1 + |z|^2)$. Show that the functional

$$\mathscr{F}[u] = \int_\Omega f(\mathrm{D}u) \,\mathrm{d}x$$

attains its minimum on $\mathrm{W}^{1,2}_{Fx}(\Omega, \mathbb{R}^m)$ for any $F \in \mathbb{R}^{m \times n}$.

3. Swlsc Show that for $\Omega = (-1, 2)^2 \subset \mathbb{R}^2$ the functional

$$\mathscr{F}[u] = \int_\Omega \det(\mathrm{D}u_j) \,\mathrm{d}x$$

is not weakly lower semi-continuous on $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^2)$ by considering the sequence

$$u_j(x, y) = \frac{(1 - |x_2|)^j}{\sqrt{j}} (\sin(jx), \cos(jx)).$$

4. Quadratic forms Prove using Plancherel's identity that every quadratic form $q \colon \mathbb{R}^{m \times n} \to \mathbb{R}$ (i.e. $q(z) = b(z, z)$ for a bilinear $b \colon \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \to \mathbb{R}$) is quasiconvex if and only if it is rank-one convex.

5. Browder's fixed point theorem Let $B = \overline{B_1(0)}$.

(i) Let $w \colon B \to \partial B$ be smooth. Use $|w(x)|^2 = 1$ for every $x \in B$ to show that $\det(\mathrm{D}w) = 0$ in B.

(ii) Argue by contradiction and using that the determinant is a null-Lagrangian that there exists at least one $x \in \partial\Omega$ with $w(x) \neq x$.

(iii) Prove the following smooth version of Browder's fixed point theorem: Let $u \colon B \to B$ be smooth. Then u has a fixed point $x \in \partial B$. (Hint: argue by contradiction and consider rays starting at $u(x)$ and passing through x to reduce to (ii)).

(iv) Extend the proof to apply for continuous u.