Problem Sheet 4

1. Caccioppoli inequality Assume $u \in \mathrm{W}^{1,p}(\Omega)$ is a Q-quasi-minimiser for the functional $\mathscr{F}[u,\Omega] = \int_\Omega f(x,u,\mathrm{D}u)$, where we assume that

$$f(x,y,z) \geq |z|^p - \theta(x,u)^p$$
$$|f(x,y,z)| \leq \Lambda(|z|^p + \theta(x,u)^p).$$

Here $\theta(x,u)^p = b(x)|u|^\gamma + a(x)$ with $1 \leq \gamma < p^*$ and $0 \leq b \in \mathrm{L}^\sigma(\Omega)$, $0 \leq a \in \mathrm{L}^s(\Omega)$ and for some $\varepsilon > 0$, $\frac{1}{s} = \frac{p}{n} - \varepsilon$, $\frac{1}{\sigma} = 1 - \frac{\gamma}{p^*} - \varepsilon$. Prove that there exists $R_0 = R_0(|u|_{\mathrm{L}^{p^*}(\Omega)}, |b|_{\mathrm{L}^\sigma(\Omega)})$ such that for all $x_0 \in \Omega$, $0 < \rho < R < \min(R_0, d(x_0, \partial\Omega)$ and $k \geq 0$,

$$\int_{A(k,\rho)} |\mathrm{D}u|^p \,\mathrm{d}x \leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p \,\mathrm{d}x + c(|a|_{\mathrm{L}^s(\Omega)} + k^p R^{-n\varepsilon})|A(k,R)|^{1-\frac{p}{n}+\varepsilon}.$$

2. Iteration lemmas

(i) Let $\alpha > 0$ and let $\{x_i\}$ be a sequence of real positive numbers such that

$$x_{i+1} \leq C B^i x_i^{1+\alpha}$$

with $C > 0$ and $B > 1$. Show that if $x_0 \leq C^{-\frac{1}{\alpha}} B^{-\frac{1}{\alpha^2}}$, we have

$$x_i \leq B^{-\frac{i}{\alpha}} x_0.$$

In particular, $x_i \to 0$ as $i \to \infty$.

(ii) Let $\phi(t)$ be a positive function and assume there exists q, $0 < \beta < \delta$ and $0 < \tau < 1$ such that for every $R < R_0$,

$$\phi(\tau R) \leq \tau^\delta \phi(R) + BR^\beta \qquad \phi(t) \leq q\phi(\tau^k R).$$

Prove that for every $\rho < R \leq R_0$,

$$]\phi(\rho) \leq C\left(\left(\frac{\rho}{R}\right)^\beta \phi(R) + B\rho^\beta\right),$$

where C depends only on q, τ, δ and β.

3. General example Let $B_1 \subset \mathbb{R}^3$ and $g \in C^\infty(\mathbb{R}^3)$. What can you say about the problem

$$\min_{u \in \mathrm{W}^{1,p}_g(B_1)} \int_{B_1} |\mathrm{D}u|^4 + |u|^3 \,\mathrm{d}x?$$