Problem Sheet 3

1. Existence Prove the following existence statement: Suppose $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain. Let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$ be Carathéodory. Assume

$$\lambda|z|^p + c_1|y|^q + c_2 \leq f(x,y,z) \tag{1.1}$$

$$|\partial_y f(x,y,z)| + |\partial_z f(x,y,z)| \leq C\left(1 + |y|^{p-1} + |z|^{p-1}\right) \tag{1.2}$$

for some $\lambda, C > 0$, $c_1, c_2 \in \mathbb{R}$, $p > q \geq 1$ and almost every $x \in \Omega$, all $(y,z) \in \mathbb{R}^m \times \mathbb{R}^{n\times m}$. Assume $f(x,y,\cdot)$ is convex for every $(x,y) \in \Omega \times \mathbb{R}^m$. Then,

$$\min_{u\in \mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^m)} \mathscr{F}[u] = \min_{u\in \mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^m)} \int_\Omega f(x,u,\mathrm{D}u)\,\mathrm{d}x$$

admits at least one solution for $g \in \mathrm{W}^{1-\frac{1}{p},p}(\partial\Omega,\mathbb{R}^m)$.

2. Quasi-minimality Suppose $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain. Let $A = (A^i_\alpha(x,u,z))$, $B = (B_\alpha(x,u,z))$ be such that

$$A^i_\alpha(x,u,z)z^\alpha_i \geq |z|^p - a_1(x)$$

$$|A(x,u,z)| \leq \Lambda|z|^{p-1} + a_2(x) \qquad |B(x,u,z)| \leq H|z|^\tau + b(x)|u|^\delta$$

for some $\Lambda > 0$, $\tau = p-1$, $\delta = p\frac{p^*-1}{p^*}$ and with $0 \leq a_1, a_2^{\frac{p}{p-1}} \in \mathrm{L}^1(\Omega)$, $0 \leq b^{\frac{p^*}{p^*-1}} \in LL^{\frac{p^*}{p^*-p}}(\Omega)$. Suppose $u \in \mathrm{W}^{1,p}_{tploc}(\Omega,\mathbb{R}^n)$ is a weak solution of the equation

$$\frac{\partial}{\partial x_i} A^i_\alpha(x,u,\mathrm{D}u) - B_\alpha(x,u,\mathrm{D}u) = 0.$$

Prove that u is a quasi-minimum of the functional

$$\mathscr{F}[u,\Omega] = \int_\Omega |\mathrm{D}u|^p + b^{\frac{p^*}{p^*-1}}|u|^p + (a_1 + a_2^{\frac{p}{p-1}})\,\mathrm{d}x.$$

3. Caccioppoli inequailty Let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$, where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain. Assume for some $\lambda, c > 0$ and $1 < p < n$,

$$\lambda|z|^p \leq F(x,y,z) \quad |F(x,y,z)| \leq c(|z|^p + \theta(x,u)^p).$$

Here $\theta(x,u)^p = b(x)|u|^\gamma + a(x)$, with $\gamma < p^* = \frac{np}{n-p}$, $0 \leq a \in \mathrm{L}^1(\Omega)$ and $0 \leq b \in \mathrm{L}^{\frac{p^*}{p^*-\gamma}}$.

Suppose $u \in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^m)$ is a quasi-minimiser of $\mathscr{F}[u] = \int_\Omega f(x,u,\mathrm{D}u)\,\mathrm{d}x$. Prove that there exists $R_0 > 0$, depending only on u, such that for $R < R_0$ and cubes $Q_R \Subset \Omega$,

$$\int_{Q_{R/2}} |\mathrm{D}u|^p + |u|^{p^*}\,\mathrm{d}x \leq c\left(\frac{1}{R^p}\int_{Q_R} |u-u_R|^p\,\mathrm{d}x + |Q_R|\left(\fint_{Q_R}|u|\,\mathrm{d}x\right)^{p^*} + \int_{Q_R} g\,\mathrm{d}x\right)$$

for some $c > 0$. Here $g = a + b^{\frac{p^*}{p^*-\gamma}}$, $\fint_{Q_R} = \frac{1}{|Q_R|}\int_{Q_R}$ and $u_R = \fint_{Q_R} u\,\mathrm{d}x$.