# Problem Sheet 2

**1. Hamilton-Jacobi** Let $f(x,u,\xi) = \sqrt{g(x,u)}\sqrt{1+\xi^2}$. Calculate the associated Hamiltonian system and find a first integral of the system when $g$ does not depend explicitly on $x$.

**2. Noether's theorem** Set for a skew-symmetric $W \in \mathbb{R}^{d\times d}$,

$$x_\tau = g(x,\tau) := \exp(\tau W)x, \quad u_\tau = H(x,\tau) := u(\exp(\tau W)x).$$

Considering the transformation defined by $(g,H)$ derive a conservation law for minimisers $u_\star \in (W^{1,2} \cap W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ of the Dirichlet integral $\int |\mathrm{D}u|^2 \,\mathrm{d}x$.

**3. Convergence of subsequences** Let $\mathscr{F}\colon X \to \mathbb{R}$, where $X$ is a complete metric space. Show that if every subsequence of $(u_j) \subset X$ with $u_j \to u$ in $X$ has a further subsequence $(u_{j(k)})_k$ such that

$$\mathscr{F}[u] \le \liminf_{k\to\infty} \mathscr{F}[u_{k(j)}],$$

then

$$\mathscr{F}[u] \le \liminf_{j\to\infty} \mathscr{F}[u_j].$$

**4. existence with lower-order terms** Let $\Omega$ be a Lipschitz domain and $g \in L^2(\Omega)$. Suppose $F\colon \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and strongly convex. Suppose it further satisfies the assumption

$$\Lambda^{-1}|\xi|^2 \le F(\xi) \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n.$$

Show that the problem

$$\min_{u\in X} \int F(\mathrm{D}u(x)) - g(x)u(x)\,\mathrm{d}x$$

admits a unique minimiser $\overline{u}$ in the class $X = W^{1,2}_0(\Omega)$. Show that for any $\phi \in W^{1,2}_0(\Omega)$,

$$\int_\Omega \langle \partial_z F(\mathrm{D}\overline{u}), \mathrm{D}\phi\rangle \,\mathrm{d}x = \int_\Omega g(x)\phi(x)\,\mathrm{d}x.$$