# Problem Sheet 1

**1. Euler-Lagrange equation** Let $\Omega \subset \mathbb{R}^n$ and $k \geq 1$. Suppose $f \colon \Omega \times \mathbb{R}^N \times \otimes^k(\mathbb{R}^n, \mathbb{R}^N)$ is given by

$$f(x, y, z) = |z|^2 - \frac{1}{2} c(x) y^2 + g(x) y$$

for some smooth functions $c$ and $g$. Derive the Euler-Lagrange equation for the functional.

$$\int_\Omega f(x, u, \mathrm{D}^k u)\, \mathrm{d}x.$$

**2. minimizers may not be $C^1$-regular** Let $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $f(u, \xi) = u^2(1 - \xi)^2$. Consider the problem

$$\inf_{u \in X} \{\mathscr{F}(u) = \int_{-1}^{1} f(u, u')\, \mathrm{d}x\} \quad \text{where} X = \{u \in C^1([-1, 1]) \colon u(-1) = 0,\ u(1) = 1\}.$$

Show that this problem has no solution in $X$. Moreover, find a solution to the problem amongst piecewise $C^1$-functions.

**3. infinitely many solutions** Show that the problem

$$\inf_{u \in X} \{\mathscr{F}(u) = \int_{-1}^{1} |u'|\, \mathrm{d}x \text{ where } X = \{u \in C^1([-1, 1]) \colon u(-1) = 0,\ u(1) = 1\}$$

has infinitely many solutions. Consider also the problem

$$\inf_{u \in X} \{\mathscr{F}(u) = \int_0^1 ((u')^2 - 1)^2\, \mathrm{d}x \text{ where } X = \{u \in C^1([0, 1] \colon u(0) = u(1) = 0\}.$$

Prove or disprove that there are infinitely many solutions. What about if

$$X = \{u \text{ piecewise } C^1([0, 1]) \colon u(0) = u(1) = 0\}?$$