Spherically Symmetric Models in General Relativity. A part III essay set by Dr Claude Warnick.

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Contents

[1]This section is not important for the rest of the essay. Nevertheless, the result presented here is of independent interest.

Abstract

In this essay, I present some recent development of the mathematical study of the weak cosmic censorship conjecture (WCCC) restricted to spherically symmetric systems. I show that for a large class of systems obeying the "Weak extension principle", the resolution of the WCCC can be reduced to the trapped surface conjecture, i.e. the conjecture that generically, the future development of suitable initial data is either geodesically complete or contains a black hole region. Finally, I show that the Einstein-Maxwell-Klein-Gordon system belongs to the class of systems considered here, potentially opening a way to understand the collapse of rotating black holes.

A short note on conventions: Claims, definitions or sections marked by *** are not necessary for the understanding of the rest of the essay but of independent interest. Moreover, this essay contains a lot of Penrose diagrams: We use the convention that dashed lines denote the topological boundary of these.

1 Introduction

Our modern understanding of gravity and how it interacts with matter is based on Einstein's theory of general relativity (GR). In it, spacetime is described as a 4-dimensional Lorentzian manifold with evolution governed by a certain system of partial differential equations (PDEs) called the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{1}$$

where $R_{\mu\nu}$,R denote the Ricci and scalar curvature of the metric $g_{\mu\nu}$ and $T_{\mu\nu}$ denotes the energy momentum tensor of matter whose evolution will also be governed by some equations of motion. In vacuum, the above equations can be shown to be of quasilinear hyperbolic type (in a suitable gauge) and have a well-posed Cauchy-problem.

One of the most remarkable predictions of GR is the prediction of its own limitations, namely the prediction of singularities. First noticed in the celebrated Schwarzschild solution, it was believed for a long time that singularities are just mathematical curiosities arising as artefacts from very high degrees of symmetries. However, in the 1960s, theorems by Hawking and Penrose [15, 22] showed some evidence that singularities are indeed generic features of gravitational collapse in GR by proving that non-compact initial data containing a closed trapped surface evolve to a future development with singularities. The natural question that has been studied[2] subsequently was whether trapped surfaces are evolutionary, i.e. if they can arise from initial data that contain no trapped surfaces. An important breakthrough in this question was published by Schoen and Yau [26]: They showed that asymptotically flat initial data containing a region where the mass density is bounded below by some constant Λ contains a trapped surface if the length scale of that

[2] Of course, observations as well as numerical simulations also played a major role in the process of understanding singularities as *physical* predictions of GR.

region is (omitting constants) larger than $\Lambda^{-1/2}$, reducing the question of trapped surface formation to the question whether such densities can form during gravitational collapse.

Studying this problem is necessarily done by studying the Einstein equations (1) for which the initial value problem is considerably richer than for a lot of other PDEs, as the global causal geometry is not a priori constrained. Unfortunately, the theory of PDEs is not yet advanced enough to allow us to tackle the completely general study of quasilinear hyperbolic PDEs in more than 1 spatial dimensions, so we either have to resort to analysing special solutions or we have to impose additional symmetries to effectively reduce the number of spatial dimensions. In 3 spatial dimensions, spherical symmetry is the only sensible symmetry that reduces the number of spatial dimensions to 1. Most importantly, spherical symmetry allows us to understand the conformal structure of a 3+1-dimensional manifold M by just drawing conformally compactified (so-called Penrose) diagrams corresponding to the 1+1-dimensional manifold $Q = M/SO(3)$. These diagrams are not only helpful for visualisation but are in fact completely rigorous mathematical tools capturing the global conformal geometry of M.

Despite the notable simplifications spherical symmetry gives to us, there is one annoying problem with it. In vacuum, Birkhoff's theorem tells us that any spherically symmetric solution of eq. (1) belongs to the 1-parameter family of *static* Schwarzschild solutions. Hence, in order to add non-trivial dynamics to the system, matter fields need be included.

With this in mind, conditions for the dynamical formation of trapped surfaces have been found for a variety of spherically symmetric matter system, e.g. for the massless Einstein-Klein Gordon system [3], the Einstein-Vlasov system [24] or the Einstein-Maxwell-Klein Gordon system [19]. It should be mentioned in this context that Christodoulou proved in 2008, without restricting to spherical symmetry, that trapped surfaces can even form from the *vacuum collapse* of gravitational waves [9], showing in complete generality that singularities are a non-generic, physical prediction of GR.

Nevertheless, we are still far from understanding *the nature of singularities*. Perhaps the most important conjecture in this context is the *weak cosmic censorship conjecture* (WCCC), first formulated by Penrose in [23]. Its basic idea is that effects of singularities arising from gravitational collapse[3] and in particular, the singularities themselves, cannot be observed by a distant observer because the singularity is hidden inside a black hole. This means that we, the distant observers, are safe from whatever happens near singularities (the answer to that question will hopefully be given by a theory of quantum gravity at some point).

A more precise formulation is that, **disregarding exceptional initial conditions, asymptotically flat data have a maximal future development which possesses a complete[4] future null infinity.**

Keeping in mind the above comments on the need for spherical symmetry, we will study the weak cosmic censorship for a large class of spherically symmetric, "weakly tame" systems. The assumptions made on these weakly tame matter systems are: i) the exclusion of anti-trapped regions, ii) non-emptiness of future null infinity, iii) matter satisfies the dominant energy condition, iv) the *weak extension principle* which roughly states that first

[3] For the big-bang singularity, the opposite is believed to be true

[4] We will give a precise definition of the meaning of "completeness" in the context of spherical symmetry in section 3.5. For the general case, see [8].

singularities not emanating from the center of symmetry are preceded by a (marginally) trapped region.

The main result of this essay is a result by Dafermos [12] that for these systems, the WCCC can be reduced to the trapped surface conjecture which states that **generically, asymptotically flat data have a maximal future development which is either geodesically complete or contains a (marginally) trapped surface.**[5]

The essay will be structured as follows:

In section 2, we give an overview over some of the historical developments of studying the WCCC.

In section 3, the main section of the essay, we thoroughly explain and discuss Dafermos' paper "Spherically symmetric spacetimes with a trapped surface" [12], where he shows, for spherically symmetric, asymptotically flat initial data with one end, that completeness of future null infinity can be inferred from the existence of a black hole region for the large class of weakly tame matter systems described above.

In section 4, we will show that the Einstein-Maxwell-Klein-Gordon [20] system belongs to this class of systems, thus ensuring that the above results hold for it. We then discuss trapped surface formation for this system in section 5. A summary of the results will be given in the final section.

2 Weak cosmic censorship: An overview

As explained in the introduction, the most promising approach to proving the WCCC is to consider simple matter systems and their collapse within *spherical symmetry*. Over the last few decades, various different spherically symmetric matter models have been studied and in particular, it was shown that the restriction to spherical symmetry still allows for various subtle phenomena: Different violations of the weak cosmic censorship have been discovered in the realm of spherical symmetry and in many cases shown to be non-generic.

The first and perhaps simplest model in this setting was the Einstein dust model by Oppenheimer and Snyder: They modelled the collapsing matter as a spherically symmetric perfect (pressureless) fluid with constant density. However, multiple numerical calculations showed that naked singularities occur in this model, i.e. that the WCCC is violated. Christodoulou later [2] proved that even for inhomogeneous (spherically symmetric) densities, naked singularities generically occur (in particular, he showed that the density in the center of the matter diverges before a black hole forms). However, the failure of the WCCC in this case is believed to come from the unrealistic assumption of vanishing pressure despite diverging densities.

Giving up on the model (the problem of collapse of a realistic fluid model is still not understood, see e.g. [6]), Christodoulou tackled a more realistic model, the model of a self-gravitating, massless scalar field (the Einstein-Klein-Gordon system with vanishing mass)

$$\nabla_\mu \nabla^\mu \phi = 0, \quad T_{\mu\nu} = 8\pi(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\xi \phi \nabla^\xi \phi)$$

[5] Precise definitions of these concept will be given later.

His motivation to choose this particular coupling, apart from its simplicity, was that on the one hand, the massless scalar model does not develop any singularities in the absence of gravity (i.e. on fixed Minkowski background) - in contrast to Einstein dust. On the other hand, spherically symmetric perturbations of Minkowski spacetime have wave character and the massless scalar model satisfies the classical wave equation in the absence of gravity. In a series of papers, he found conditions for the matter to disperse in the infinite future if suitably small initially (in contrast to Einstein dust, which always collapses) and, more importantly, he found conditions on the initial data to guarantee trapped surface formation [3].

He then started constructing counter-examples to the WCCC: Taking initial data of bounded variation (he proved that the IVP is still well-posed for such initial data), he moved on to the construction of interesting examples with additional symmetries in [5]. Analysing the global behaviour of these examples, he managed to show that there exist asymptotically flat initial data such that the future development possesses a naked singularity and an incomplete future null infinity, seemingly violating weak cosmic censorship. Furthermore, he showed that there exist initial data such that future null infinity is complete but the singularity is not covered by a black hole.[6] However, he then finally showed in [7] that initial data leading to the above cases can be, even in the context of spherical symmetry, regarded as exceptional. More precisely, he showed that any deviation from these initial data leads to a future development that is either geodesically complete or possesses a singularity covered by a black hole, and in particular has a complete future null infinity. A summary of his findings is given in [8], where he also gives his reformulation of completeness of future null infinity which we shall restate in the context of spherical symmetry in section 3.5.

So Christodolou showed 3 things: First, the word "generic" in the WCCC cannot be avoided. Secondly, the restriction to spherical symmetry can still lead to rich dynamics that can potentially allow us to understand the more general problem. Finally, although restricted to a certain system, he proved the first weak cosmic censorship theorem.[7]

However, as of now, it is the only non-trivial system that is fully understood w.r.t. the WCCC (in fact, even w.r.t. the strong cosmic censorship conjecture). For various different systems, like the Einstein-Vlasov [13] or the Einstein-Maxwell-Klein-Gordon [20] system, it has been proved that if a trapped surface forms, then the WCCC holds. It has also been shown for these systems that under certain circumstances, trapped surfaces *do* form [19, 24]. What remains to be shown for these systems, however, is the trapped surface conjecture mentioned in the beginning, stating that generically, either trapped surfaces form, or the future development is geodesically complete.

We should mention that to understand the WCCC, there is also the approach to search for physically reasonable counter-examples: In this context, the ideas of e.g. "supercharging" [18, 1] a near extremal Reissner-Nordström black hole or "overspinning" [25] a near extremal Kerr(-Newman) black hole, thus destroying their event horizon, have been given attention. These possibilities have been shown via heuristic arguments, but are hoped to be eliminated by e.g. taking gravitational backreaction effects [17] of the matter used to

[6] So naked singularities are not equivalent to incompleteness of future null infinity! Nevertheless, in these examples, the singularity itself cannot be observed from infinity. Points arbitrarily close to it can, however!

[7] Indeed, Christodoulou himself said the understanding of this system was a "milestone" in proving the possibility of trapped surface formation due to focussing of gravitational waves.

“supercharge”/”overspin” the black holes into account. Indeed, for the case of Reissner-Nordström, the results of section 4 will show that there is no way of super-charging a RN black hole.

Lastly, we mention that the search for counter-examples becomes simpler in AdS-spacetimes. Such (mostly numerical) considerations are often motivated by results from String Theory. In AdS, it is generally harder to form black holes while it is generally easier to form singularities making the WCCC more vulnerable. Moreover, the AdS-CFT correspondence gives a non-perturbative way of describing such systems! In this context, Hertog et al. [16] gave heuristic arguments for possibly creating naked singularities in AdS spacetimes in the model of a self-gravitating Higgs field. However, some of these possibilities have been ruled out by Dafermos in [11], using mainly the results of the next section.

3 Spherically symmetric spacetimes with a trapped surface

In this section, I will thoroughly review and explain Dafermos' publication "Spherically symmetric spacetimes with a trapped surface" [12] where he proves weak cosmic censorship for a large class of spherically symmetric systems spacetimes containing a black hole or[8] a (marginally) trapped region[9] or having a bounded area-radius on the event horizon. For this class of systems, this work therefore reduces the proof of weak cosmic censorship to the trapped surface conjecture.

The section is structured as follows: First, I will state, motivate and explain the assumptions we make on our spacetime in sections 3.1-3.4. These assumptions will mostly remain quite general, except for the so-called *weak extension principle* (WEP) which is a statement about first singularities having to be preceded by a (marginally) trapped region, unless they arise from the center of the spacetime. During the introduction of the assumptions, we will also give all the necessary definitions, in particular the definition of completeness of null infinity.

For proving the completeness of future null infinity in 3.5, we will then assume the existence of a black hole region to show that the area-radius function on the event horizon of the black hole is bounded by the final Bondi mass, which will be sufficient to derive the completeness of future null infinity. From the proof, we will infer that indeed, even if the black hole region is empty: as long as the extension of the area-radius function onto the event-horizon is bounded, null infinity is complete. In 3.6, we give a brief extension of the results. We conclude our findings in 3.7.

3.1 First assumptions

I will here state the first assumptions we make on our system. Since we are dealing with spherically symmetric systems, we will formulate the assumptions directly at the level of a $1+1$-dimensional Lorentzian submanifold Q^+ (a subset of two-dimensional Minkowski

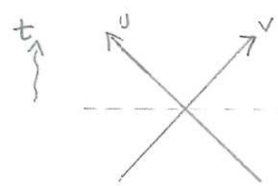
[8] Indeed, we shall see that the existence of a (marginally) trapped region implies the existence of a black hole region.

[9] Precise definitions of these and other concept mentioned here will be given at a later point.

space) which should be thought of as the conformal compactification of the quotient manifold of a four-dimensional spherically symmetric Lorentzian manifold factored out by the group action of $SO(3)$. I will go into more detail on this after having stated the following assumption. In general, whenever an assumption is introduced, it will then immediately be assumed throughout this section.

3.1.1 The quotient manifold Q^+

Let $(\mathbb{R}^2, -dudv)$ denote Minkowski space with double null coordinates s.t. u, v both increase with time. We choose the v-axis to be at 45 degrees to the horizontal and the u-axis at 135 degrees. Whenever we mention chronological or causal future (or similar concepts), we will mean it in the context of $(\mathbb{R}^2, -dudv)$. So, for instance, the chronological past of $p = (u', v')$ is $I^-(p) = \{(u, v) : u < u', v < v'\}$ and the causal past is $J^-(p) = \{(u, v) : u \le u', v \le v'\}$

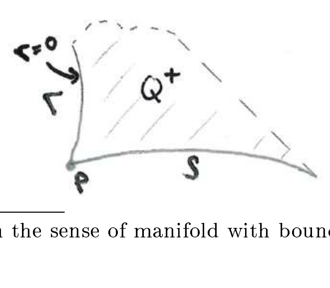


Assumption A. *Assume we have a bounded two-dimensional submanifold $Q^+ \subset \mathbb{R}^2$ with boundary*[10] *$\Gamma \cup S$, where Γ (S) is a connected, non-empty, timelike (spacelike) curve and $\Gamma \cap S = \{p\}$, where p is a single point. We assume that Q^+ is foliated by lines of constant u and lines of constant v with past-endpoint on $\Gamma \cup S$, S, respectively.*

We further assume that on Q^+, we have C^1-functions $r \ge 0$, $\Omega > 0$ with $r(q) = 0$ if and only if $q \in \Gamma$ and metric $-\Omega^2 dudv$. Finally, we assume that the so-called Hawking mass m defined by

$$m(u, v) := \frac{r}{2}\left(1 + \frac{4}{\Omega^2}\partial_u r \partial_v r\right) \tag{2}$$

satisfies $|m|_S| < c$ for some constant c, i.e. it is uniformly bounded along S.



[10] Note that this is a boundary in the sense of manifold with boundary, not in the sense of topological boundary.

Remark 1. As stated before, this should thought of as coming from some spherically symmetric 4-dimensional Lorentzian manifold with metric $-\Omega^2 dudv + r^2(u,v)\gamma$, where γ is the usual metric on a 2-sphere and

$$r(u,v) = (\text{area}(u,v)(4\pi)^{-1})^{1/2}$$

is the area-radius function of the 2-sphere corresponding to (u,v) (and area(u,v) denotes this sphere's area). More precisely, if we have an appropriate[11] spherically symmetric data set with initial Cauchy hypersurface Σ such that there exists a maximal Cauchy development $(M, -\Omega^2 dudv + r^2(u,v)\gamma)$ then Q^+ can be regarded as the conformal compactification of $(M \cap J^+(\Sigma))/SO(3)$. S then corresponds to $\Sigma/SO(3)$. The assumption that Q^+ is foliated by lines of null rays with endpoint on $\Gamma \cup S$ is exactly the assumption of global hyperbolicity of $M(= D(\Sigma))$.

Γ (in view of $r(q) = 0$ iff $q \in \Gamma$) corresponds to all $q \in M \cap J^+(\Sigma)$ that are invariant under $SO(3)$, i.e. it corresponds to all the 2-spheres in M of vanishing volume. It is called the center of symmetry of Q^+. We therefore call rays of constant v "ingoing" and rays of constant u "outgoing".

The assumption of connectedness and non-emptiness of Γ and S puts restrictions on possible initial data. For example, if the initial data had topology of $\mathbb{R} \times \mathbb{S}^2$, as is the case for the Einstein-Rosen bridge for the Kruskal spacetime, then the center would be empty. If the topology of the initial data set were that of $\mathbb{S}^3$, then the center Γ would consist of two connected components (corresponding to the two poles of the three-sphere). The assumption holds if Σ has topology of $\mathbb{R}^3$ and *one* asymptotically flat end.

Remark 2. In general, we will need every single part of each assumption. However, it is possible to drop the assumption $r|_\Gamma = 0$ and still keep most of our results. I will clarify in footnotes when we use this assumption and how it can be avoided.

3.1.2 Structure equations on Q^+

The following assumption regards the structure equations on Q^+ which should naturally be thought of as coming from the Einstein equations on M. Again, I will give more details after the assumption.

Assumption B. *On Q^+, we have a symmetric tensor with bounded components T_{uu}, T_{vv}, $T_{uv} = T_{vu}$ which are locally integrable along lines of constant u or constant v. Then the following equations hold at each $p \in Q^+$:*

$$\partial_u(\Omega^{-2}\partial_u r) = -4\pi r \Omega^{-2} T_{uu}, \tag{3}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -4\pi r \Omega^{-2} T_{vv}, \tag{4}$$

$$\partial_u m = 8\pi r^2 \Omega^{-2}(T_{uv}\partial_u r - T_{uu}\partial_v r), \tag{5}$$

$$\partial_v m = 8\pi r^2 \Omega^{-2}(T_{vu}\partial_v r - T_{vv}\partial_u r), \tag{6}$$

[11] For vacuum, for example, a theorem by Choquet-Bruhat and Geroch tells us that if the initial data satisfies certain constraint equations then there exists a maximal Cauchy development, as seen in the black holes course. Similar theorems will have to be shown on a case-by-case basis when one is dealing with matter systems.

Remark 3. Equations (3) - (6) are equations that would be implied by the Einstein equations (1) (with the metric g again given by $-\Omega^2 dudv + r^2(u,v)\gamma$) and a spherically symmetric energy-momentum tensor T that due to the symmetry has to take the form

$$T_{ab}dx^a dx^b + f(u,v)r^2(u,v)\gamma$$

for some function f, where I adopted the convention that I denote the coordinates on Q^+ by lower case Latin letters. (I will denote the remaining "angular" ones by upper case Latin letters.) I will here not write down the full calculations and instead only briefly sketch how to get the above equations from Einstein's equations: Using an analytical software package, one can easily calculate the Ricci tensor. One can then substitute the AB-components of Einstein's equation into the trace (with respect to the metric g_{ab} on Q^+) of the ab-components, and then substitute that trace back into the ab-components of the Einstein equations to get[12]:

$$\nabla_a \nabla_b r = \frac{1}{2r}(1 - \partial^c \partial_c r)g_{ab} - 4\pi r(T_{ab} - g_{ab}g_{cd}T^{cd}) \tag{7}$$

From this, one can infer equations (3) - (6). For example, for eq. (3), simply compute that $\Gamma^v_{uu} = 0$ and $\Gamma^u_{uu} = 2\partial_u \Omega/\Omega$ (note that the only non-vanishing components of the inverse metric are $g^{uv} = g^{vu} = -2/\Omega^2$). However, deriving eq. (7) requires that r, Ω are C^2-functions. To weaken the regularity restrictions on r, Ω, we here assumed eqns. (3) - (6) directly.

Assumption B implies the following equations which we shall need later in this section:

Claim 1. *From the assumption above, we get, if* $1 - \frac{2m}{r}, \partial_u r, \partial_v r \neq 0$*:*

$$\partial_u \frac{\partial_v r}{1 - \frac{2m}{r}} = \frac{4\pi r T_{uu}}{\partial_u r} \frac{\partial_v r}{1 - \frac{2m}{r}} \tag{8}$$

$$\partial_v \frac{\partial_u r}{1 - \frac{2m}{r}} = \frac{4\pi r T_{vv}}{\partial_v r} \frac{\partial_u r}{1 - \frac{2m}{r}} \tag{9}$$

Proof. Using the definition of $1 - 2m/r = -4\Omega^{-2}\partial_u r \partial_v r$ (cf. eq. (2)), the LHS of eq. (8) reads

$$\partial_u(-\frac{\Omega^2}{4\partial_u r}) = \frac{1}{4}\partial_u(\Omega^{-2}\partial_u r)\left(\frac{\Omega^2}{\partial_u r}\right)^2 = 4(-4\pi r \Omega^{-2} T_{uu})\frac{-\Omega^2}{4\partial_u r}\frac{\partial_v r}{1 - \frac{2m}{r}}$$

where we used equation (3) in the second step.

We thus proved eq. (8). The proof of eq. (9) is analogous and will be omitted here. However, it can be simply obtained by copying the proof above and just swapping the u- and v-indices. ☐

3.1.3 Dominant energy condition

We further assume the energy momentum tensor to be non-negative, i.e.:

[12]For more details on this, see chapter 3 in [6] or the appendix of [14]

Assumption C.

$$T_{uu} \geq 0,\, T_{vv} \geq 0,\, T_{uv} \geq 0 \tag{10}$$

Remark 4. Equation (10) is a consequence of the dominant energy condition. To see why, consider the vector $j^\mu := -T^\mu_\nu V^\nu$. Choose V s.t. only its u- and v-components are non-zero. Then V is causal if $V^u V^v \geq 0$.[13] The first part of the dominant energy condition, i.e. that j is causal, then implies[14]

$$j^a j_a = g^{ad}(T_{db}V^b T_{ac}V^c) = -4\Omega^{-2}(T_{ub}T_{vc}V^bV^c) \leq 0$$

from which we infer

$$T_{uu}T_{uv} \geq 0 \text{ and } T_{vv}T_{uv} \geq 0 \tag{11}$$

by choosing $V^v = 0$ or $V^u = 0$, respectively. The other requirement from the dominant energy condition, namely that j is future-directed if V is, becomes

$$j^a V_a = -T_{ab}V^aV^b \leq 0$$

and analogously (choosing $V^v = 0$ or $V^u = 0$) implies $T_{uu} \geq 0$ and $T_{vv} \geq 0$. So, looking back at eq. (11), we see that T_{uv} can only be negative if $T_{uu} = 0 = T_{vv}$. In this case, however, we get that $j^aV_a = -2T_{uv}V^uV^v > 0$ for all timelike vectors V (these have $V^vV^u > 0$). So T_{uv} has to be non-negative.

So far, the assumptions introduced remain very general, with the only real restrictions being made on the topology of the initial Cauchy surface. In the following subsections, we shall analyse how much we can get out of just these assumptions and where we need additional assumptions.

3.2 Regular, trapped and marginally trapped regions

In this subsection, we will make some statements on the behaviour of the functions r, its derivatives and m in different regions of Q^+. We introduce the following very tractable definitions[15] of trapped, marginally trapped, regular and anti-trapped regions:

Definition 3.1. *We define the regular region as*

$$\mathcal{R} := \{q \in Q^+ : \partial_v r(q) > 0, \partial_u r(q) < 0\},$$

the trapped region as

$$\mathcal{T} := \{q \in Q^+ : \partial_v r(q) < 0, \partial_u r(q) < 0\},$$

the marginally trapped region[16] *as*

[13] $g_{ab}V^aV^b = -\Omega^2 V^uV^v$

[14] Again, we take T to be spherically symmetric, so, in particular, $T_{aA} = 0$.

[15] One computes that in spherical symmetry, the null expansions of spheres are given by $\theta_i = \frac{4}{\Omega^2 r}\partial_i r$, $i = u, v$. This is yet another simplification spherical symmetry gives to us, as we have very good control over $\partial_u r, \partial_v r$ via the eqns. 3,4.

[16] The wording Dafermos chooses in his paper might suggest that $\mathcal{A}$ is 1-dimensional. This, in general, is not the case.

$$\mathcal{A} := \{q \in Q^+ : \partial_v r(q) = 0, \partial_u r(q) < 0\},$$

and the anti-trapped region as

$$\dashv\mathcal{T} := \{q \in Q^+ : \partial_v r(q) < 0, \partial_u r(q) < 0\}.$$

For the Kruskal spacetime, we have seen in the Black Holes course that region 2, the interior of the black hole region is trapped, region 1 is regular and the event horizon is marginally trapped. We have also seen that region 3, the white hole region is "anti-trapped". To exclude such white-hole like regions from our spacetime, we introduce

Assumption D. *We assume that $\partial_u r < 0$ along S i.e. there are no anti-trapped surfaces initially.*

This assumption will, inter alia, allow us to deduce that $\partial_u r < 0$ will hold everywhere on Q^+ (so white holes/anti-trapped surfaces are non-evolutionary). We will now show this statement and several other very useful properties of the regions $\mathcal{R}$, $\mathcal{A}$ and $\mathcal{T}$.

Proposition 1. *i) $Q^+ = \mathcal{R} \cup \mathcal{T} \cup \mathcal{A}$, i.e. Q^+ contains no anti-trapped regions.*
ii) $1 - \frac{2m}{r} = 0$ on $\mathcal{A}$, $1 - \frac{2m}{r} < 0$ on $\mathcal{T}$ and $1 - \frac{2m}{r} > 0$ on $\mathcal{R}$
iii) If $(u, v) \in \mathcal{T}$, then $(u, v') \in \mathcal{T}$ for all $v' > v$ as long as $(u, v') \in Q^+$
iv) If $(u, v) \in \mathcal{T} \cup \mathcal{A}$, then $(u, v') \in \mathcal{T} \cup \mathcal{A}$ for all $v' > v$ with $(u, v') \in Q^+$
v) In $\mathcal{R}$, we have $\partial_v m \geq 0$, $\partial_u m \leq 0$ and $m \geq 0$. The first two inequalities hold also in $\mathcal{A}$.

Proof. i) By assumption A, for every $p = (u, v) \in Q^+$, the ingoing null ray ($v = const.$) through it has past endpoint $q = (u_b, v)$ on the boundary S. There, the quantity $\Omega^{-2}\partial_u r$ is negative (by assumptions A and D). Now, integrating[17] eq. (3) from q to p along an ingoing null ray gives, in view of the non-negativity of T_{uu} and Ω, that

$$(\Omega^{-2}\partial_u r)(u, v) - \underbrace{(\Omega^{-2}\partial_u r)(u_b, v)}_{\leq 0} = \int_{u_b}^{u} -4\pi r \Omega^{-2} T_{uu}(\bar{u}, v)\, d\bar{u} \leq 0.$$

Hence, $\partial_u r(u, v) < 0$ for all $(u, v) \in Q^+$ and the statement follows.

ii) Since $1 - \frac{2m}{r} = -4\Omega^{-2}\partial_u r \partial_v r$ and $\partial_u r < 0$ in Q^+, we have $\text{sign}(1 - \frac{2m}{r}) = \text{sign}(\partial_v r)$ on Q^+. Thus, the statement follows.

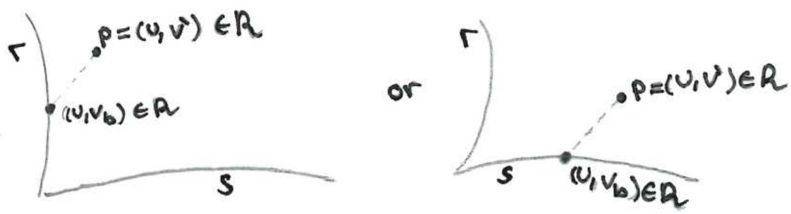
iii), iv) From eq. (4) and the positivity of T_{vv}, we see that along outgoing null rays ($u = const.$), $\Omega^{-2}\partial_v r$ is non-increasing towards the future, i.e., for $v_2 > v_1$:

[17] This, of course, requires the local integrability of the components of T and the continuity of Ω, assured by assumptions A, B.

$$(\Omega^{-2}\partial_v r)(u, v_2) - (\Omega^{-2}\partial_v r)(u, v_1) = \int_{v_1}^{v_2} -4\pi r \Omega^{-2} T_{vv}(u, \bar{v})\, d\bar{v} \leq 0.$$

(the same assumptions as for i) are needed here, apart from the foliation of Q^+ with null rays that have endpoint on the boundary of Q^+). So, if $\partial_v r(u, v_1) < 0$, then $\partial_v r(u, v_2) < 0$ for all $v_2 > v_1$ as long as we are still in Q^+. Clearly, the same statement still holds if we replace $<$ by $\leq$. Thus, the statement follows.

v) The inequalities $\partial_u m \leq 0$, $\partial_v m \geq 0$ on $\mathcal{R}$ are immediate in view of eqns. (5), (6) and the positivity of the energy momentum tensor. To now deduce that $m \geq 0$ on $\mathcal{R}$, consider a point $p = (u, v') \in \mathcal{R}$. Then, from iv), (u, v) will also be in $\mathcal{R}$ (if it is in Q^+) for all $v < v'$. Let v_b denote the (unique, since Γ, S are timelike/spacelike) point s.t. (u, v_b) is on the (geometrical) boundary of Q^+. This exists because of the foliation of Q^+ by null rays with past endpoint on the boundary of Q^+.



Then the inequality $\partial_v m \geq 0$ we just showed to hold in $\mathcal{R}$ implies $m(u, v') \geq m(u, v_b)$. So we only need to show the following

Subclaim. $m \geq 0$ on $(\Gamma \cup S) \cap \mathcal{R}$

Proof. Note that by Assumption A, $m|_\Gamma = 0$ (because $r|_\Gamma = 0$). To show positivity on $S \cap \mathcal{R}$, we will first show that m increases along $S \cap \mathcal{R}$. The (by assumption A spacelike) unit tangent vector along S (away from Γ), $\boldsymbol{k} = k^u \partial_u + k^v \partial_v$, has k^v positive and therefore k^u negative (since $1 = k^a k_a = -\Omega^2 k^u k^v$). Now, using eqns. (5) and (6) and the positivity of the energy-momentum tensor, we get

$$\boldsymbol{k} \cdot m = 8\pi r^2 \Omega^{-2} (\underbrace{k^u T_{uv} \partial_u r}_{\geq 0} - \underbrace{k^u T_{uu} \partial_v r}_{\leq 0} + \underbrace{k^v T_{uv} \partial_v r}_{\geq 0} - \underbrace{k^v T_{vv} \partial_u r}_{\leq 0}) \geq 0$$

on $S \cap \mathcal{R}$. So indeed, m increases along $S \cap \mathcal{R}$. Now, let s be the coordinate along S defined by $k \cdot s = 1$ and $s = 0$ at $\Gamma \cap S$ (so s is increasing away from Γ). If $S \cap \mathcal{R}$ is empty, we have nothing to show. If $S \cap \mathcal{R}$ is non-empty, there exists an $s_\mathcal{R} \in S \cap \mathcal{R}$. Now, either $[0, s_\mathcal{R}] \in S \cap \mathcal{R}$, in which case we are done because $m(s = s_\mathcal{R}) \geq m(s = 0) = 0$. Or, by continuity of $\partial_v r$, there exists a $0 \leq s_\mathcal{A} < s_\mathcal{R}$ with $s_\mathcal{A} \in S \cap \mathcal{A}$ s.t. $(s_\mathcal{A}, s_\mathcal{R}) \in S \cap \mathcal{R}$. But we now from ii) that $m(s = s_\mathcal{A}) = r(s = s_\mathcal{A})/2$. Since r is by assumption A non-negative, we have $m(s = s_\mathcal{R}) \geq (m(s = s_\mathcal{A}) \geq 0$ and thus proved the subclaim.

□

Note that the subclaim we just proved shows that $m(u, v_b) \geq 0$ and we therefore get, by $\partial_v m \geq 0$ in $\mathcal{R}$, that $m(u, v') \geq 0$. Hence, $m(p) \geq 0$ for all $p \in \mathcal{R}$. This completes the proof. □

In view of $m|_{\mathcal{R}} \geq 0$ and $m|_{\mathcal{A}\cup\mathcal{T}} \geq r|_{\mathcal{A}\cup\mathcal{T}}/2 \geq 0$, we get the following corollary:

Corollary 3.1. *The Hawking mass m is non-negative everywhere on Q^+.*

Remark 5. Referring to remark 2, it is clear that the positivity of the mass in $\mathcal{R}$ needs the assumption $r|_{\Gamma} = 0$. This is the only time we explicitly use the assumption. Nevertheless, we will often use the positivity of the mass in $\mathcal{R}$. I shall mark when we do this using footnotes, also explaining how this can be circumvented.

The behaviour of r, m in $\mathcal{R}$ will, as I mentioned, play a crucial role in the upcoming proofs, so I recommend the reader to familiarise himself with the situation which I summarised in figure 1

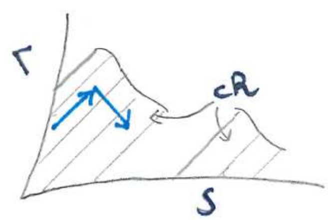


Figure 1: The blue arrows show the directions of increasing r/non-decreasing m. Also captured in the figure is the fact that if $(u, v') \in \mathcal{R}$, then $(u, v) \in \mathcal{R}$ for $v < v'$.

3.3 Future null infinity

Since the weak cosmic censorship conjecture essentially claims the completeness of future null infinity for realistic spacetimes, we better define future null infinity. For that, we first define $\mathcal{U}$, the set of coordinates u for which $r(u, v)$ goes to infinity for some v and then define future null infinity as the the set of points (u, v') s.t. $u \in \mathcal{U}$ and (u, v') lies on the topological boundary of Q^+. More precisely:

Definition 3.2. *S acquires a unique limit point i^0 in $\overline{Q^+} \setminus Q^+$. We will call this point spacelike infinity.*

Further, we define $\mathcal{U} := \{u : \sup_{v:(u,v)\in Q^+} r(u, v) = \infty\}$.

Then, for each $u \in \mathcal{U}$, there is a unique $v'(u)$ such that $(u, v'(u)) \in \overline{Q^+} \setminus Q^+$.[18]

We then define future null infinity as

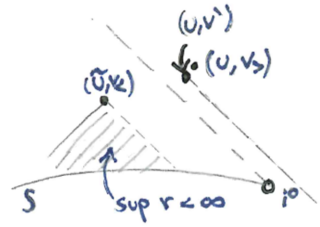
$$\mathcal{I}^+ := \bigcup_{u\in\mathcal{U}} (u, v'(u)).$$

[18]Existence follows straight from the definition of $\mathcal{U}$.

A priori, there is no reason why $\mathcal{I}^+$ (or equivalently, $\mathcal{U}$) should not be empty. Since u strictly decreases along S (in the direction away from Γ), this does not change even if r diverges along S. (i^0 does not belong to $\mathcal{I}^+$.) However, if $\mathcal{I}^+$ is non-empty, we have the following

Proposition 2. *If $\mathcal{I}^+$ is non-empty, it is a connected, ingoing null ray with past limit point i^0.*

To show that future null infinity has to lie on an ingoing null ray emanating from i^0 the idea is that points "to the right of" that null ray would not satisfy the foliation assumption from A whereas points "to the left of" can be shown to be finite in view of $\partial_u r < 0$ on Q^+. The connectedness will then follow from that same inequality.



Proof. Deviating from Dafermos' notation, we will call the coordinates of i^0 $(U_\infty, V_{\mathcal{I}^+})$[19]. Now, assume there is $v_> > V_{\mathcal{I}^+}$ such that $(u, v_>) \in \mathcal{I}^+$. Then, there is a $v' \in (V_{\mathcal{I}^+}, v_>)$ with $(u, v') \in Q^+$. Then, the ingoing null ray though this point will not have past endpoint on S, in contradiction to the global hyperbolicity assumption of assumption A. So let us instead consider $(u, v_<) \in \mathcal{I}^+$ with $v_< < V_{\mathcal{I}^+}$. In view of $\partial_u r < 0$ on Q^+, $\sup_{Q^+ \cap \{v < v_<\}} r \leq \sup_{S \cap \{v < v_<\}} r < \infty$ (by the regularity assumption we made on r), so we find that no points on $\mathcal{I}^+$ can have $v < V_{\mathcal{I}^+}$. Hence, $\mathcal{I}^+$ is a subset of $\{v = V_{\mathcal{I}^+}\}$. To prove connectedness, we remark that because $\partial_u r < 0$ on Q^+, we have that r cannot be increasing along $\mathcal{I}^+$ (away from i^0). More precisely, $r(u, v) > r(u', v)$ for all $v < V_{\mathcal{I}^+}$, $u < u'$ and thus $\lim_{v \to V_{\mathcal{I}^+}} r(u, v) \geq \lim_{v \to V_{\mathcal{I}^+}} r(u', v)$ for all $u < u'$. Hence, if $(u', V_{\mathcal{I}^+}) \in \mathcal{I}^+$, so is $(u, V_{\mathcal{I}^+})$ for all $u < u'$. This proves the proposition.

□

Note that we needed assumption D for both the connectedness of $\mathcal{I}^+$ and the fact that $\mathcal{I}^+$ lies on an ingoing null ray.

In the following, we will make

Assumption E. *$\mathcal{I}^+$ is non-empty.*

Remark 6. This assumption holds true for initial data with matter of compact support. Because then, points arbitrarily close to both the initial hypersurface and to the topological boundary emanating from i^0 will not notice the initial matter. So for these points, the metric will be given (by Birkhoff's theorem) by the Schwarzschild metric, which has

[19] This is perhaps in bad style because we do not know yet that $\mathcal{I}^+$ has a constant v-coordinate.

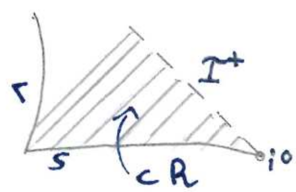
non-empty future null infinity. We could also add an electromagnetic field that does not have compact support on the initial hypersurface. Then the same argument as above applies but with the Reissner-Nordström metric instead of the Schwarzschild metric (this is of relevance to section 4).

In the proof of the above proposition, we have not made use of any of the specific inequalities on $\mathcal{R}, \mathcal{A}$ or $\mathcal{T}$ from prop. 1, we instead only made use of the general inequality $\partial_u r < 0$ on Q^+. This is because we do not yet know anything about where in Q^+ the regular, trapped and marginally trapped regions lie. But since we now know something about $\mathcal{I}^+$, this will change:

Proposition 3. *We have* Not shown in [12]

$$J^-(\mathcal{I}^+) \cap Q^+ \subset \mathcal{R} \tag{12}$$

Proof. Consider $(u, V_{\mathcal{I}^+}) \in \mathcal{I}^+$. Then, $\lim_{v \to V_{\mathcal{I}^+}} r(u,v) = \infty$ implies that in each set $\{u\} \times [v_<, V_{\mathcal{I}^+})$, there must be, for all $v_< < V_{\mathcal{I}^+}$, a point (u, v') with $\partial_v r(u, v') > 0$. But we know from prop. 1iv) that if $\partial_v r(u, v') > 0$ then $\partial_v r(u, v) > 0$ for all $v < v'$ (so long as we don't leave Q^+). Hence, $\partial_v r > 0$ in $J^-(\mathcal{I}^+) \cap Q^+$ and thus the first statement follows. □



Proposition 4. *The Hawking mass m can be extended to a function on $\mathcal{I}^+$ that is non-negative*[20] *and non-increasing away from i^0.* Not shown in [12]

Proof. The extendibility of m to $\mathcal{I}^+$ is a consequence of the assumption that m is uniformly bounded on S (assumption A). For, by the previous proposition, $J^-(\mathcal{I}^+) \cap Q^+ \subset \mathcal{R}$ and so, by prop. 1v), we have $\partial_u m \leq 0$ in $J^-(\mathcal{I}^+) \cap Q^+$. This means that m is in fact uniformly bounded in $J^-(\mathcal{I}^+) \cap Q^+$ and thus extendible to $\mathcal{I}^+$. To see that it is non-negative and non-increasing, we use the other inequalities from prop. 1v): $m \geq 0$, $\partial_v m \geq 0$. We then find that for all $(u, V_{\mathcal{I}^+}) \in \mathcal{I}^+$, $\lim_{v \to V_{\mathcal{I}^+}} m(u,v) \geq 0$. Similarly, $\partial_u m \leq 0$ implies that $\lim_{v \to V_{\mathcal{I}^+}} m(u', v) \leq \lim_{v \to V_{\mathcal{I}^+}} m(u,v)$ for $U_\infty < u' < u$. This completes the proof. □

Since we just showed its existence, we now introduce the definition:

Definition 3.3. *For $(u, V_{\mathcal{I}^+}) \in \mathcal{I}^+$, we define the Bondi mass as*
$m(u, V_{\mathcal{I}^+}) := \lim_{v \to V_{\mathcal{I}^+}} m(u,v)$*, where the limit is of course taken from $v < V_{\mathcal{I}^+}$. Further, we define the final Bondi mass as the infimum of the Bondi mass, $M_f := \inf_{\mathcal{I}^+} M$.*

[20] For the non-negativity of the Bondi mass, the non-negativity of the Hawking mass is, a priori, needed. However, we shall see later that this can be avoided.

3.4 The weak extension principle

To finally start proving the completeness of $\mathcal{I}^+$, we need one last assumption about the nature of singularities in our spacetime.

Assumption F. *If $p \in \overline{\mathcal{R}} \setminus \bar{\Gamma}$ and $q \in I^-(p) \cap \overline{\mathcal{R}}$ such that $J^-(p) \cap J^+(q) \setminus \{p\} \subset \mathcal{R} \cup \mathcal{A}$. Then $p \in \mathcal{R} \cup \mathcal{A}$.*

This principle is depicted below:

Remark 7. Note that since $q \in I^-(p)$ (as opposed to $J^-(p)$), the rectangle $J^-(p) \cap J^+(q)$ always has non-vanishing volume.[21] The content of this assumption is basically that all possible singularities, unless on the center or on certain null parts of the boundary, have to be preceded by some trapped region. This will become clearer through the consequences of this assumption which I show below. We will discuss it a bit more in section 4.2.

This assumption, called the *weak extension principle* (WEP), is by far the most restricitive assumption we make. The other assumptions were either fairly general assumptions on the initial data, or even more general assumptions on their evolution (namely the Einstein equations and the dominant energy condition). In contrast, this assumption is a statement on the development of the initial data and the kinds of singularities arising in these developments. To name a few[22], it was shown to be true for the Einstein-Vlasov system [13], the self-gravitating Higgs [11] and the Einstein-Maxwell-Klein-Gordon system [20]. For the latter, we will present the proof in the next section.

Dafermos now makes a minor claim regarding the topological boundary of Q^+ that is supposedly implied by this and prop. 1.

Claim 2. **** If there is a $p \in \overline{\mathcal{R}} \setminus Q^+$, then p lies either on the ingoing null ray going through the limit point i^+ of $\mathcal{I}^+$ or on the outgoing null ray emanating from the limit point of Γ, which we shall refer to as b_Γ in the following.* Not shown in [12]

I do not think that this statement is correct, at least I do not think it can be proved from just prop. 1 and assumption F. Instead, I could only show this weaker claim:

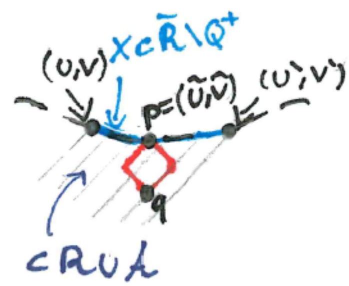
Claim 3. ****If there is a non-empty, connected set $X \subset \overline{\mathcal{R}} \setminus Q^+$ that is not just a point, $X \neq \{p\}$, then X lies either on the ingoing null ray going through the limit point i^+ of $\mathcal{I}^+$, or on any other ingoing null ray on the boundary of Q^+, or on the outgoing null ray emanating from the limit point of Γ.* Not stated in [12]

[21] Our convention is that $p \notin I^-(p)$ but $p \in J^-(p)$.
[22] See chapter 1.7.9 in [20] for a more comprehensive list

Neither of these claims will be used for the proof of completeness of $\mathcal{I}^+$, but I still think that a discussion of these claims may assist the reader in developing a better understanding of the WEP and how we will use it in the next few proofs.

Proof. The above claim states connected subsets of $\overline{\mathcal{R}} \setminus Q^+$ that are not just a point cannot lie on a spacelike part of the boundary of Q^+ or on any outgoing null ray parts of the boundary of Q^+, unless they end on the future limit point of Γ. We will prove this by contradiction:

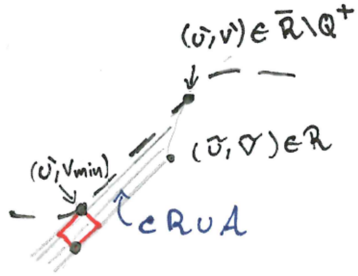
Assume first that X is contained in a spacelike part of the boundary. Take two points (u, v) and (u', v') in X and take, WLOG, $v' > v$, $u' < u$ (because we are in **spacelike** part of the boundary). Because of the result from prop. 1iv) that if $(\bar{u}, \bar{v}') \in \mathcal{R}$ then $(\bar{u}, \bar{v}) \in \mathcal{R}$ for all $\bar{v} < \bar{v}'$, we then get, by the assumption that X is connected, that the set $((u', u) \times \{v < v'\}) \cap Q^+$ is contained in $\mathcal{R}$. So clearly, for each point $p = (\tilde{u}, \tilde{v})$ in X with $v < \tilde{v} < v'$ and thus $u > \tilde{u} > u'$, there is a $q \in ((u', u) \times \{v < v'\}) \cap Q^+ \subset \mathcal{R}$ such that $q \in I^-(p)$ and $J^-(p) \cap J^+(q) \setminus \{p\} \subset ((u', u) \times \{v < v'\}) \cap Q^+ \subset \mathcal{R}$ (where we used that we are one a spacelike part of the boundary). Hence, by assumption F, p is not on the boundary of Q^+ after all, a contradiction. The diagram below illustrates the proof.



For the other possibility, assume X is contained in an outgoing null ray part of the boundary of Q^+ with $u = u'$. Let v_{min} be the minimal[23] v s.t. (u', v) is still on the boundary. Now, assume that there is a point $p = (u', v')$ on this null part of the boundary, with $v' > v_{min}$, which is in $\overline{\mathcal{R}}$. Then, by prop. 1iv), we get that $\{u'\} \times [v_{min}, v'] \subset \overline{\mathcal{R}}$. By continuity, we can now find a $\tilde{v} \in (v_{min}, v')$ such that for some $\tilde{u} < u'$, the set $[\tilde{u}, u') \times \{\tilde{v}\}$ is fully contained in $\mathcal{R}$. We then again have, by prop. 1 iv), that the stripe $[\tilde{u}, u') \times [v_<, \tilde{v}]$ is fully contained in $\mathcal{R}$ for some $v_< < v_0$ large enough s.t. we haven't left Q^+ yet (this is why we must demand the null ray to not have the future limit point of Γ as endpoint). We can now repeat the argument above: We have that $[\tilde{u}, u'] \times [v_<, v_{min}] \setminus (u', v_{min}) \subset \mathcal{R} \cup \mathcal{A}$[24] so assumption F tells us that the point (u', v_{min}) is in $\mathcal{R} \cup \mathcal{A}$, a contradiction. The diagram below illustrates this.

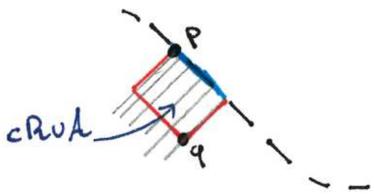
[23]This is minimal and not infimal because the set $(\overline{Q^+} \setminus Q^+) \cap \{u = u_{const.}\}$ is closed for all $u_{const.}$

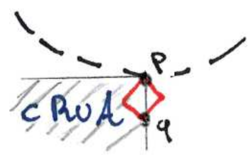
[24]Since $\bar{R} \cap Q^+ = (\mathcal{R} \cup \mathcal{A}) \cap Q^+$



□

So the scenarios of Claim 2 we could not exclude are exactly the following:





The scenario on the left is a scenario where there is a subset on an ingoing null part of the boundary, in which case any rectangle as in assumption F will intersect the boundary, so assumption F is not applicable here. The right scenario is a scenario where there is only one point in $\overline{\mathcal{R}} \cap (\overline{Q^+} \setminus Q^+)$ such that any rectangle as in assumption F is not fully contained in $\mathcal{R} \cup \mathcal{A}$.

The reason we cannot exclude these scenarios is that, in general, we have no way of saying that if $(u', v') \in \overline{\mathcal{R}}$, then (u, v') in $\overline{\mathcal{R}}$ for $u < u'$. This motivates the following proposition which will be of major importance in the sequel.[25]

Proposition 5. *If $p = (u', v') \in \overline{\mathcal{R}} \setminus Q^+$ with $v' < V_{\mathcal{I}^+}$, $u_{min} = \min\{u : (u, v') \in \overline{\mathcal{R}} \setminus Q^+\}$ and $[u_<, u_{min}) \times \{v'\} \subset \mathcal{R}$ for some $u_< < u_{min}$. Then $u' = u_{b_\Gamma}$, i.e. p lies on the outgoing null ray emanating from b_Γ, the future limit point of Γ.*

Not stated in [12]

Proof. If $[u_<, u_{min}) \times \{v'\} \cap Q^+ \subset \mathcal{R}$ we find that

$$[u_<, u_{min}) \times [v_<, v'] \subset \mathcal{R} \tag{13}$$

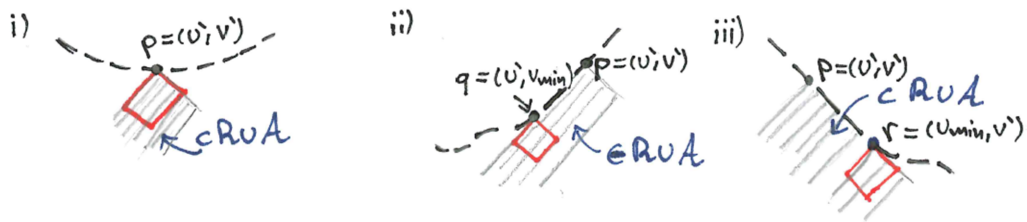
by prop. 1 iv) (for $v_< < v'$ large enough so that we do not leave Q^+). Now, there are three cases which are also depicted in the diagram below. (Note that the topological boundary cannot be timelike due to the foliation assumption from assumption A):

i) If $p = (u', v')$ is on a spacelike part of the boundary, we have $u' = u_{min}$: Then, we get from eq. (13) that

$$[u_<, u'] \times [v_<, v'] \setminus \{p\} \subset \mathcal{R}$$

So, by assumption F, $p \in Q^+$, a contradiction.

[25] Dafermos does not explicitly state this, but I deem it necessary for the following proofs.



ii) If $p = (u', v')$ is on an outgoing null part of the boundary where $u = const = u'$, we also have $u' = u_{min}$. Let's assume $u' < u_{b_\Gamma}$. Pick the minimal v_{min} such that $q = (u', v_{min})$ is still on the boundary of Q^+. Because $u' < u_{b_\Gamma}$, it is clear that we can adapt equation (13) to hold for a $\tilde{v}_< < v_{min}$ large enough s.t. we have not left Q^+. Then, we get from eq. (13) that

$$[u_<, u'] \times [\tilde{v}_<, v_{min}] \setminus \{q\} \subset \mathcal{R} \cup \mathcal{A}$$

and hence $q \in Q^+$, a contradiction. So, if we are on an outgoing null part of the boundary, then $u' = u_{b_\Gamma}$.

iii) If $p = (u', v')$ is on an ingoing null part of the boundary where $v = const = v'$: Because $v' < V_{\mathcal{I}^+}$, this cannot be on the ingoing null ray emanating from the future limit point of $\mathcal{I}^+$. We then have that $r = (u_{min}, v')$ is on the boundary of Q^+. Then, we get from eq. (13) that

$$[u_<, u_{min}] \times [v_<, v'] \setminus \{r\} \subset \mathcal{R} \cup \mathcal{A}$$

and hence $r \in Q^+$, a contradiction.

Thus, the proposition follows. □

Although not important in our context, we will briefly mention the following

Corollary 3.2. *If the topological boundary is not just the union of two null rays emanating from the future limit points of* Γ *and* $\mathcal{I}^+$*, respectively, then, if a black hole region exists, it cannot be regular everywhere.*

For proving the completeness of $\mathcal{I}^+$, Dafermos makes use of an equivalent reformulation of assumption F which replaces the condition $p \notin \overline{\Gamma}$ with the assumption that $p \in \overline{Q^+} \setminus \{v = V_{\mathcal{I}^+}\}$ and $r(p) > 0$. I will avoid using it in this section but it will become relevant again in section 4.

3.5 Proof of completeness of future null infinity

In this subsection, we will infer the completeness of $\mathcal{I}^+$ assuming the existence of a (marginally) trapped region. What we will prove is a reformulation of the weak cosmic censorship conjecture due to Christodolou [8], but slightly adapted. To state this reformulation in this context, it is helpful to introduce this

Definition 3.4. *We call the future boundary of* $J^-(\mathcal{I}^+) \cap Q^+$*, excluding* i^+*, the event horizon,* $\mathcal{H}$*. We let* $U_{\mathcal{H}}$ *denote the u-coordinate of the event horizon* $\mathcal{H}$*.*

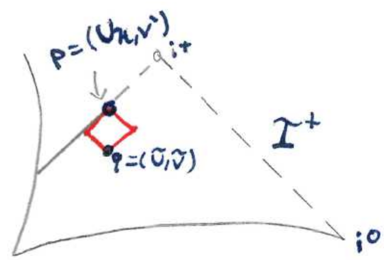
Compared to [12], I changed the definition of $\mathcal{H}$ slightly.

However, the wording "event horizon" only really makes sense if $\mathcal{H} \subset Q^+$. To see when this is the case, we show the following

Proposition 6. *If $Q^+ \setminus (J^-(\mathcal{I}^+) \cap Q^+)$ is non-empty (i.e. if the black hole region is non-empty), then $\mathcal{H} \subset Q^+$.* Not shown in [12]

Proof. If $Q^+ \setminus (J^-(\mathcal{I}^+) \cap Q^+)$ is non-empty, there is a non-empty open interval $\{U_\mathcal{H}\} \times (v, v') \in \mathcal{H} \cap Q^+$. Because of the assumption that Q^+ is foliated by outgoing null rays with past endpoint on $\Gamma \cup S$, we immediately get that the half-open interval $\{U_\mathcal{H}\} \times [V_{\Gamma\cup S}, v') \in \mathcal{H} \cap Q^+$, where $V_{\Gamma\cup S}$ is the unique v s.t. $(U_\mathcal{H}, V_{\Gamma\cup S}) \in \Gamma \cup S$.

Now, if $v' = V_{\mathcal{I}^+}$, then the statement $\mathcal{H} \subset Q^+$ follows. Therefore, assume that $v' < V_{\mathcal{I}^+}$ and $p = (U_\mathcal{H}, v') \notin Q^+$. But then, any point $q = (\tilde{u}, \tilde{v}) \in J^-(\mathcal{I}^+) \cap Q^+ \cap I^-(p)$ with $\tilde{v} < v'$ satisfies $J^-(p) \cap J^+(q) \setminus \{p\} \in \mathcal{R} \cup \mathcal{A}$ in view of equation 12.



So, by assumption F, we obtain $p \in \mathcal{R} \cup \mathcal{A}$, a contradiction. This completes the proof. □

Corollary 3.3. *In particular, equation 12 then implies: If $Q^+ \setminus (J^-(\mathcal{I}^+) \cap Q^+)$ is non-empty, then*

$$\mathcal{H} \subset \mathcal{R} \cup \mathcal{A}. \tag{14}$$

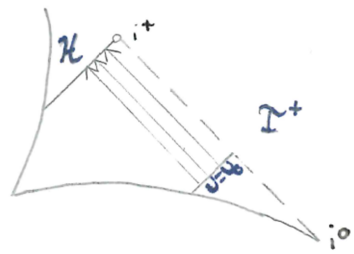
Note that $\mathcal{H}$ can have its past endpoint on S *or* Γ, however the latter case is much more important for the issue of evolutionary black hole formation.

Our formulation of the weak cosmic censorship now takes the following form:

Reformulation of weak cosmic censorship Fix $u_0 < U_\mathcal{H}$ s.t. there exists a $v_0 < V_{\mathcal{I}^+}$ s.t. $\{u_0\} \times [v_0, V_{\mathcal{I}^+}) \in Q^+$. Let $\boldsymbol{X}(u, v)$ on $J^-(\mathcal{I}^+) \cap Q^+$ be the ingoing null vector field defined by demanding it to be parallely transported along $u = u_0$ and along any ingoing null ray. We specify[26] the initial condition $\boldsymbol{X}(u_0, v_0) = \partial_u$.

Then $\mathcal{I}^+$ is future complete if, as $v \to V_{\mathcal{I}^+}$, the affine length of $\boldsymbol{X}$ from u_0 to $U_\mathcal{H}$ diverges. In other words, take the geodesic generators parallel to those of $\mathcal{I}^+$ and normalise them along some outgoing null ray $u = u_0$. Then, the affine length from u_0 to $U_\mathcal{H}$ of these geodesic generators diverges as $\mathcal{I}^+$ is approached.

[26] In [8], the vector field is normalised on S instead.



We are now ready to formulate the main result of Dafermos' paper:

Theorem 3.1. *If $Q^+ \setminus (J^-(\mathcal{I}^+) \cap Q^+)$ is non-empty, then $\mathcal{I}^+$ is future complete.*

Eq. 12 then yields the following

Theorem 3.2. *If $\mathcal{A} \cup \mathcal{T} \neq \emptyset$, $\mathcal{I}^+$ is future complete.*

Proof. We will conduct the proof in 3 steps. First, we find the vector field $\boldsymbol{X}(u,v)$. To show that its affine length diverges, we then need a Penrose inequality that bounds r on the event horizon: $r|_{\mathcal{H}} \leq 2M_f$. Prop. 5 will play a major role for this. Having shown this Penrose inequality will allow us to introduce estimates on r, m with the help of which we will be able to show that the affine length diverges.

3.5.1 Finding $X(u,v)$

We need to solve the initial value problem

Not done in [12]

$$\nabla_u \boldsymbol{X}(u,v) = 0,\ \nabla_v \boldsymbol{X}(u_0,v) = 0,\ \boldsymbol{X}(u_0,v_0) = \partial_u$$

Using that only the off-diagonal components of the metric g_{ab} are non-vanishing, one finds

$$\Gamma^u_{bc} = \frac{1}{2} g^{uv}(g_{vb,c} + g_{vc,b} - g_{bc,v})$$
$$\Gamma^v_{bc} = \frac{1}{2} g^{uv}(g_{ub,c} + g_{uc,b} - g_{bc,u})$$

from which one can read off that $\Gamma^u_{uu} = \frac{2\partial_u \Omega}{\Omega}$, $0 = \Gamma^u_{uv} = \Gamma^v_{uv}$.
Now, since it is parallely transported, we can write $\boldsymbol{X}(u,v) = f(u,v)\partial_u$ for some function f. Using the above result for the Christoffel symbols, we get

$$\nabla_u \boldsymbol{X} = (\partial_u f + f \cdot \Gamma^u_{uu})\partial_u = (\partial_u f + f \cdot \frac{2\partial_u \Omega}{\Omega})\partial_u$$

So, for this to vanish, we require $\partial_u \log(f) = -2\partial_u \log(\Omega)$ and hence $f(u,v) = g(v)\Omega^{-2}(u,v)$ for some function g. Parallel transport along $u = u_0$ on the other hand requires:

$$\nabla_v \boldsymbol{X} = \partial_v f|_{u=u_0} \partial_u = 0.$$

Also using the initial condition, we find

$$\boldsymbol{X}(u,v) = \frac{\Omega^2(u_0,v)}{\Omega^2(u,v)} \partial_u \tag{15}$$

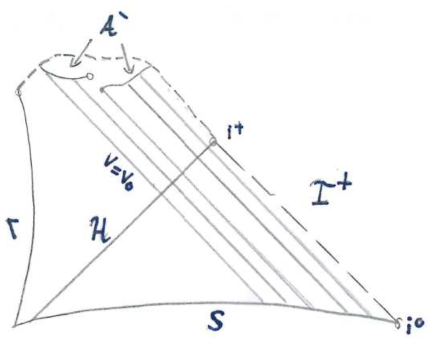
3.5.2 ***The outermost apparent horizon[27]

The following lemma is claimed in [12] to be necessary for his proof. However, in the context Dafermos refers to it, it is not even helpful. It can be used for a part of the proof of the Penrose inequality, but we will instead use prop. 5 for that, as it is the more general result (as we shall see in the proof of this lemma). We still include the proof because it is of independent interest, in particular for numerical studies.

Definition 3.5. ****We define the outermost apparent horizon as the set*

$$\mathcal{A}' := \{(u,v) \in \mathcal{A} : (u^*,v) \in \mathcal{R}\, \forall u^* < u \text{ and } (u',v) \in J^-(\mathcal{I}^+) \cap Q^+ \text{ for some } u'\}$$

Lemma 3.1. **** If $\mathcal{A} \cup \mathcal{T}$ is non-empty, then $\mathcal{A}'$ is a non-empty, achronal[28] curve intersecting all ingoing null curves for $v \geq v_0$ for some $v_0 < V_{\mathcal{I}^+}$.*



Proof. To show that $\mathcal{A}'$ is achronal, let $p = (u,v) \in \mathcal{A}'$ and $q = (u^*,v^*) \in \mathcal{A}' \cap I^-(p)$, i.e. $u^* < u$, $v^* < v$. By definition of $\mathcal{A}'$, $(u^*,v) \in \mathcal{R}$. But now, from prop. 1iv) we get that $(u^*,v^*) \in \mathcal{R}$ since $v^* < v$, a contradiction.

For the second part of the lemma, let now $(u',v') \in \mathcal{A} \cup \mathcal{T}$.[29] Then, choose $v_0 > v'$ s.t. $S \cap \{v \geq v_0\} \in \mathcal{R}$. This exists because of equation (12) and the fact that $\mathcal{I}^+$ is non-empty by assumption D and equation (12). Now, pick any point (u'',v'') on S with $v'' \geq v_0$. **If we assume that** $[u'',u'] \times \{v''\} \in Q^+$, then, since $(u',v') \in \mathcal{A} \cup \mathcal{T}$, $(u',v'') \in \mathcal{A} \cup \mathcal{T}$ as well (by prop. 1iv)). So, in view of eq. (12), there has to be a $\tilde{u} < u'$ such that $(\tilde{u},v'') \in \mathcal{A}'$. (To see why this is the case, note that if there is no $u > U_{\mathcal{H}}$ with $(u,v'') \in \mathcal{A}'$, then we still have $(U_{\mathcal{H}},v'') \in \mathcal{A}'$.)

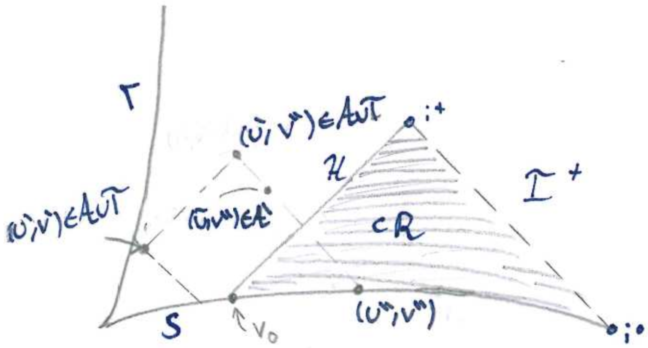
So we get the intermediate result which is also depicted in the diagram below:

$$\forall v'' \geq v_0 \text{ with } [u'',u'] \times \{v''\} \in Q^+,\ \exists \tilde{u} \in [u'',u'] \text{ with } (\tilde{u},v'') \in \mathcal{A}'.$$

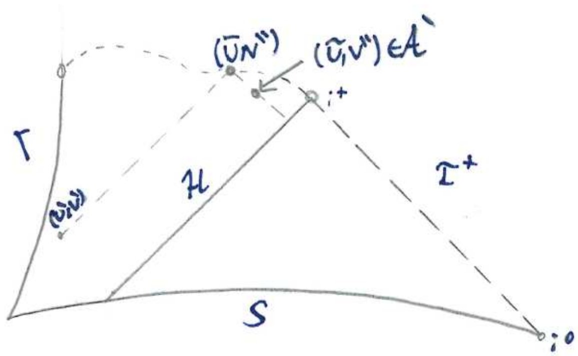
[27] This section is not important for the rest of the essay. Nevertheless, the result presented here is of independent interest.

[28] Meaning no two points on it have a timelike separation

[29] In view of equation (12), $u' \geq U_{\mathcal{H}}$.



We now need to discuss the assumption we made that $[u'', u'] \times \{v''\} \in Q^+$. This assumption fails if and only if $(u', V_{\mathcal{I}^+})$ is not on the boundary of Q^+, or, equivalently, if the null ray $u' = const$ intersects the boundary at $v < V_{\mathcal{I}^+}$. (See the diagram below)



To see that in the lemma still holds in this case, as suggested by the diagram above, we prove the following

Subclaim. If there exists a $\bar{u} \leq u'$ s.t. $(\bar{u}, v'') = p \in (\overline{Q^+} \cap \{v < V_{\mathcal{I}^+}\}) \setminus Q^+$, then the null ray $v'' \times [u'', \bar{u})$ intersects $\mathcal{A}$.

So in particular, the subclaim implies that again there can be found a $\tilde{u} < \bar{u}$ s.t. $(\tilde{u}, v'') \in \mathcal{A}'$.

Proof. The subclaim is an immediate consequence of prop. 5. For assume it is wrong. Then $(\bar{u}, v'') = p \in (Q^+ \cap \{v < V_{\mathcal{I}^+}\}) \setminus Q^+$ and we get for the minimal u_{min} s.t. (u_{min}, v'') is still on the boundary that there exists a $u_<$ such that $[u_<, u_{min}) \times \{v''\} \in \mathcal{R}$ (this would be true for all $u_<$ as long as $(u_<, v'')$ is still in Q^+. So prop. 5 tells us that $u_{min} = u_{b_\Gamma}$, which is in contradiction to $u_{min} \leq \bar{u} < u_{b_\Gamma}$. This completes the proof □

This completes the proof. □

3.5.3 Penrose inequalities on $\mathcal{A}'$ and $\mathcal{H}$

In this subsubsection, we will prove the following *Penrose inequality*[30]:

Lemma 3.2. *On $\mathcal{H}$, we have that $r \leq 2M_f$.*

Remark 8. Remind yourself of the definition of the final Bondi mass: $M_f = \inf_{\mathcal{I}^+} M$.

Proof. We will prove the lemma by contradiction. The strategy will roughly consist of the following steps: We will assume that the lemma is wrong. Then, using mainly the results from prop. 1 and eqns. (14),(12), we will show some basic inequalities for m, r in the region $[u_0, U_{\mathcal{H}}] \times [v_0, V_{\mathcal{I}^+})$ which would allow us to infer that r diverges as $v \to V_{\mathcal{I}^+}$ for all $u \in [u_0, U_{\mathcal{H}}]$ (using that it diverges for $u = u_0$). (We will not do this explicitly but it will become clear in the proof.) Of course, this alone would not be a contradiction. However, we will then see that we can define a bigger region $[u_0, u'] \times [v_0, V_{\mathcal{I}^+})$ for some $u' > U_{\mathcal{H}}$ which will both be fully contained in $\mathcal{R}$ and fulfil very similar inequalities for m, r (in fact, the region will be defined by imposing such inequalities). With a bit of work, we will then again be able to infer that r in fact diverges for all $u \in [u_0, u']$ as v approaches $V_{\mathcal{I}^+}$. That however implies that $\mathcal{I}^+$ has u-coordinates larger than $U_{\mathcal{H}}$. But $\mathcal{H}$ is the future boundary of $J^-(\mathcal{I}^+) \cap Q^+$, a contradiction. The reader may already wish to refer to the two diagrams below for an illustration of the idea.

The proof will assume positivity of m as shown in prop. 1, but in view of remark 5, I will mention when we assume it and how to avoid the assumption using footnotes with lower case Latin letters. I recommend to skip these footnotes on first reading.

So let's start with the proof:

Suppose the lemma is not true. Then, there exists a point $(U_{\mathcal{H}}, v_0) \in \mathcal{H}$ with

$$r(U_{\mathcal{H}}, v_0) = R > 2M_f$$

Furthermore, there exists a point $(u_0, V_{\mathcal{I}^+}) \in \mathcal{I}^+$ s.t.[a]

$$m(u_0, V_{\mathcal{I}^+}) \leq M \text{ with } M \in (M_f, R/2)$$

This is in fact the only point in this proof where we use the definition of M_f.[31]

Note that this implies $1 - \frac{2M}{R} > 0$. From eqns. (12), (14), we have

$$W := [u_0, U_{\mathcal{H}}] \times [v_0, V_{\mathcal{I}^+}) \subset (J^-(\mathcal{I}^+) \cap Q^+) \cup \mathcal{H} \subset \mathcal{R} \cup \mathcal{A},$$

so we see from prop. 1v) that we have the following inequalities:

$$\partial_v r|_W \geq 0,\ \partial_u r|_W < 0,\ \partial_v m|_W \geq 0,\ \partial_u m|_W \leq 0$$

[30] *A* comprehensive review of the general relevance of Penrose inequalities and recent progress is given in [21]

[31] Imagine we only had $r(U_{\mathcal{H}}, v_0) = R > 2M_f - \epsilon$. Then $m(u_0, V_{\mathcal{I}^+}) \leq M$ with $M \in (M_f - \epsilon/2, R/2)$ would not necessarily be true anymore, for it would then be possible that $R < 2M_f$.

[a] Even if the Bondi mass were negative, we could still pick $M > 0$, for $R/2 > 0$.

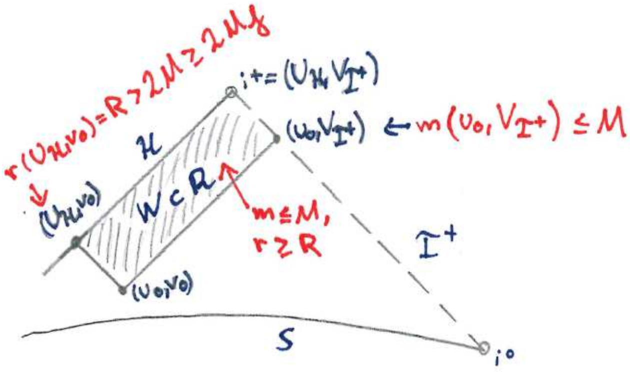
We infer that, on W, we have $r|_W \geq R$ and $m|_W \leq M$, and so, in particular[b]

$$\frac{m(u,v)}{r(u,v)} \leq \frac{M}{R} < \frac{1}{2} \Rightarrow 1 - \frac{2m(u,v)}{r(u,v)} \geq 1 - \frac{2M}{R} > 0 \quad \forall (u,v) \in W$$

But prop. 1 tells us that in $\mathcal{A}$, $1 - \frac{2m}{r} = 0$, so we conclude that in fact $W \subset \mathcal{R}$
So in summary,

$$W = [u_0, U_{\mathcal{H}}] \times [v_0, V_{\mathcal{I}^+}) \subset \mathcal{R}, \tag{16}$$
$$r|_W \geq R, m|_W \leq M \text{ and } (1 - \frac{2m}{r})|_W \geq 1 - \frac{2M}{R} > 0$$

These results are depicted in the diagram below.



Note, in particular, that $\mathcal{H} \subset \mathcal{R}$. As mentioned before, we can show that an even larger region fulfilling similar inequalities is contained in $\mathcal{R}$:

Subclaim. There exists a $u' > U_{\mathcal{H}}$ and a $R' \in (0, R)$, such that

$$Z := [u_0, u'] \times [v_0, V_{\mathcal{I}^+}) \subset \mathcal{R} \tag{17}$$
$$r|_Z \geq R', m|_Z \leq M \text{ and } (1 - \frac{2m}{r})|_Z \geq 1 - \frac{2M}{R'} > 0$$

Proof. As I outlined in the beginning of the proof, we will take these inequalities as starting point, i.e. we will intersect Z with these inequalities and with $\mathcal{R}$ and then see that these intersections, if u' is chosen appropriately, do not cut anything away from Z. So let us start the proof:

First, since $\mathcal{H} \subset \mathcal{R}$ and thus $\partial_v r > 0$ on $\mathcal{H}$, and since $\partial_v r$ is continuous, there exists a $u' > U_{\mathcal{H}}$ with $(u', v_0) \in Q^+$ and $\partial_v r > 0$ along $[U_{\mathcal{H}}, u'] \times \{v_0\}$ and hence $[U_{\mathcal{H}}, u'] \times \{v_0\} \in \mathcal{R}$. We can put a stronger restriction on u': Let it additionally be small enough so that $r(u, v_0) > R'$ for all $u \in [U_{\mathcal{H}}, u']$ where R' satisfies $R > R' > 0$ and $1 - \frac{2M}{R'} > 0$ (this exists because $1 - \frac{2M}{r(U_{\mathcal{H}}, v_0)} = 1 - \frac{2M}{R} > 0$ and the continuity of r). However, what is not a priori clear is that we can also find a u' such that all the above still holds in the entire region $[U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+})$ (in particular, it is not clear at all why this should still be in

[b] Since we can pick M positive, this equation is still valid for negative m, for then we have $1 - 2m/r > 1$.

Q^+). But it turns out that the u' as specified above already suffices to guarantee exactly this, as we will show now: Define the sets

$$Y = [U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+}), \qquad Y_\cap = Y \cap Q^+, \tag{18}$$
$$Y_{\cap\cap} = Y_\cap \cap \{(\bar{u}, \bar{v}) : r(u,v) > R'' \,, m(u,v) < M' \text{ for all } U_{\mathcal{H}} \le u \le \bar{u},\ v_0 \le v \le \bar{v}\}$$

for some $M' > M$, $R'' < R'$ with $1 - \frac{2M'}{R''} > 0$.

We will now show that neither of the intersections cut something from the set Y, i.e. we will show that $Y_{\cap\cap} = Y_\cap = Y$.

We first show the first equality: Notice that $Y_{\cap\cap}$ is clearly open in $Y_\cap$. Furthermore, since[c] $1 - \frac{2m}{r} > 1 - \frac{2M'}{R''} > 0$ in $Y_{\cap\cap}$, $Y_{\cap\cap}$ can neither have non-empty intersection with $\mathcal{A}$ nor with $\mathcal{T}$ (by prop. 1ii)). So

$$Y_{\cap\cap} \subset \mathcal{R}$$

Now, the only limit points of $Y_{\cap\cap}$ w.r.t. $Y_\cap$ which could potentially not be contained in $Y_{\cap\cap}$ would be points $(u,v) \in V_\cap$ with $r(u,v) = R''$ or $m(u,v) = M'$. But since $Y_{\cap\cap} \subset \mathcal{R}$, it follows from the standard inequalities in $\mathcal{R}$ from prop. 1v), that $r(\bar{u}, \bar{v}) > R' > R''$ and $m(\bar{u}, \bar{v}) \le M < M'$ for all $(\bar{u}, \bar{v}) \in Y_{\cap\cap}$. So we see that $Y_{\cap\cap}$ has no limit points w.r.t. $Y_\cap$ that are not contained in $Y_{\cap\cap}$, hence $Y_{\cap\cap}$ is closed in $Y_\cap$ and the second intersection does not cut anything away from the rest set, i.e. $Y_{\cap\cap} = Y_\cap$ (since $Y_\cap = [U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+}) \cap Q^+$ is connected and $Y_{\cap\cap} \neq \emptyset$ by construction).

So we indeed have

$$Y_{\cap\cap} = Y_\cap = [U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+}) \cap Q^+ \subset \mathcal{R} \tag{19}$$

Now, let's show that to intersect with Q^+ is redundant as well. For that, we want to show, analogously to the above procedure, that $Y_\cap$ is open and closed in $Y = [U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+})$. Open-ness is again clear. To show closure, we want to show that

$$[U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+}) \cap \overline{Q^+} = [U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+}) \cap Q^+ (= Y_\cap)$$

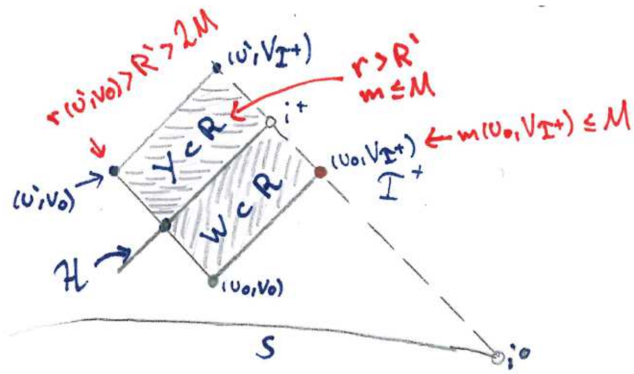
for the LHS is precisely the closure of the RHS in $Y = [U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+})$. Assume the equation is wrong. That would mean that, in view of equation (19), there are points in $(\overline{\mathcal{R}} \setminus Q^+) \cap ([U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+}))$, let's call them (u_c, v_c), such that $[U_{\mathcal{H}}, u_c) \times \{v_c\}$ would be in $\mathcal{R}$. Then prop. 5 tells us that indeed $u_c = u_{b_\Gamma}$. So, in particular, $(u_c, v_0) \in \overline{\mathcal{R}} \setminus Q^+$. This contradicts the initial construction assumption that $[U_{\mathcal{H}}, u'] \times \{v_0\} \in \mathcal{R}$. We learn that, since Y is connected, $Y_\cap = Y$.

We summarise:

$$Y = [U_{\mathcal{H}}, u'] \times [v_0, V_{\mathcal{I}^+}) \subset \mathcal{R}, \tag{20}$$
$$r|_Y > R', m|_Y \le M \text{ and } (1 - \frac{2m}{r})|_Y > 1 - \frac{2M}{R'} > 0$$

This is also depicted in the diagram below.

[c] Again, M' can be chosen to be positive in which case the inequality is trivial for m negative.



Looking back at eq. (16), we find that $Z = Y \cup W$ satisfies the subclaim. $\square$

Now that this subclaim is established, we can use these inequalities on Z to show that $\mathcal{I}^+$ extends beyond $u = U_{\mathcal{H}}$: First, we integrate eq. (5), for any $(u^*, v^*) \in Z$:

$$
\begin{aligned}
m(u_0, v^*) - m(u^*, v^*) &= -\int_{u_0}^{u^*} \partial_u m(u, v^*)\, du \qquad (21)\\
&= -\int_{u_0}^{u^*} \left(8\pi r^2 \Omega^{-2} (\underbrace{T_{uv}\partial_u r}_{\leq 0} - T_{uu}\partial_v r) \right)(u, v^*)\, du \\
&\geq \int_{u_0}^{u^*} \left(8\pi r^2 \Omega^{-2} (T_{uu}\partial_v r) \right)(u, v^*)\, du \\
&= \int_{u_0}^{u^*} \left(8\pi r^2 T_{uu} (\underbrace{-\frac{4}{\Omega^2}\partial_v r \partial_u r}_{1-2m/r} \cdot \frac{1}{-4\partial_u r}) \right)(u, v^*)\, du \\
&= \frac{1}{2}\int_{u_0}^{u^*} \left(\frac{4\pi r T_{uu}}{-\partial_u r}(r - 2m) \right)(u, v^*)\, du \geq 0
\end{aligned}
$$

In the third line, we used $T_{uv}\partial_u r \leq 0$ and in the last, we inserted the definition of m (eq. (2)). Now, we make use of the subclaim: The subclaim implies[d] $m(u_0, v^*) - m(u^*, v^*) < M$ and $(r - 2m)|_Z \geq R' - 2M > 0$. So we indeed have

$$
\frac{2M}{R' - 2M} \geq \int_{u_0}^{u^*} \frac{4\pi r T_{uu}}{-\partial_u r}(u, v^*)\, du \quad \forall (u^*, v^*) \in Z \qquad (22)
$$

The integrand is familiar from eq. (8) which I repeat here for better readability[32]:

$$
\partial_u \frac{\partial_v r}{1 - \frac{2m}{r}} = \frac{4\pi r T_{uu}}{\partial_u r} \frac{\partial_v r}{1 - \frac{2m}{r}}
$$

[32] Note that the assumptions made to get this equation, i.e. $\partial_u r \neq 0 \neq (1 - 2m/r)$, are true in $\mathcal{R}$ and thus in Z.

[d] If we allow the Hawking mass to be negative, then $m(u_0, v^*) - m(u^*, v^*) < 2M$ by the triangle equality. Since we only want it to estimate it against some constant, this would also suffice.

It has the form $\partial_u f = gf$, so $\partial_u \log(f) = g$. Integrating such an equation from u_0 to u^*, we get

$$\log\left(f(u^*)/f(u_0)\right) = \int_{u_0}^{u^*} g(u)du$$

where I omitted the v-argument. Applying this to our case (and exponentiating), we obtain

$$\begin{aligned}\frac{\partial_v r}{1-\frac{2m}{r}}(u^*,v^*) &= \frac{\partial_v r}{1-\frac{2m}{r}}(u_0,v^*)\exp\left(\int_{u_0}^{u^*}\frac{4\pi r T_{uu}}{\partial_u r}(u,v^*)\,du\right) \\ &\geq \frac{\partial_v r}{1-\frac{2m}{r}}(u_0,v^*)\exp\left(-\frac{2M}{R'-2M}\right)\end{aligned} \tag{23}$$

where I used (22) and the positivity of the LHS (the subclaim guarantees that $\partial_v r$ and $1-\frac{2m}{r}$ are positive in $Z \subset \mathcal{R}$) for the estimate. Finally, we have that $(1-\frac{2m}{r})(u^*,v^*) > 1-\frac{2M}{R'}$, and because m is non-negative[e] in $Z \subset \mathcal{R}$, $0 < (1-\frac{2m}{r})(u_0,v^*) \leq 1$. We arrive at

$$\partial_v r(u^*,v^*) \geq \left(1-\frac{2M}{R'}\right)\exp\left(-\frac{2M}{R'-2M}\right)\partial_v r(u_0,v^*) \quad \forall (u^*,v^*) \in Z \tag{24}$$

Integrating this inequality (along the v-direction), we thus find that, if $\lim_{v^*\to V_{\mathcal{I}^+}} r(u_0,v^*) = \infty$ (which is the case since $u_0 < U_{\mathcal{H}}$), then $\lim_{v^*\to V_{\mathcal{I}^+}} r(u^*,v^*) = \infty$ for all $u^* \in [u_0,u']$. Hence, since $u' > U_{\mathcal{H}}$, we reached a contradiction to $\mathcal{H}$ being the event horizon (remember that $U_{\mathcal{H}}$, the event-horizon's u-coordinate is the u-coordinate of the future limit point of $\mathcal{I}^+$). This completes the proof. □

Remark 9. In view of the inequality $\partial_u m \leq 0$ on $\mathcal{H} \subset \mathcal{R} \cup \mathcal{A}$, it would have sufficed to show that $r|_{\mathcal{H}\cap\{v\geq v_0\}}$ is bounded by M_f for sufficiently large v_0. That means that we could have used lemma 3.1 for proving $Y_\cap = Y$ instead of prop. 5.

Since $r > 0$ on $\mathcal{H} \setminus \Gamma$, we get the corollary:

Corollary 3.4. *In particular, the final Bondi mass is positive.*

Furthermore, we get:

Corollary 3.5. *On $\mathcal{A}'$, we have that $r \leq 2M_f$.*

To see that this is a corollary of lemma 3.2, remember that $\partial_u r < 0$ on Q^+. So in view of eq. (14), we get $\sup_{\mathcal{H}} r \geq \sup_{\mathcal{A}'}$ (because $\mathcal{A}'$ cannot have u-coordinates smaller than$U_{\mathcal{H}}$).

In view of footnotes (a)-(e), we see that we in fact did not need the positivity of the mass to derive the Penrose inequality (see remarks 2,5). The Penrose inequality is thus still valid if we do not assume $r|_\Gamma = 0$! In particular, in view of corollary 3.4, we get the following

Lemma 3.3. *Even without the proposition that $m \geq 0$, the final Bondi mass and thus the Bondi mass is positive. Moreover, lemma (3.2) still holds.* Not shown in [12]

[e] If $m(u^*,v^*)$ is negative, then, in view of $\partial_v m \geq 0$, m either becomes positive for large enough v. Or it approaches some finite negative value, in which case $1-2m/r \to 1$ as v approaches $V_{\mathcal{I}^+}$ so we can for instance choose a v^* large enough such that $(1-2m/r)(u^*,v^*) < 2$.

3.5.4 Proof of completeness of future null infinity

We can now use the Penrose equality on $\mathcal{H}$ to show that $\mathcal{I}^+$ is complete in the sense of the reformulation in the beginning of the subsection. What we have to show is that the affine length of $\boldsymbol{X}(u,v)$ from eq. (15) diverges, i.e. we have to show:

Lemma 3.4.

$$\lim_{v \to V_{\mathcal{I}^+}} \int_{u_0}^{U_{\mathcal{H}}} (\boldsymbol{X}(u,v) \cdot u)^{-1}\, du = \infty \tag{25}$$

Proof. This proof will mostly recycle the ideas we used in the proof of the Penrose inequality. (I will again use Latin letter footnotes to make comments on how to avoid using positivity of the mass.)

At the same time, we use the Penrose inequality itself only once, and that is in the very beginning: Pick $R > 2M_f$ s.t., for a $u_0 < U_{\mathcal{H}}$, the Bondi mass $M = m(u_0, V_{\mathcal{I}^+})$ has

$$R > 2M \tag{26}$$

Then, for sufficiently large $v_0 < V_{\mathcal{I}^+}$, all ingoing null rays with $v \geq v_0$ take the value R exactly once in $J^-(\mathcal{I}^+)$, i.e. they intersect $J^-(\mathcal{I}^+) \cap \{r = R\}$ exactly once. We call these points $(u^*(v), v)$. To see why these exist and are unique, let $v_0 < V_{\mathcal{I}^+}$ be large enough such that $r(u_0, v_0) > R$. This is possible because $r(u_0, v)$ diverges towards $V_{\mathcal{I}^+}$ for all $u_0 < U_{\mathcal{H}}$. In view of $J^-(\mathcal{I}^+) \cap Q^+ \subset \mathcal{R}$ (eq. (12)) and the inequality $\partial_v r > 0$ in $\mathcal{R}$, we then know that $r(u_0, v) > R$ for all $v \in [v_0, V_{\mathcal{I}^+})$. So from the Penrose inequality $r|_{\mathcal{H}} \leq 2M_f < R$ and the continuity of r, there is clearly at least one $u^*(v) > u_0$ such that $r(u^*(v), v) = R$ for all $v \geq v_0$. But this point is also unique by the global inequality $\partial_u r < 0$ on Q^+ (prop. 1i)).[33]

However, at no point in this argument did we use the definition of M_f. We could have replaced it with any other constant C as long as $\sup_{\mathcal{H}} r < C$.

The existence of the points $v^*(u)$ allows us to make the following crucial estimate (the integrand is positive) on the integral in eq. (25).

$$\int_{u_0}^{U_{\mathcal{H}}} (\boldsymbol{X}(u,v) \cdot u)^{-1}\, du \geq \int_{u_0}^{u^*(v)} (\boldsymbol{X}(u,v) \cdot u)^{-1}\, du \tag{27}$$

Why is this estimate so helpful? On the set

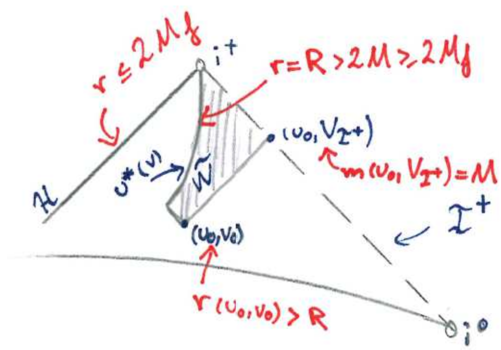
$$\tilde{W} = \{(u,v) : v \in [v_0, V_{\mathcal{I}^+}),\, u_0 \leq u \leq u^*(v)\}$$

(which is contained in $\mathcal{R}$ by equation 12), the inequalities $\partial_u r < 0$, $\partial_v r > 0$, $\partial_u m \leq 0$, $\partial_v m \geq 0$ and the definition of $u^*(v)$ imply:

$$m|_{\tilde{W}} \leq M,\; r|_{\tilde{W}} \geq R \Rightarrow (r - 2m)|_{\tilde{W}} \geq R - 2M > 0 \tag{28}$$

See the diagram below for a depiction of the situation:

[33] The unique existence of this point is where Dafermos claimed that Lemma 3.1 is needed.



The positivity of the RHS comes simply from eq. (26). This inequality is reminiscent of equation (20) in the previous proof and we will in fact make very similar use of it:

$$
\begin{aligned}
\int_{u_0}^{u^*(v)} (\boldsymbol{X}(u,v)\cdot u)^{-1}\,du &= \int_{u_0}^{u^*(v)} \overbrace{\frac{\partial_u r \partial_v r}{1-\frac{2m}{r}}(u,v)}^{-\Omega^2(u,v)/4}\,\overbrace{\frac{1-\frac{2m}{r}}{\partial_u r\partial_v r}(u_0,v)}^{-4\Omega^{-2}(u_0,v)}\,du \\
&= \frac{1}{\partial_u r(u_0,v)}\int_{u_0}^{u^*(v)} \exp\left(\int_{u_0}^{u}\frac{4\pi r T_{uu}}{-\partial_u r}(\bar{u},v)\,d\bar{u}\right)\partial_u r(u,v)\,du \\
&= \frac{r(u^*(v),v)-r(u_0,v)}{\partial_u r(u_0,v)}\exp\left(\int_{u_0}^{u}\frac{4\pi r T_{uu}}{\partial_u r}(\bar{u},v)\,d\bar{u}\right) \\
&= \frac{r(u_0,v)-R}{-\partial_u r(u_0,v)}\exp\left(\int_{u_0}^{u}\frac{4\pi r T_{uu}}{\partial_u r}(\bar{u},v)\,d\bar{u}\right)
\end{aligned}
\tag{29}
$$

where I have used eq. (15) and the definition of the Hawking mass (eq. (2)) in the first step, eq. (23) for $\frac{\partial_v r}{1-\frac{2m}{r}}(u,v)$, having replaced $u^*\to u, v^*\to v$, in the second step and simply performed the integral in the penultimate one. In the last one, I used $r(u^*(v),v)=R$. Now, recall eq. (21), replacing $v^*\to v$, $u^*\to u$:

$$
m(u_0,v)-m(u,v)\geq \frac{1}{2}\int_{u_0}^{u}\frac{4\pi r T_{uu}}{-\partial_u r}(r-2m)(\bar{u},v)\,d\bar{u}>0
$$

The positivity of the RHS follows from $\partial_u r<0$ and eq. (28). Eq. (28) then also implies that the LHS is smaller than[f] M and that the $r-2m$ term in the integral can be estimated against $r-2m>R-2M$. We therefore get, just as in eq. (22),

$$
\frac{2M}{R-2M}\geq \int_{u_0}^{u}\frac{4\pi r T_{uu}}{-\partial_u r}(\bar{u},v)\,d\bar{u}>0 \quad \forall (u,v)\in \tilde{W}
$$

Plugging this back into eq. (29)

$$
\int_{u_0}^{u^*(v)} (\boldsymbol{X}(u,v)\cdot u)^{-1}\,du \geq \frac{r(u_0,v)-R}{-\partial_u r(u_0,v)}\exp\left(-\frac{2M}{R-2M}\right). \tag{30}
$$

[f] Or smaller than $2M$, if the Hawking mass can be negative. See footnote (d).

We know that the numerator on the RHS diverges as $v \to V_{\mathcal{I}^+}$ ($r(u_0, v) \to \infty$), so to show that the LHS diverges, all that is left to do is to show that the denominator of the RHS remains finite as $V_{\mathcal{I}^+}$ is approached. Recall how we used eq. (8) to show that $\partial_v r$ is bounded from below on ingoing null rays (eq. (24)). It turns out that we can use a very similar argument to show that $-\partial_u r$ is bounded from below on outgoing null rays[34].

Note that in analogy to how eq. (8) implied eq. (23), we have that eq. (9) implies that

$$\frac{\partial_u r}{1-\frac{2m}{r}}(u_0, v) = \frac{\partial_u r}{1-\frac{2m}{r}}(u_0, v_0) \exp\left(\int_{v_0}^{v} \frac{4\pi r T_{vv}}{\partial_v r}(u_0, \bar{v})\, d\bar{v}\right)$$

and thus

$$-\partial_u r(u_0, v) = -\partial_u r(u_0, v_0) \frac{\left(1-\frac{2m}{r}\right)(u_0, v)}{\left(1-\frac{2m}{r}\right)(u_0, v_0)} \exp\left(\int_{v_0}^{v} \frac{4\pi r T_{vv}}{\partial_v r}(u_0, \bar{v})\, d\bar{v}\right) \tag{31}$$

where we added the minus sign so that both sides become positive. Finding estimates for the $1-\frac{2m}{r}$ terms is easy: For the term in the numerator, we have[g] $\left(1-\frac{2m}{r}\right)(u_0, v) \le 1$, whereas for the term in the denominator we have by equation (28) $\left(1-\frac{2m}{r}\right)(u_0, v_0) \ge 1-\frac{2M}{R} > 0$. Inserting these estimates yields

$$-\partial_u r(u_0, v) \le -\partial_u r(u_0, v_0) \frac{1}{1-\frac{2M}{R}} \exp\left(\int_{v_0}^{v} \frac{4\pi r T_{vv}}{\partial_v r}(u_0, \bar{v})\, d\bar{v}\right) \tag{32}$$

So we only need to find an estimate for the integral on the RHS. For this, we proceed just as in eq. (21): We use equation (6) to write:

$$\begin{aligned} m(u_0, v) - m(u_0, v_0) &= \int_{v_0}^{v} \partial_v m(u_0, \bar{v})\, d\bar{v} \qquad (33)\\ &= \int_{v_0}^{v} \left(8\pi r^2 \Omega^{-2} (\underbrace{T_{vu}\partial_v r}_{\ge 0} - T_{vv}\partial_u r)\right)(u_0, \bar{v})\, d\bar{v} \\ &\ge \int_{v_0}^{v} \left(8\pi r^2 \Omega^{-2}(-T_{vv}\partial_u r)\right)(u_0, \bar{v})\, d\bar{v} \\ &= \frac{1}{2}\int_{v_0}^{v} \left(\frac{4\pi r T_{vv}}{\partial_v r}(r-2m)\right)(u_0, \bar{v})\, d\bar{v} \ge 0 \end{aligned}$$

In the third step, we used positivity of $T_{vu}\partial_v r$ in $\mathcal{R}$ and in the last step we plugged in the definition of m, just as in equation (21).

Finally, by equation (28), we have that the LHS is smaller than[h] M and that the $r-2m$ term on the RHS and hence the entire RHS is positive (by $\partial_v r > 0$ in $\mathcal{R}$ and the positivity of the energy-momentum tensor). More concretely, it implies we can estimate the term in the integral by $r - 2m > R - 2M > 0$ to arrive at what we wanted:

$$\frac{2M}{R-2M} \ge \int_{v_0}^{v} \frac{4\pi r T_{vv}}{\partial_v r}(u_0, \bar{v})\, d\bar{v}. \tag{34}$$

[34] The reason why we could show boundedness below for $\partial_v r$ whereas we will now show boundedness above for $\partial_u r$ is, of course, the difference in sign between $\partial_v r$ and $\partial_u r$.

[g] See footnote (e) for how this works when the Hawking mass is allowed to be negative.

[h] Or smaller than $2M$, if the Hawking mass can be negative. See footnote (d).

Plugging this back into equation (32), we get

$$-\partial_u r(u_0, v) \leq -\partial_u r(u_0, v_0) \frac{1}{1 - \frac{2M}{r(u_0,v_0)}} \exp\left(\frac{2M}{R - 2M}\right) = const. \qquad (35)$$

Since $-\partial_u r$ is always positive in Q^+, we thus showed that $-\partial_u r(u_0, v)$ is bounded. Therefore, the RHS of eq. (30) indeed diverges and so does the LHS. This completes the proof of the claim □

and hence the proof of Theorem 3.1.

□

As mentioned in the proof, we did not need the specific Penrose inequality, we only required to have some bound on $r|_{\mathcal{H}}$. So we in fact have the following

Theorem 3.3. *Even if the black hole region is empty, i.e. if $Q^+ = J^-(\mathcal{I}^+) \cap Q^+$: As long as the extension of the area radius onto $\mathcal{H}$ is bounded, i.e. if $\sup_{\mathcal{H}} r < C$ for some constant C, then $\mathcal{I}^+$ is complete.*

Not mentioned in [12]

This is a weaker statement than Theorem 3.1, however, and is independent of the WEP.

3.6 Extension of the results

Not shown in [12]. Statement slightly changed.

Finally, in view of remarks 2,5 and of footnotes a-h, we also have that the positivity of the mass from prop. 1v) was nowhere needed for any of the results of this paper. We can thus extend them to apply also to "reasonable" spherically symmetric initial data with *two* asymptotically flat ends for which S is a connected piece of the quotient of one of the ends and Γ is an ingoing null curve intersecting S only at its endpoing ($r|_\Gamma = 0$ can be dropped).

3.7 Conclusion and Outview

The results of this section show that for this broad class of systems the proof of WCCC can be approached locally, by either showing that generically, either the future development is complete or trapped regions form. Alternatively, in view of Theorem 3.3, it would also suffice to show that generically, if the black hole region is empty, the supremum of the area-radius on $\mathcal{H}$ is bounded. The two main directions of interest that might bring us closer to the resolution of the WCCC are: i) Search for physically relevant (spherically symmetric) systems for which the WEP (or other assumptions made in this section) does not apply; and understand the WCCC for them. This might give some intuition to how the results of this section could be shown without use of the WEP. ii) Consider spherically symmetric systems for which the WEP (and the other assumptions) hold and understand the Trapped surface conjecture for these. This has so far only been done for Christodoulou's scalar field model. Understanding this conjecture for more complicated systems like the EMKG system considered in the next section might give hints how to complete the proof of the WCCC for the class of systems considered here.

4 The Einstein-Maxwell-Klein-Gordon system

In this section, I will discuss the previous results in the context of a specific spacetime: I will review Kommemi's publication and PhD thesis [20, 19] where he shows that the results from the previous section are applicable to the the spherically symmetric Einstein-Maxwell-Klein-Gordon (EMKG) system with self-gravitating charged, massive and spinless particles. This system is governed by the Einstein equations (1) with $T_{\mu\nu} = T^{EM}_{\mu\nu} + T^{KG}_{\mu\nu}$ and

$$T^{EM}_{\mu\nu} = \frac{1}{4\pi}(g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}) \tag{36}$$

$$T^{KG}_{\mu\nu} = D_{(\mu}\phi(D_{\nu)}\phi)^\dagger - \frac{1}{2}g_{\mu\nu}\left(g^{\alpha\beta}D_\alpha\phi(D_\beta\phi)^\dagger + \mathfrak{m}^2\phi\phi^\dagger\right) \tag{37}$$

$$\nabla^\nu F_{\mu\nu} = 2\pi\mathfrak{e}i\left(\phi(D_\mu\phi)^\dagger - \phi^\dagger D_\mu\phi\right) \tag{38}$$

$$D^\mu D_\mu\phi = \mathfrak{m}^2\phi \tag{39}$$

Here, $\mathfrak{m}^2 \in \mathbb{R}_{\geq 0}$ is the real, non-negative (otherwise the dominant energy condition would be violated) mass of the field, $\mathfrak{e} \in \mathbb{R}$ its charge[35] and $D = \nabla + i\mathfrak{e}A$ the gauge derivative where A is the 1-form with $dA = F$.

This is a generalisation of Christodoulou's self-gravitating real and massless scalar field ($F_{\mu\nu} = \Im(\phi) = 0 = \mathfrak{e} = \mathfrak{m}^2$) and of Dafermos' model [10]with $\Im(\phi) = 0 = \mathfrak{e} = \mathfrak{m}^2$. The main difference to the former model is that this model allows for Cauchy horizons on the null ray emanating from i^+ which makes it a much richer object to study with regards to strong cosmic censorship[36]. While this was also possible in Dafermos' model, that model can only have one asymptotically flat end if $F_{\mu\nu} = 0$, as there, the charge must arise from non-trivial topology[37]. The study of charged black holes is also motivated by the similarity between rotating and charged black holes, i.e. it can be seen as a way to leave the realm of spherical symmetry. However, in contrast to Christodoulou's model, WCCC and SCCC are still unsolved for the EMKG system.

Nevertheless, important steps towards settling these questions have been made by Kommemi. In particular, he has shown that this model obeys a stronger extension principle than the WEP from assumption F, thus showing applicability of the results from section 3 to the EMKG system. In this section, I will focus on presenting the proof of this extension principle to the reader.

First, some necessary groundwork will be set in subsection 4.1, where I set up the IVP such that assumptions A-E from the previous sections hold.

In subsection 4.2, I will then introduce the generalised extension principle (GEP) and show that the GEP implies the WEP.

I will prove the GEP for th EMKG system in subsection 4.3, thus establishing applicability of the results from the previous section.

Finally, I will summarise the results.

[35]For a discussion of why we can ignore the quantisation of the charge, see chapter 2.2.2 in Kommemi's paper.

[36]A topic which I will not expand on in this essay.

[37]Heuristically, for a *fixed* spacetime charge Q, we see that the usual formula $F_{uv} = \frac{Q}{r^2}\Omega^2$ blows up as $r \to 0$.

4.1 The setup

Using standard methods, one can show that for the EMKG system, a generalisation of Choquet-Bruhat and Geroch's theorem holds for smooth spherically symmetric initial data sets $(\Sigma, h, k_{ij}, E_i, B_i, A_i, \phi, \phi')$ satisfying certain constraint equations and Σ having trivial topology. Here, E_i, B_i denote electric and magnetic fields, respectively. In particular, one shows that the spherical symmetry is preserved if imposed initially.[38]

With this in mind, we can state the following theorem the proof of which will form the rest of this section:

Theorem 4.1. *For smooth, spherically symmetric and asymptotically flat initial data with one end satisfying assumption D (i.e. $\partial_u r < 0$ initially), there exists a maximal future development $(M = Q^+ \times_r \mathbb{S}^2, g_{\mu\nu}, F_{\mu\nu}, \phi)$, where r denotes the area-radius function. If we additionally assume that the initial data are chosen such that assumption E holds*[39]*, then the results of section 3 apply.*

As discussed in the remarks following their introduction, if we choose coordinates s.t. the metric is given by $g = -\Omega^2 du dv + r^2\gamma$, assumptions A and B are trivially satisfied. Assumption C, the dominant energy condition, can easily be checked to be satisfied from eqns. (36) and (37) if $\mathfrak{m}^2 \geq 0$.[40] So to prove the above theorem, we only need to show that assumption F is satisfied, i.e. that the spacetime obeys the weak extension principle.

Before we show this, however, we need to fix the electromagnetic gauge freedom. For $\mathfrak{e} \neq 0$, the EKGM system is invariant under the local gauge transformations $\phi \to e^{-i\mathfrak{e}\chi}$, $A_\mu \to A_\mu + \partial_\mu \chi$. We can globally[41] fix this gauge freedom by choosing

$$A_u(u,v)|_{\pi^{-1}(\Gamma\cup S)} = 0, \;\; A_v(u,v) = 0$$

(The other components of A vanish by spherical symmetry.) The only non-vanishing component of F is then $F_{uv} = -\partial_v A_u$. We define $Q_e = 2r^2\Omega^2 F_{uv}$.

This gauge leads to the reduced system of eqns.:

---

[38] For more details on e.g. the meaning of spherical symmetry in this context, refer to Kommemi's paper.

[39] The reader should go back to the discussion of the assumption to see when this holds. It is not the goal of this essay to state the weakest possible assumption on the initial data such that assumption E holds.

[40] Without this assumption, only the null energy condition is obeyed.

[41] A consequence of spherical symmetry. Again, refer to Kommemi's paper for more details.

$$r\partial_v\partial_u r = -\frac{1}{4}\Omega^{-2} - \partial_v r\partial_u r + \mathfrak{m}^2\pi r^2\Omega^2\phi\phi^\dagger + \frac{1}{4}\Omega^2 r^{-2}Q^2 \tag{40}$$

$$r^2\partial_u\partial_v \log\Omega = -2\pi r^2\left(D_u\phi(\partial_v\phi)^\dagger + \partial_v\phi(D_u\phi)^\dagger\right) - \frac{\Omega^2 Q^2}{2r^2} + \frac{\Omega^2}{4} + \partial_u r\partial_v r \tag{41}$$

$$\partial_u(\Omega^{-2}\partial_u r) = -4\pi r\Omega^{-2}D_u\phi(D_u\phi)^\dagger \tag{42}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -4\pi r\Omega^{-2}\partial_v\phi(\partial_v\phi)^\dagger \tag{43}$$

$$\partial_u Q_e = 2\pi\mathfrak{e}ir^2\left(\phi(D_u\phi)^\dagger - \phi^\dagger D_u\phi\right) \tag{44}$$

$$\partial_v Q_e = 2\pi\mathfrak{e}ir^2\left(\phi^\dagger\partial_v\phi - \phi(\partial_v\phi)^\dagger\right) \tag{45}$$

$$-\frac{1}{4}\Omega^2\mathfrak{m}^2\phi = \partial_u\partial_v\phi + r^{-1}(\partial_u r\partial_v\phi + \partial_v r\partial_u\phi) + \mathfrak{e}i\Psi(A) \tag{46}$$

$$\Psi(A) = A_u(\phi r^{-1}\partial_v r + \partial_v\phi) - \frac{1}{4}\Omega^2 r^{-2}Q_e\phi \tag{47}$$

Here, $Q^2 = Q_e^2 + Q_m^2$ where Q_m denotes the magnetic charge coming from $F_{\theta\phi}$. For the purpose of this paper, the reader should ignore its inclusion[42]. A full derivation of these equations is not given here, but I shall state where they come from: Eq. (40) follows from eq. (7). Eq. (41) follows from the Einstein equations and relates the Gaussian curvature[43] to the components $T_{\theta\theta}$,$T_{\phi\phi}$. The next two equations are simply eqns. (3),(4), whereas eqns. (44)-(46) come from the equations of motions (38),(39) for the matter fields.

4.2 The generalised extension principle

The extension principle Kommemi proves is (cf. the WEP, assumption F)

> **The generalised extension principle (GEP)** Let $p \in \overline{Q^+}$, $q \in (I^-(p) \cap Q^+)$ s.t. $\mathcal{D} := (J^-(p) \cap J^+(q)) \setminus \{p\} \subset Q^+$: If $\mathcal{D}$ has finite spacetime volume, and if there exists constants r_0, R s.t.
>
> $$0 < r_0 \le r(p') \le R < \infty \ \forall p' \in \mathcal{D}$$
>
> Then $p \in Q^+$.

Remark 10. While the statement of the WEP is that *first singularities*[44] emanating from the closure of the regular region can only arise on the center $\overline{\Gamma}$, the GEP roughly states that first singularities can only arise on parts of the boundary where r tends to 0.

Before proving it for the EMKG system in the next subsection, I shall show that the GEP implies the WEP, assuming the dominant energy condition (in fact, the null energy condition suffices). For that, we need a

Proposition 7. *Let $p \in \overline{Q^+} \setminus \{v = V_{\mathcal{I}^+}\}$ and assume assumptions A- D. Then the region $J^-(p) \cap Q^+$ has finite spacetime volume.*

[42]Its inclusion is related to the fact that we may treat $\mathfrak{e}$ as a continuous parameter. $Q_m = 0$ whenever $\mathfrak{e} \neq 0$.

[43]$K = 4\Omega^{-2}\partial_u\partial_v \log\Omega$ in our coordinates. See [6], chapter 3, for example.

[44]For a precise definition, see paragraph 3.4, definition 6 in [20]

Proof. Using that Q^+ is a bounded subset of $\mathbb{R}^2$ and that the coordinates thus have finite ranges, writing the volume form as $\Omega^2/2 = (-\Omega^{-2}\partial_u r)^{-1}(-\partial_u r)/2$ and finally recalling equation[45] (3) and $\partial_u r < 0$ on Q^+, this is easily proved. See chapter 3.2 in Kommemi. □

Claim 4. *Assuming assumptions A- D, the GEP implies the WEP.*

Proof. Choose a point $(U, V) = p \in \overline{Q^+} \setminus \overline{\Gamma}$ which satisfies the conditions of the WEP, i.e. $\mathcal{D} = (J^-(p) \cap J^+(q)) \setminus \{p\} \subset \mathcal{R} \cup \mathcal{A}$ for some $(U', V') = q \in (I^-(p) \cap Q^+)$. This implies that $V < V_{\mathcal{I}^+}$, so the proposition above tells us that $\mathcal{D}$ has finite spacetime volume. Now, let $r_0 = r(U, V')$ and $R = r(U', V)$. Recalling $\partial_u r < 0, \partial_v r \geq 0$ on $\mathcal{R} \cup \mathcal{A}$, we get that $0 < r_0 \leq r(p') \leq R < \infty$, for all $p' \in \mathcal{D}$. So, from the GEP, $p \in Q^+$ and hence the WEP is established. □

4.3 Proof of the GEP for the EMKG system

The previous disccusion of the GEP in section 4.2 was completely general. We now move back to the EMKG system and prove that the GEP holds for it.

The proof will make use of a local existence result which will reduce the proof to showing that the norm (defined for any subset $Y \subset Q^+$)

$$N(Y) := \sup\{|\Omega|_1, |\Omega^{-1}|_0, |r|_2, |r^{-1}|_0, |\phi|_1, |A_u|_0, |Q_e|_0\} \tag{48}$$

is bounded for $Y = \mathcal{D}$ with $\mathcal{D}$ as defined in the GEP. Here, $|f|_n$ is the restriction of the C^n-norm on Q^+ to Y.

We will now first show this claim in 2 propositions and then show the boundedness of $N(\mathcal{D})$.

With standard methods[46] one proves the following

Proposition 8. *Let X be the lightcone boundary $[0, d] \times \{0\} \cup \{0\} \times [0, d]$. Let $k \geq 0$ and let $r > 0, A_u$ be C^{k+2} functions on X and $\Omega > 0, \phi$ be C^{k+1}functions on X such that eqns. (42),(43) hold on $[0, d] \times \{0\}$, $\{0\} \times [0, d]$ respectively. Let $|\cdot|_n, |\cdot|_{n,u}, |\cdot|_{n,v}$ denote the C^n-norm on Q^+ restricted to X, the $C^n(u)$-norm on $[0, d] \times \{0\}$ and the $C^n(v)$-norm on $\{0\} \times [0, d]$ respectively and define*

$$\tilde{N}(X) = \sup\{|\Omega|_{1,u}, |\Omega|_{1,v}, |\Omega^{-1}|_0, |r|_{2,u}, |r|_{2,v}, |r^{-1}|_0, |\phi|_{1,u}, |\phi|_{1,v}, |A_u|_0, |A_u|_{2,v}, |\partial_v A_u|_{1,u}\} \tag{49}$$

Then, there exist $\delta(\tilde{N})$ and C^{k+2}-functions r, A_u -which are unique amongst C^2-functions- and C^{k+1}-functions Ω, ϕ -which are unique amongst C^1-functions- such that r, A_u, Ω, ϕ satisfy the reduced system of equations (40) − (47) on the set $[0, \delta^*] \times [0, \delta^*]$ where $\delta^* = \min(\delta, d)$ and the restrictions of the these functions to X yield the initial data.

[45] The proof uses the monotonicity of $\Omega^2\partial_u r$ which still holds if only the null energy condition is assumed.

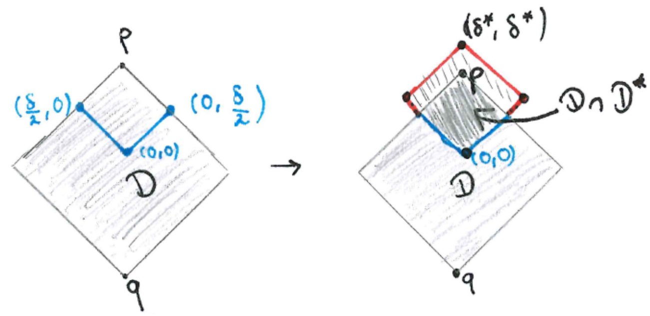
[46] See for example the appendix of [13]

Proposition 9. *Let* $(U,V) = p \in \overline{Q^+} \setminus Q^+$*,* $(U',V') = q \in I^-(p) \cap Q^+$ *s.t.* $\mathcal{D} = (J^+(q) \cap J^-(p)) \setminus \{p\} \subset Q^+$*. Then* $N(\mathcal{D}) = \infty$*.*

Proof. We will assume $N(\mathcal{D}) < \infty$ and use local existence and openness of Q^+ to show that we can extend Q^+ uniquely past p and thus infer $p \in Q^+$.
Take $2N(\mathcal{D}) = \tilde{N}$ and take $\delta(\tilde{N}) > 0$ as in the proposition above. Let $\delta(\tilde{N})$ be small enough such that $(U - \frac{\delta}{2}, V - \frac{\delta}{2})$ is still in Q^+. For convenience, translate the coordinates such that this point becomes $(0,0)$.[47] Clearly, $X = [0, \frac{\delta}{2}] \times \{0\} \cup \{0\} \times [0, \frac{\delta}{2}] \subset Q^+$, so by openness of Q^+ and continuity of all the functions and their derivatives occuring in $N(\mathcal{D})$, $\exists \delta^* \in (\frac{\delta}{2}, \delta)$ such that

$$X^* = [0, \delta^*] \times \{0\} \cup \{0\} \times [0, \delta^*] \subset Q^+$$

and $N(X^*) < \infty$. In particular, eqns. (44),(45) hold in X^*, which shows that the finiteness of $N(X^*)$ implies the finiteness of $\tilde{N}(X^*)$.[48] We now are in a situation where the local existence result from prop. 8 applies, i.e. there exists a unique solution in $\mathcal{D}^* = [0, \delta^*] \times [0, \delta^*]$ which coincides with the previous solution in $\mathcal{D} \cap \mathcal{D}*$.



But $\mathcal{D}^*$ arose from initial data on a subset of Q^+ and hence by maximality of Q^+ (which comes from the generalisation of Choquet-Bruhat), we have $\mathcal{D}^* \subset Q^+$. Since $p \in \mathcal{D}^*$, this contradicts the assumption $p \in \overline{Q^+} \setminus Q^+$.

□

We can now prove the GEP for the EMKG system as specified in Theorem 4.1.

Theorem 4.2. *Let* $(U,V) = p \in \overline{Q^+}$*,* $(U',V') = q \in (I^-(p) \cap Q^+) \setminus \{p\}$ *s.t.* $\mathcal{D} := (J^-(p) \cap J^+(q)) \setminus \{p\} \subset Q^+$ *and s.t.*

$$W = \int_{V'}^{V} \int_{U'}^{U} \Omega^2 \, du dv < \infty \quad \text{and} \quad 0 < r_0 \leq r(p') \leq R < \infty \; \forall p' \in \mathcal{D} \tag{50}$$

Then $N(\mathcal{D}) < \infty$ *and* $p \in Q^+$*.*

[47] Then $p = (\frac{\delta}{2}, \frac{\delta}{2})$.
[48] The only norms within $\tilde{N}(X^*)$ where this is not immediately obvious are the second derivatives in $|A_u|_{2,v}, |\partial_v A_u|_{1,u}$. But these appear precisely on the LHS's of eqns. (44),(45), where the RHS's only contain bounded terms by the finiteness of $N(X^*)$. (Recall $Q_e = -2r^2\Omega^{-2}\partial_v A_u$) So these terms are bounded as well.

Proof. In view of prop. 8, showing that $N(\mathcal{D}) < \infty$ suffices to infer $p \in Q^+$.

Since $X = [U', U] \times \{V'\} \cup \{U'\} \times [V', V] \subset Q^+$ is compact and the solution as in Theorem 4.1 is sufficiently regular, we can give several finite bounds on $|r|, |\phi|, |A|, |\Omega|$ and derivatives of them, restricted to X. Using these bounds in combination with eq. (50) and the reduced system of equations (40)-(47), we can then show a priori integral estimates on several quantities in the region $D = \mathcal{D} \cup \{p\}$. The hardest part of the proof consists of finding a uniform bound for $|r\phi|$ in D.[49] Once this is done, one can systematically use eqns. (40)-(47) to show the other bounds.[50] I will only cover parts of the proof and will e.g. give only the bounds within X which we will make use of. For more details, the reader is referred to section 4.3 in Kommemi's paper.

So let's begin: One can easily show from its definition (eqns. (36),(37)) that $4T_{uv} = \Omega^2(\mathfrak{m}^2\phi\phi^\dagger + Q^2/(4\pi r^4))$. Eq. (40) then gives

$$0 \leq \int_{V'}^{V}\int_{U'}^{U} 4\pi r^2 T_{uv} du dv = \int_{V'}^{V}\int_{U'}^{U} \frac{1}{2}\partial_u\partial_v r^2 + \frac{1}{4}\Omega^2 du dv \leq (R^2 - r_0^2) + \frac{W}{4} = const.,$$

where the last step comes directly from eq. (50). From this, one can see that the same integral estimates hold for each of the summands of T_{uv} (they are both positive). Then, writing $F_{uv}^2 = \frac{\Omega^2 r^2 Q_e^2}{4} \cdot \frac{\Omega^2}{r^2}$ and using Cauchy-Schwarz one infers that a similar integral estimate holds for $|F_{uv}|(= |-\partial_v A_u|)$. Introducing now the bound $|A_u|_X \leq A$ we get

$$\int_{U'}^{U} \sup_{V' \leq v \leq V} |A_u|\, du \leq A(U - U') + \int_{U'}^{U}\int_{V'}^{V} |\partial_v A_u|\, dv du = const.$$

If we further introduce the bounds $|r\partial_v r|_X \leq N, |r\partial_u r|_X \leq \Lambda$ for some constants N, Λ, one can also show the boundedness of the integrals

$$\int_{U'}^{U} \sup_{V' \leq v \leq V} |r\partial_u r|\, du, \quad \int_{V'}^{V} \sup_{U' \leq u \leq U} |r\partial_v r|\, dv$$

We now use these a priori bounds to show the uniform boundedness of $r\phi$ in $\mathcal{D}$. For that, we introduce the final bound $|r\phi|_X \leq \Phi$. We will do this by partitioning $\mathcal{D}$ into a a set of smaller subregions: Given an $\epsilon > 0$, define a partition $\mathcal{D}_{jk} = ([u_j, u_{j+1}] \times [v_k, v_{k+1}]) \cap \mathcal{D}$ with $j, k = 0, ..., I(\epsilon)$ and $(u_0, v_0) = (U', V')$, $(u_{I+1}, v_{I+1}) = (U, V)$ such that, for all j, k

$$\int_{u_j}^{u_{j+}}\int_{v_k}^{v_{k+1}} \Omega^2 + |F_{uv}| + 4\pi r^2 T_{uv}\, dv du < \epsilon \tag{51}$$

$$\int_{v_k}^{v_{k+1}} \sup_{u_j \leq u \leq u_{j+1}} |r\partial_v r|\, dv < \epsilon$$

$$\int_{u_j}^{u_{j+}} \sup_{v_k \leq v \leq v_{k+1}} |r\partial_u r| + \sup_{v_k \leq v \leq v_{k+1}} |A_u|\, dv < \epsilon$$

This can be done because we just showed global a priori estimates on these integrals over $\mathcal{D}$. Clearly, for $j + k < 2I$, the quantities $P_{jk} = \sup_{\mathcal{D}_{jk}} |r\phi|$ are finite (in view of the

[49] The uniform boundedness of r is already given by eq. (50)

[50] In [13, 11], where only the WEP is proved, the proofs mainly rely on helpful properties of the Hawking mass m in $R \cup \mathcal{A}$. These are obviously not applicable for the GEP.

regularity of the solution in Q^+). We will show that P_{II} is also bounded by proving a recursive estimate on the P_{jk} for $j+k<2I$ and then using an argument similar to the arguments in the proof of the Penrose inequality (when we showed that $Y_{\cap\cap} = Y_\cap = Y$). First, re-write eq. (46) as

$$\begin{aligned}\partial_u\partial_v(r\phi) =&\phi\partial_u\partial_v r - \mathfrak{e} ir\Psi(A) - \frac{1}{4}\mathfrak{m}^2\Omega^2 r\phi \\ =&\phi\partial_u\partial_v r - \frac{1}{4}\mathfrak{m}^2\Omega^2 r\phi - \mathfrak{e} i\partial_v(A_u r\phi) - \frac{\mathfrak{e} i}{2} r\phi F_{uv}\end{aligned} \tag{52}$$

where we inserted the definitions of $\Psi(A), Q_e$ in the second step. Integrating this for $(u^*, v^*) \in \mathcal{D}_{jk}$ gives

$$(r\phi)(u^*, v^*) = \int_{u_j}^{u^*}\int_{v_k}^{v^*} (\phi\partial_u\partial_v r + ...)\, dvdu + (r\phi)(u^*, v_k) + (r\phi)(u_j, v^*) - (r\phi)(u_j, v_k) \tag{53}$$

The ... denotes the other three terms in eq. (52). To give a recursive estimate, we need bounds for the integral: Eq. (40) gives, using the triangle inequality and the expression for T_{uv}:

$$|\partial_u\partial_v r| \le \frac{1}{4}\Omega^2 r^{-1} + |r\partial_u r||r\partial_v r| r^{-3} + 4\pi r^2 T_{uv} r^{-1}$$

We infer

$$\left|\int_{u_j}^{u*}\int_{v_k}^{v*} (\phi\partial_u\partial_v r)\, dvdu\right| \le \int_{u_j}^{u*}\int_{v_k}^{v*} \frac{P_{jk}}{r_0}|\partial_u\partial_v r|\, dvdu \le P_{jk} r_0^{-1}(\frac{1}{4}\epsilon r_0^{-1} + r_0^{-3}\epsilon^2 + r_0^{-1}\epsilon)$$

where we used eq. (51). The other 3 terms in the integral in eq. (52) are also bounded by constants proportional to ϵP_{jk} which follows directly[51] from eq. (51). Since this and eq. (53) hold for all $(u^*, v^*) \in \mathcal{D}_{jk}$, we get

$$P_{jk} \le \mathcal{O}(\epsilon) P_{jk} + \sup_{[u_j,u_{j+1}]\times\{v_k\}} |r\phi| + \sup_{\{u_j\}\times[v_k,v_{k+1}]} |r\phi| + |(r\phi)(u_j, v_k)| \tag{54}$$

where the $\mathcal{O}(\epsilon)$ term denotes terms at least linear in ϵ and otherwise only depending on $\mathfrak{m}^2, \mathfrak{e}, r_0$. For $j+k<2I$, P_{jk} is finite: we can then choose ϵ small enough such that

$$P_{jk} \le \overbrace{\sup_{[u_j,u_{j+1}]\times\{v_k\}} |r\phi|}^{\le P_{j,k-1}} + \overbrace{\sup_{\{u_j\}\times[v_k,v_{k+1}]} |r\phi|}^{\le P_{j-1,k}} + 2|(r\phi)(u_j, v_k)| \le 2(P_{j,k-1} + P_{j-1,k})$$

Since I is finite, it follows by induction that all the P_{jk} for $j+k<2I$ are bounded by a constant depending only on the initial data and, more importantly, that for some finite constant[52] $C(\epsilon)$

$$\sup_{[u_I,u_{I+1}]\times\{v_I\}} |r\phi| + \sup_{\{u_I\}\times[v_I,v_{I+1}]} |r\phi| \le P_{I,I-1} + P_{I-1,I} \le C(\epsilon)\Phi \tag{55}$$

[51] The way I wrote eq. (52) makes this slightly easier compared to Kommemi's way.

[52] Recall $|r\phi|_X \le \Phi$

To now make a statement about P_{II}, we define

$$\mathcal{D}_{II\cap} = \{s \in \mathcal{D}_{II} : |(r\phi)(\bar{u}, \bar{v})| < 4C(\epsilon)\Phi \quad \forall(\bar{u}, \bar{v}) \in (J^-(s) \setminus \{s\}) \cap \mathcal{D}_{II}\}$$

The claim is that $\mathcal{D}_{II\cap} = \mathcal{D}_{II}$ which would establish the boundedness of P_{II}. Note that $\mathcal{D}_{II}$ is connected and $\mathcal{D}_{II\cap}$ is open and by continuity non-empty. To show closure, take eq. (54) and let $P_{II\cap} = \sup_{\mathcal{D}_{II\cap}} |r\phi| \leq 4C(\epsilon)\Phi$ (the bound comes from the definition of $P_{II\cap}$ and continuity). We get from eq. (54):

$$P_{II\cap} \leq \overbrace{\sup_{[u_I, u_{I+1}]\times\{v_I\}} |r\phi| + \sup_{\{u_I\}\times[v_I, v_{I+1}]} |r\phi|}^{\leq C(\epsilon)\Phi \text{ from eq.(55)}} + \overbrace{|(r\phi)(u_I, v_I)|}^{\leq C(\epsilon)\Phi} + \mathcal{O}(\epsilon) \cdot \overbrace{P_{II\cap}}^{\leq 4C(\epsilon)\Phi}$$

If we take ϵ small enough such that the $\mathcal{O}(\epsilon)$-term is smaller than 1/4, we obtain $P_{II\cap} \leq 3C(\epsilon)\Phi$. So $\mathcal{D}_{II\cap}$ does contain its limit points (w.r.t. $\mathcal{D}_{II}$) and we conclude $\mathcal{D}_{II} = \mathcal{D}_{II\cap}$. The result that $|r\phi|$ is bounded follows (since $P_{II\cap} = P_{II}$ is finite).

The bounds on the other quantities are now established by showing a bound on $|\int\int \left(D_u\phi(\partial_v\phi)^\dagger + \partial_v\phi(D_u\phi)^\dagger\right) dudv|$ and inferring the boundedness of $|\Omega|, |\Omega^{-1}|$ from eq. (41). The other bounds follow from straightforward calculations.

We thus get $N(\mathcal{D}) < \infty$ hence prop. 8 ensures $p \in Q^+$. This completes the proof.

□

This shows that Theorem 4.1 is true.

4.4 Conclusion and Outview

We established that the EMKG system belongs to the class of systems considered in section 3. A point we did not expand on for this discussion is the question: What are the least restrictive conditions under which the assumption of non-emptiness of $\mathcal{I}^+$can be guaranteed? Apart from that, the main difficulty was to establish the GEP for this system, which we showed to imply the WEP if the matter obeys the null energy condition.[53]

So in particular, we showed that if either the black hole region is non-empty, or $\sup_{\mathcal{H}} r < \infty$, then the WCCC holds true for the EMKG system. Remarkably, the results of this section rule out the possibility of creating naked singularities by "super-charging" nearly extremal black holes above extremality and thus destroying the event horizon. For the constructed scenarios all have $\mathcal{A}$ non-empty, in which case we know WCCC to hold true (and the event horizon cannot be destroyed in view of prop. 6).

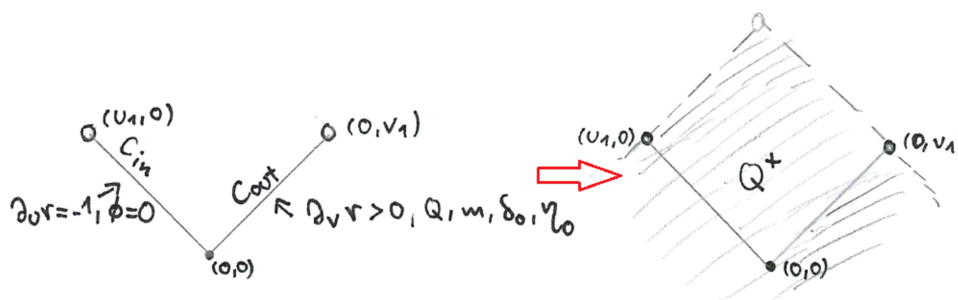
For the final resolution of WCCC for this system, the results of this section suggest to study conditions for trapped surface formation and perhaps establish generalisations of Christodoulou's trapped surface theorem for the massless Einstein-Klein-Gordon system. An alternative approach would be to show that generically, if no black hole regions form, then $\sup_{\mathcal{H}} r < \infty$. Work in both of these directions has been done by Kommemi in his PhD Thesis [19]. We will outline some of it in the next section.

[53]The GEP plays an important role in the discussion of SCCC for this system. It might thus be of interest to discuss when systems obeying the WEP also obey the GEP.

5 Trapped Surface formation for the EMKG system

Due to the additional repulsion coming from charge and mass of the scalar field, the analysis of trapped surface formation in the EMKG model can be expected to be more difficult than in Christodoulou's model. In this section, we present a result due to Kommemi [19] which establishes a criterion for trapped surface formation. The idea is motivated by Christodoulou's corresponding result in [3].

Kommemi considers initial data (at the level of Q^+) on a light cone $\mathcal{C}_{in} \cup \mathcal{C}_{out}$ where the (truncated) null rays are parametrised by $\mathcal{C}_{in} = [0, u_1)$, $\mathcal{C}_{out} = [0, v_1)$. Initally, we have two C^2-functions $r > 0, \phi$ with the properties that $\partial_v r|_{\mathcal{C}_{out}} > 0, \partial_u r|_{\mathcal{C}_{in}} = -1$ and $\phi|_{\mathcal{C}_{in}} = 0$. (So we just have Minkowski data on $\mathcal{C}_{in}$.)



He then defines[54] a mass function $m(0, v)$ (with $m(0, 0) = 0$) and a charge function $Q(0, v)$ in terms of r and ϕ and the two quantities

$$\delta_0(v) = \frac{r(0, v)}{r(0, 0)} - 1, \quad \eta_0(v) = \frac{2(m(0, v) - m(0, 0))}{r(0, v)}$$

The mass function will indicate the formation of trapped surfaces: If $\mu = \frac{2m}{r} = 1$, then a marginally trapped surface forms. The theorem he then shows is:

Theorem 5.1. *Let $\alpha \in (0, 1)$ and $\delta_0(v) \ll 1$ sufficiently small. If*[55]

$$\lim_{u \to u_1} r(u, 0) \sim r(0, v)\delta_0(v)^\alpha \quad \textit{and} \quad \eta_0(v) \sim \delta_0(v)^\alpha \log \delta_0(v) \tag{56}$$

Then there exists a unique (globally hyperbolic, spherically symmetric) ***black hole*** *spacetime $(M = Q^+ \times_r \mathbb{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ such that $\mathcal{C}_{in} \cup \mathcal{C}_{out} \subset Q^+$ embeds into Q^+ as*[56] *$D^+(\mathcal{C}_{in} \cup \mathcal{C}_{out}) \cap Q^+ = J^+(\mathcal{C}_{in} \cup \mathcal{C}_{out}) \cap Q^+$ and such that the functions r, ϕ, Q, m, where m denotes the Hawking mass on Q^+, restrict properly to $\mathcal{C}_{in} \cup \mathcal{C}_{out}$.*

The proof consists roughly of the following steps. First, Kommemi introduces $x(u, v) = \frac{r(u,v)}{r(0,v)}$ (so $x(0, v) = 1$) . He then assumes that no marginally trapped regions exist in

[54] For the details, refer to Kommemi's work [19].

[55] The notation: $x \sim y$ means there exist constants a, b s.t. $by < x < ay$.

[56] I assume that Kommemi meant in fact $D^+(\mathcal{C}_{in} \cup \mathcal{C}_{out}) = J^+(\mathcal{C}_{in} \cup \mathcal{C}_{out}) \cap Q^+$. However, if the boundary is spacelike, we know from the previous results (corollary 3.2) that there exists a (marginally) trapped region in $J^+(\mathcal{C}_{in} \cup \mathcal{C}_{out}) \cap Q^+$ anyway. We can therefore ignore this case and for the proof assume the boundary to be null.

the region $\mathcal{D} = [0, u_1) \times [0, v]$ for some $0 < v < v_1$, i.e. $\partial_v r|_{\mathcal{D}} > 0$. This and the first part of eq. (56) can be used to take $\lim_{u \to u_1} r(u, v) \geq \lim_{u \to u_1} r(u, 0) = \frac{1}{2}\delta_0^\alpha r(0, v)$, so $x(u_1, v) \geq \delta_0^\alpha/2$. With this result, he derives the following estimate on the evolution of μ:

$$\frac{d}{dx}(x\mu(x)) \leq \delta_0^\alpha/x \tag{57}$$

The proof of this estimate uses several bootstrap arguments (as in the final part of the proof of Theorem 4.2) and the reduced system of equations (40)-(47) as well as the fact that ϕ vanishes on $\mathcal{C}_{in}$. Now, he uses the second estimate from eq. (56): Assuming $(\eta_0(v) =)\mu(0, v) \geq -\alpha\delta_0^\alpha \log \delta_0 + \delta_0^\alpha$, integration of the equation above easily gives, for sufficiently small δ_0, that $\mu(x_{max}) \geq 1$ for some $x(0, v) = 1 > x_{max} \geq \delta_0^\alpha > x(u_1, v)$, a contradiction!

This shows that (marginally) trapped surfaces do form under certain circumstances for the EMKG system which in turn adds more weight to the previous results which showed that spacetimes with a marginally trapped surface obey the WCCC.

6 Conclusion

In this essay, we considered a large class of spherically symmetric, asymptotically flat initial data with one end, with the most restrictive assumption on it being the *weak extension principle*. For these systems, we showed in section 3 that completeness of future null infinity can be inferred from either the existence of a trapped region (or more generally, a black hole region) or, even if the black hole region is empty, from the boundedness of the area-radius function r on the event horizon $\mathcal{H}$. In fact, the main work that had to be done in the case of the black hole region being non-empty was establishing a *Penrose inequality* on $\mathcal{H}$, namely $\sup_{\mathcal{H}} r \leq 2M_f$, where M_f denotes the final Bondi mass. We showed that the results can also be applied to appropriate initial data with two ends.

To add weight to these results, we showed in section 4 that the Einstein-Maxwell-Klein-Gordon system belongs to the class of systems considered before. This model is of interest since it generalises the well-understood massless Einstein-Klein-Gordon system studied by Christodoulou and is in fact the simplest generalisation of it where the charge is not induced from non-trivial topology. Due to the similarity between charge and angular momentum, it is hoped that the understanding of the EMKG system with regards to the cosmic censorship conjectures will play an important role in understanding *non-spherically symmetric collapse*.

We showed that the EMKG system obeys the *generalised extension principle* which is stronger than the WEP. This in turn showed that the results from section 3 apply to the EMKG system.

Finally, we presented a criterion for the formation of trapped surfaces for the EMKG system in section 5.

The WCCC for this system thus depends on showing that solutions for it with an empty black hole region and $\sup_{\mathcal{H}} r = \infty$ are non-generic. A possible approach to this problem would be to show that for the EMKG system, similar results as for Christodoulou's chargeless and massless system in [4] hold, i.e. that the IVP for initial data of *bounded* *variation* is still well-posed, a similar *sharp extension* criterion holds and a trapped surface formation criterion stronger than the one formulated in section 5 holds. If this can be shown, work by Kommemi [19] will ensure that the WCCC is *generically true* for the EMKG system.

As a final remark, we note that establishing a trapped surface theorem stating that generically, either $\mathcal{A}$ is non-empty or the future development is geodesically complete would not only ensure that the WCCC holds: If one could, in addition to its non-emptiness, show that $\mathcal{A}$ has a limit point on b_Γ, the future limit point of the center of symmetry Γ, then the C^2-formulation of the SCCC would hold as well [20]!

References

[1] Enrico Barausse, Vitor Cardoso, and Gaurav Khanna. Test bodies and naked singularities: Is the self-force the cosmic censor? Phys. Rev. Lett., 105:261102, Dec 2010.

[2] Demetrios Christodoulou. Violation of cosmic censorship in the gravitational collapse of a dust cloud. Comm. Math. Phys., 93(2):171–195, 1984.

[3] Demetrios Christodoulou. The formation of black holes and singularities in spherically symmetric gravitational collapse. Communications on Pure and Applied Mathematics, 44(3):339–373, 1991.

[4] Demetrios Christodoulou. Bounded variation solutions of the spherically symmetric einstein-scalar field equations. Communications on Pure and Applied Mathematics, 46(8):1131–1220, 1993.

[5] Demetrios Christodoulou. Examples of naked singularity formation in the gravitational collapse of a scalar field. Annals of Mathematics, 140(3):607–653, 1994.

[6] Demetrios Christodoulou. Self-gravitating relativistic fluids: A two-phase model. Archive for Rational Mechanics and Analysis, 130:343–400, December 1995.

[7] Demetrios Christodoulou. The instability of naked singularities in the gravitational collapse of a scalar field. Annals of Mathematics, 149(1):183–217, 1999.

[8] Demetrios Christodoulou. On the global initial value problem and the issue of singularities. Classical and Quantum Gravity, 16(12A):A23, 1999.

[9] Demetrios Christodoulou. The formation of black holes in general relativity. EMS Publishing House, 2009.

[10] Mihalis Dafermos. The interior of charged black holes and the problem of uniqueness in general relativity. Communications on Pure and Applied Mathematics, 58(4):445–504.

[11] Mihalis Dafermos. On naked singularities and the collapse of self-gravitating higgs fields. Adv. Theor. Math. Phys., 9(4):575–591, 08 2005.

[12] Mihalis Dafermos. Spherically symmetric spacetimes with a trapped surface. Classical and Quantum Gravity, 22(11):2221, 2005.

[13] Mihalis Dafermos and Alan D. Rendall. An extension principle for the einstein-vlasov system in spherical symmetry. Annales Henri Poincaré, 6(6):1137–1155, Dec 2005.

[14] Mihalis Dafermos and Alan D. Rendall. Strong cosmic censorship for surface-symmetric cosmological spacetimes with collisionless matter. Communications on Pure and Applied Mathematics, 69(5):815–908, 2007.

[15] Stephen Hawking and Roger Penrose. The singularities of gravitational collapse and cosmology. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 314(1519):529–548, 1970.

[16] Thomas Hertog, Gary T. Horowitz, and Kengo Maeda. Update on cosmic censorship violation in ads. 2004.

[17] Shahar Hod. Weak cosmic censorship: As strong as ever. Phys. Rev. Lett., 100:121101, Mar 2008.

[18] Veronika E. Hubeny. Overcharging a black hole and cosmic censorship. Phys. Rev. D, 59:064013, Feb 1999.

[19] Jonathan Kommemi. The global structure of spherically symmetric charged scalar field spacetimes. PhD Thesis, 2013.

[20] Jonathan Kommemi. The global structure of spherically symmetric charged scalar field spacetimes. Communications in Mathematical Physics, 323(1):35–106, Oct 2013.

[21] Marc Mars. Present status of the penrose inequality. Classical and Quantum Gravity, 26(19):193001, 2009.

[22] Roger Penrose. Gravitational collapse and space-time singularities. Phys. Rev. Lett., 14:57–59, Jan 1965.

[23] Roger Penrose. Gravitational Collapse: the Role of General Relativity. Nuovo Cimento Rivista Serie, 1, 1969.

[24] A D Rendall. Cosmic censorship and the vlasov equation. Classical and Quantum Gravity, 9(8):L99, 1992.

[25] Maurício Richartz and Alberto Saa. Overspinning a nearly extreme black hole and the weak cosmic censorship conjecture. Phys. Rev. D, 78:081503, Oct 2008.

[26] Richard Schoen and S. T. Yau. The existence of a black hole due to condensation of matter. Communications in Mathematical Physics, 90(4):575–579, Dec 1983.