Horn Formulas as Linear Inequalities and Nonlocality Paradoxes

Tobias Fritz
tobias.fritz@icfo.es

OASIS seminar, Oxford Computing Laboratory, 21 Jan 2011

Abstract

A Horn formula can be interpreted as a system of linear inequalities over the Boolean semiring. Similarly, a union-closed set system is the Boolean version of a polytope. I will show how a resolution-like Fourier-Motzkin algorithm can be used to compute a Horn formula determining a union-closed set system from its generating sets and describe the output of some sample computations. In particular, one can use the software to calculate all Hardy-type nonlocality paradoxes as Bell inequalities over the Boolean semiring. Everything is work in progress, and I will mention the problem of redundancy elimination as a major obstacle towards further progress.

Disclaimer: This is work in progress and (hopefully) subject to improval. In particular, the presentation in this preliminary article is certainly far from optimal.

Bell scenarios and Hardy's nonlocality paradox. We begin with a thought experiment. Imagine two physicists, commonly taken to be *Alice* and *Bob*, in their respective labs, in, say, Cambridge and Oxford. They are connected via optical fibre cables to a photon source in Milton Keynes. They conduct many runs of the following kind of experiment: at a preassigned time, the photon source sends one photon each to Alice and Bob; each photon has two properties that can be measured: the colour, which may be red or blue, and the polarisation, which may be horizontal or vertical:

$$\text{colour} \in \{\text{red}, \text{blue}\}, \qquad \text{polarisation} \in \{\text{horizontal}, \text{vertical}\}$$

(Unfortunately, since a measurement destroys the photon, only one of these measurements can be conducted in each run.) Every time, Alice and Bob are free to choose which of these two properties they are going to measure. Such a kind of thought-experimental setting is known as a *Bell scenario*.

Now suppose that the following happens:

(a) When they both measure colour, it sometimes happens that they both obtain red.

(b) When one of them measures colour and the other measures polarisation, it never happens that they obtain red and horizontal together.

(c) When they both measure polarisation, they never obtain both vertical.

What do these properties imply about the colour and the polarisation of the photons emitted by the source? Well, by property (a) there is a subensemble of photon pairs where both photons in the pair are red. Consider this subensemble. By property (b), the polarisation of the pairs in this subensemble cannot be horizontal, therefore it has to be vertical. However, this is in contradiction with property (c)! Therefore, there is no experiment in which the properties (a), (b) and (c) all hold at the same time. Right?

No! It turns out that the kind of behaviour (a), (b) and (c) together is actually predicted to occur by quantum mechanics [2] and there are experiments where these quantum-mechanical predictions have been confirmed [3]. This fact is known as *Hardy's nonlocality paradox.* So why does quantum mechanics not lead to the logical contradiction described in the previous paragraph? As in every apparent paradox, that reasoning contains some hidden assumption which have not been made explicit. These assumptions are,

		c		p	
		r	b	r	b
c	r	1	?	0	?
	b	?	?	?	?
p	r	0	?	?	?
	v	?	?	?	0

Figure 1: The table of measurements and possible outcomes in Hardy's nonlocality paradox. An entry "1" means that this outcome does occur sometimes, while an entry "0" means that this outcome is known to never occur. The questions marks are unspecified entries. c = colour, p = polarisation, r = red = horizontal, b = blue = vertical.

- *Realism*: The quantities colour and polarisation do have definite values, even when not subjected to a measurement.
- *Locality*: The colour of Alice's photon does not depend on which measurement Bob conducts. In other words, Bob's measurement does not instantaneously change the colour of Alice's photon. Likewise for the colour of Bob's photon, and the polarisation of both photons.

Note that the concept of Locality makes sense only when assuming Realism. For in the definition of Locality, we speak about the colour of Alice's photon without mentioning any measurement conducted by Alice.

Hence in order to explain Hardy's nonlocality paradox, one has to acknowledge that quantum mechanics, and consequently also the world we live in, violates either Realism or Locality. Due to the presence of uncertainty relations, it seems that Realism is that which is violated by quantum mechanics. However it is possible to modify the theory such that all predictions are retained and Realism is satisfied; this is the case e.g. in Bohmian mechanics. The price one has to pay is the sacrifice of Locality.

Towards possibilistic Bell inequalities. Usually, what one considers in a Bell scenario is not merely which outcomes do sometimes occur and which outcomes never occur, but one considers the outcome *probabilities*. These evidently contain more information than a *possibility* table like figure 1. Then there are well-known criteria for when a conditional probability distribution coming from a Bell scenario is consistent with Realism and Locality; these criteria are known as *Bell inequalities*. They are linear inequalities in the entries of the probability table. Then a given conditional probability distribution is consistent with Realism and Locality if and only if it satisfies all Bell inequalities given for the Bell scenario under consideration. For each Bell scenario, these inequalities are the facets of the *Bell polytope*, and hence they can all be calculated at least in principle, and in particular there is only a finite number of them.

So far the well-known story. What we would like to do here is to transfer these ideas and results to the *possibilistic* setting where, as in Hardy's nonlocality paradox, we consider only possibilities instead of probabilities. We will find that logical paradoxes of the above type are the possibilistic analogue of Bell inequalities, and that there is actually a way in which these can be written as linear inequalities in the entries of a possibility table like figure 1. Before starting with this however, we need a small mathematical digression.

The Boolean semiring. Probabilities are positive real numbers that can be added, mulitplied and compared. The addition is relevant for computing the probability that one event or another one occurs, while the multiplication is important for determining the joint probability of independent events. Finally, the order relation is crucial for saying when some event occurs more frequently than another. So, a probability value is an element of the *ordered semiring*

$$\mathbb{R}_{\geq 0} \equiv ([0,\infty), +, \cdot, \leq) \ .$$

(A *semiring* is the same thing as a unital associative ring, except that we do not require the existence of additive inverses. Since this is a "ri*n*g without *n*egatives", such an algebraic structure is sometimes also dubbed a "*rig*". Concerning ordered semirings, we will not specify the exact compatibility which we would like to have between the ordering relation and the semiring structure, since at the present stage we only consider two concrete examples of semirings and no general theory.)

So now that we know what probabilities are, what are *possibilities*? Clearly, an event is either possible to occur, corresponding to a possibility value of 1, or impossible to occur, so that we assign to it a possibility value of 0. As for the probabilities, we will now find that the inpretation of 0 and 1 as (im-)possibility induces on $\{0,1\}$ naturally the structure of an ordered semiring. The addition operation should correspond to the possibility that some event or another event occurs; clearly, the total event is possible if and only if at least one of the subevents is possible; this forces us to define

$$0+0=0\,,\qquad 0+1=1\,,\qquad 1+0=1\,,\qquad 1+1=1\,.$$

This gives the set $\{0,1\}$ the structure of a commutative monoid. Since the idempotency relation $x+x=x$ holds in this commutative monoid, it is actually a *semilattice*. In particular, subtraction is strictly forbidden: the equation $y+x=z+x$ does not imply that $x=y$!

Now what about multiplication? The product of the possibility values for two events should be the possibility value that both events occur, given that these two events are in some sense independent. Clearly, two events appear jointly if and only if each event appears for itself; this forces us to take

$$0\cdot 0=0\,,\qquad 0\cdot 1=0\,,\qquad 1\cdot 0=0\,,\qquad 1\cdot 1=1\,.$$

Then it is easy to check that this commutative multiplication on $\{0,1\}$ distributes over the addition, so that the set $\{0,1\}$ now carries the structure of a commutative semiring.

Finally, we can also compare possibilities. If we want the order relation $\leq$ on $\{0,1\}$ capture when some event is at most as possible than another one, we necessarily need to set

$$0\leq 0\,,\qquad 0\leq 1\,,\qquad 1\not\leq 0\,,\qquad 1\leq 1\,.$$

Taking it all together, we now have another ordered semiring governing the possibility theory, which we will denote by $\mathbb{B}$:

$$\mathbb{B}\equiv(\{0,1\},+,\cdot,\leq)\ .$$

There are at least two intimately related contexts in which this semirings or occurs. Firstly, we can interpret "0" as "false" ($\bot$) and "1" as "true" ($\top$); then, the operation $+$ can be identified with logical or ($\vee$), while multiplication $\cdot$ becomes logical and ($\wedge$), while the order relation $\leq$ coincides with logical implication ($\rightarrow$). So we have the isomorphism

$$\mathbb{B}\cong(\{\bot,\top\},\vee,\wedge,\rightarrow)\ . \qquad (1)$$

While this is an interpretation of $\mathbb{B}$ in terms of logic and also the reason why $\mathbb{B}$ is known as the *Boolean semiring*, there is another interpretation of $\mathbb{B}$ in terms of combinatorics. In order to explain this interpretation, let $\{*\}$ be any one-element set. Then on the power set $\mathcal{P}(\{*\})=\{\emptyset,\{*\}\}$, we have the operations of intersection ($\cap$) and union ($\cup$) as well as the binary relation of set containment ($\subseteq$). Identifying $\emptyset\in\mathcal{P}(\{*\})$ with $0\in\mathbb{B}$ and $\{*\}\in\mathcal{P}(\{*\})$ with $1\in\mathbb{B}$, we obtain the isomorphism

$$\mathbb{B}\cong(\{\emptyset,\{*\}\},\cap,\cup,\subseteq)\ . \qquad (2)$$

Finally, as a curious aside, note that $\mathbb{B}$ can also be thought of as the initial element in the category of bounded lattices.

To conclude this paragraph, we have identified the underlying algebraic structure of possibility in terms of the Boolean semiring $\mathbb{B}$. The different models of the Boolean semiring suggests intimate relations to logic and to combinatorics; these will be fruitfully exploited in the following. For example, we can now define a *possibility measure* on a measurable space Ω in complete analogy to a probability measure by using the Kolmogorov axioms and changing the coefficients from $\mathbb{R}_{\geq 0}$ to $\mathbb{B}$; by the isomorphism (2), a possibility measure on Ω is the same thing as a (measurable) subset of Ω, namely the subset of possible outcomes.

Systems of inequalities over $\mathbb{B}$, anti-Horn formulas, and multicategories. Let us consider any inequality over $\mathbb{B}$,

$$\sum_{i\in L} y_i \leq \sum_{j\in R} y_j \ , \tag{3}$$

where $L, R \subseteq [n]$ are again just subsets of the set of variable indices. Going now back to the language of logic, this inequality states precisely

$$\bigvee_{i\in L} y_i \to \bigvee_{j\in R} y_j \ .$$

Now this propositional formula can in turn be equivalently rewritten as

$$\bigwedge_{i\in L} \left(y_i \to \bigvee_{j\in R} y_j \right) \ ,$$

so that we see that the inequality (3) is equivalent to the inequality system

$$y_i \leq \sum_{j\in R} y_j \quad \forall i \in L \ .$$

Logically, each such inequality $y_i \to \bigvee_{j\in R} y_j$ is an anti-Horn clause. Hence, a system of inequalities over $\mathbb{B}$ is precisely nothing but an *anti-Horn formula*.

Another reformulation of the concept of inequality systems over $\mathbb{B}$ uses the concept of *multicategories* [4]. The following is simpler to formulate in terms of the negated variables $z_i \equiv \neg y_i$. Then, the an inequality is given by a Horn clause

$$\bigwedge_{j\in R} z_j \to z_i \ . \tag{4}$$

We will now define a (symmetric) multicategory as follows: take $[n]$, the set of variable indices, as the set of objects. Any hom-set is taken to be either empty or a one-element set; From an ordered multiset of objects $R \subseteq [n]$ to an object $i \in [n]$, we define a morphism to exist iff the Horn clause (4) is logically implied by the given system of inequalities; this is supposed to mean that a violation of (4) in terms of $z_j = 1$ for all $j \in R$ and $z_i = 0$ is not consistent with the system of inequalities. Then it is a routine exercise to check that the multicategory axioms are satisfied. We may think of the given Horn formulas constituting the inequality system as a kind of generators of the multicategory.

Conversely, a finite symmetric multicategory in which each hom-set contains at most one element [4, 2.1.7] is very similar to the collection of Horn clauses implied by a Horn formula. In fact, such a multicategory arises from the previous construction precisely whenever the hom-sets of the form $C(a, a; a)$ are non-empty.

Checking Realism and Locality. So suppose that we are given a Bell scenario and a possibility distribution over the outcomes for each choice of pairs of measurements; more concretely, this would be specified by a *conditional possibility distribution* $\Pi(a, b|x, y)$. Here, x stands for the measurement setting of Alice (*colour* or *polarisation* in the above example), and likewise y for Bob's measurement setting, while a and b are the measurement outcomes of Alice and Bob, respectively. For each choice of x, y, a and b, the value $\Pi(a, b|x, y) \in \mathbb{B}$ specifies whether the joint outcome (a, b) is possible to occur given the joint measurement settings (x, y). The data $\Pi(a, b|x, y)$ can conveniently be visualized in a possibility table like figure 1, where each question mark should be replaced by a concrete value in $\mathbb{B}$.

Now the crucial question is, how is is possible to decide whether a given $\Pi(a, b|x, y)$ is consistent with Realism and Locality? This is certainly the case when each of the four relevant quantities (colour and polarisation of Alice's photon and Bob's photon) has a definite value which gets detected by the measurement. Mathematically, this would mean that there are some functions

$$f : \{c, p\} \longrightarrow \{r, b\}, \qquad g : \{c, p\} \longrightarrow \{r, b\} \tag{5}$$

such that Π has the form

$$\Pi(a,b|x,y) = \delta_{a,f(x)} \cdot \delta_{b,g(y)}$$

which is to be regarded as a formula over $\mathbb{B}$. For each choice of measurement, there is only one outcome which is possible, hence this outcome actually occurs with certainty.

Moreover, Realism and Locality also hold whenever there is a possibility distribution over these definite assignments of values; this would be the case whenever there is some statistical variation in the properties of the photon pairs emitted by the light source. Then the quantities of colour and polarisation would be uncertain in each run of the experiment, but still they would have "hidden" definite values; this would be classical statistical uncertainty, in contrast to quantum uncertainty. So we would like to assume the existence of a possibility distribution $\Lambda(f,g)$ over the set of functions f and g as in (5). Then we obtain

$$\Pi(a,b|x,y) = \sum_{f,g} \Lambda(f,g)\, \delta_{a,f(x)}\delta_{b,g(y)} \ . \tag{6}$$

Hence the question now reads as follows: how is it possible to decide, given some conditional possibility distribution $\Pi(a,b|x,y)$, whether it can be written in the form (6)? This is what we would like to answer in the following.

Union-closed set systems. In order to now move from possibility theory to combinatorics, we identity a conditional probability distribution $\Pi(a,b|x,y)$ with the set of tuples (a,b,x,y) for which $\Pi(a,b|x,y) = 1$. Then a deterministic model like (5) coinicdes with the set

$$\bigcup_{x,y} \{f(x), g(y), x, y\} \ . \tag{7}$$

Likewise, a model like 6 coincides with the bigger set

$$\bigcup_{f,g \text{ with } \Lambda(f,g)=1} \bigcup_{x,y} \{f(x), g(y), x, y\} \ .$$

In this language, the question is the following: given a certain set of tuples (a,b,x,y), when is it a union of sets of the form (7)? This is now a question amenable to immediate generalization:

Problem 1. Given a finite set $[n] \cong \{1,\ldots,n\}$ and subsets $x_1,\ldots,x_k \subseteq [n]$, when can a given subset $y \subseteq [n]$ be written as a union of the x_i?

Hence what we are studying is a a *union-closed set system*: all y which can be written as a union of the x_i form a family of subsets of $[n]$ which is closed under union.

We identify a subset of $[n]$ with its indicator function, which is an element of $\mathbb{B}^n$. Then the union of subsets is addition of indicator functions. We think of this as a $\mathbb{B}$-valued analogue of convex combinations of points in $\mathbb{R}^n$. This leads us to the following guiding principle:

Idea: Union-closed set systems are $\mathbb{B}$-valued versions of convex polytopes.

We will see in the following that this principle is quite powerful, and in particular lets us answer the question of the previous paragraph in complete analogy to the same question for conditional probability distribution, where the answer is that $P(a,b|x,y)$ is consistent with Realism and Locality iff all Bell inequalities are satisfied.

Fourier-Motzkin elimination for union-closed set systems. Using the arithmetic structure of $\mathbb{B}$ and again thinking of a subset $y \subseteq [n]$ in terms of its indicator function $y \in \mathbb{B}^n$, we reformulate problem 1 as follows:

$$\exists \lambda_1,\ldots,\lambda_k \in \mathbb{B} \ \text{ s.t. } \ y = \sum_{j=1}^{k} \lambda_j x_j \quad ? \tag{8}$$

Of course, here the equation $y = \sum_{i=1}^{k} \lambda_i x_i$ has to hold in each of the n components, i.e. $y_i = \sum_{j=1}^{k} \lambda_j x_{j,i}$ for all $i \in [n]$. We may also think of each equation $y_i = \sum_{j=1}^{k} \lambda_j x_{j,i}$ as the two inequalities $y_i \leq \sum_{j=1}^{k} \lambda_j x_{j,i}$ and $\sum_{j=1}^{k} \lambda_j x_{j,i} \leq y_i$; due to the curious addition in $\mathbb{B}$, or thinking of $+$ as logical or and $\leq$ as logical implication, the latter inequality is actually equivalent to $\lambda_j x_{j,i} \leq y_i$ for all j. Taking this all together, equation (8) gets reformulated as

$$\exists \lambda_1, \ldots, \lambda_k \in \mathbb{B} \text{ s.t. } \begin{pmatrix} y_i \leq \sum_{j=1}^{k} x_{j,i} \quad \forall i \\ \\ \lambda_j \leq y_i \quad \forall i, j \text{ with } x_{j,i} 1 \end{pmatrix}$$

Here, it makes most sense to think of the $x_{j,i}$ as simply some matrix with fixed coefficients in $\mathbb{B}$. The λ_j are variables that should be eliminated, while the y_i are variables that need to be kept. Geometrically, we are given a system of linear inequalities which defines a subset of $\mathbb{B}^{k+n}$, and we would like to determine a description of the projection of this subset onto $\mathbb{B}^n$ which we obtain by dropping the first k coordinates. In the case of $\mathbb{R}_{\geq 0}$-coefficients, this feat could be accomplished by Fourier-Motzkin elimination, which is a technique for combining the inequalities in the set such that the unwanted variables drop out and the resulting inequalities are necessary and sufficient conditions for containment in the projected set. We will now describe an analogous procedure which works in the present case.

Since the $\lambda_1, \ldots, \lambda_k$ can be eliminated one at a time, we will consider more generally a system of inequalities over $\mathbb{B}$ in which we want to eliminate exactly one variable, say λ, and we would like to keep the other variables, say $y_1, \ldots, y_n$. In terms of some coefficient matrices $A_{i,j} \in \mathbb{B}$, $B_{r,s} \in \mathbb{B}$ and $C_{v,t,u}$, this problem is of the following form:

$$\exists \lambda \in \mathbb{B} \text{ s.t. } \begin{pmatrix} y_i \leq \lambda + \sum_j A_{i,j} y_j \quad \forall i \\ \\ \lambda \leq \sum_s B_{r,s} y_s \quad \forall r \\ \\ y_t \leq \sum_u C_{v,t,u} y_u \quad \forall t, v \end{pmatrix} \tag{9}$$

We now claim that a solution for λ exists if and only if the system of linear inequalities

$$\begin{aligned} y_i &\leq \sum_s B_{r,s} y_s + \sum_j A_{i,j} y_j \quad \forall i, r \\ y_t &\leq \sum_u C_{v,t,u} y_u \quad \forall t, u, v \end{aligned} \tag{10}$$

is satisfied. This system has been obtained from (9) by the following procedure:

- Keep all the inequalities which do not contain λ.
- For each inequality which contains λ on the right-hand side (say, $y_i \leq \lambda + \sum_j A_{i,j} y_j$) and each inequality for which λ is the left-hand side (say, $\lambda \leq \sum_s B_{r,s} y_s$), replace the λ in the first inequality by the right-hand side of the second inequality.

The crucial step is to replacement of λ by combining the inequalities. This clearly yields a valid inequality; in logical terms, this corresponds to an application of the *cut rule*. That this step yields a valid inequality corresponds to the logical property of *soundness*. The slightly non-trivial part is checking that the resulting system is sufficient for the existence of such a λ (logical *completeness*). So we need to show that if $y_1, \ldots, y_k$ satisfies (10), then it also satisfies (9). In order to achieve this, we distinguish two cases: when $\min_r \sum_s B_{r,s} y_s = 0$, then the system (10) implies that $y_i \leq \sum_j A_{i,j} y_j$ holds for all i. In this case, setting $\lambda = 0$ provides a consistent solution to (9). The other case is $\sum_s B_{r,s} y_s = 1$ for all r. Then $\lambda = 1$ is a consistent solution to the system (9).

Eliminating redundancy. In general, the projected system (10) will be highly redundant. One simple redundancy check that has been implemented is to make sure that no inequality is trivially dominated by some other, in the sense that these two inequalities have the same left-hand side, while the variables occuring on the right-hand side of the dominating inequality is a subset of the variables occuring on the right-hand side of the dominated inequality.

However, redundancies still remain in the final output. One kind of redundancy comes from *equations* over $\mathbb{B}$: given the variables $y_1, \ldots, y_n$, an equation over $\mathbb{B}$ is a statement of the form

$$\sum_{i\in L} y_i = \sum_{i\in R} y_i \tag{11}$$

for some sets $L, R \subseteq [n]$. This equation is equivalent to the pair of inequalities obtained by replacing "=" by "$\leq$" and "$\geq$", respectively. This pair of inequalities in turn is equivalent to

$$y_i \leq \sum_{j\in R} y_j \quad \forall i \in L\,, \qquad y_i \leq \sum_{j\in L} y_j \quad \forall i \in R\,.$$

So when some system of linear inequalities contains all these, then in particular this system of inequalities has the equation (11) as a consequence.

Now given that a system of inequalities has (11) as a consequence, the set of variables appearing on the left-hand side of (11) are freely interchangeable with those on the right-hand side. In particular, when an inequality contains the set $L \subseteq [n]$ of variables on the right-hand side, it is equivalent to the inequality where these variables are replaced by the set of variables indexed by $R \subseteq [n]$. However in general, the output of the present algorithm will contain *both* of these inequalities and not detect the redundancy.

Actually, given a system of inequalities over $\mathbb{B}$, generating an irredundant subset is relatively simple. Whether some given inequality in the system is redundant or not can be efficiently checked using linear-time algorithms for the satisfiability problem of Horn formulae [5] (which is basically linear programming over $\mathbb{B}$). In this way eliminating redundant inequalities one by one, it is possible to create an irredundant subsystem. However, the problem with this is that this irredundant subset is not unique, it rather may depend on the order in which the inequalities are subjected to the redundancy check. There are examples where even the cardinality of an irredundant subset is not unique. Because of these problems, and general uneasiness about a lack of theoretical understanding of the redundancy issue, no redundancy elimination feature has been implemented as yet.

First results. The algorithm has been implemented, and some computational results for Bell scenarios with a low number of parties, measurement settings and measurement outcomes are available. More concretely, the computation has been feasible for the following triples of (number of parties, number measurement settings, number of measurement outcomes):

$$(2,2,2)\,; \quad \ldots \quad (2,2,6)\,; \quad (2,3,2)\,; \quad (3,2,2)\,.$$

In general, the output is difficult to interpret for a human mind due to the high level redundancy; e.g. for the $(2,3,2)$ case, the list of inequalities with a fixed variable on the left-hand side spans 7 pages. Surprisingly, no possibilistic Bell inequalities going beyond the Hardy paradox [2] and its generalisation to the ladder paradoxes [3] have so far been identified in the output of the calculations for the bipartite scenarios. Thanks to these computational results, we ahve also been able to *prove* that no such additional paradoxes exist at least in the $(2,2,n)$ case.

Potential other applications. Many structures in combinatorics involve union-closed set systems. For example, the collection of all Hamiltonian graphs on a given set of vertices is a union-closed set system. In the philosophy advocated in this work, such a combinatorial structure should be regarded as a polytope over $\mathbb{B}$, which can be either be expressed in terms of its "extreme points"—in this case, the collection of Hamiltonian cycles on the given set of vertices—or in terms of its "facets", which can be calculated using the Fourier-Motzkin elimination algorithm presented above (modulo the caveat of the redundancy issues mentioned previously).

Now we can also regard a graph $G = (V, E)$ as an assignment of $\mathbb{B}$-valued edge weights λ_e to the edges $e \in K(V)$ of the complete graph $K(V)$, where the edges in the graph are exactly those which carry the weight 0. (We will see in a second why we select those with weight 0 rather than those with weight 1.) Now we consider for each edge $x \in K(V)$ some variable $x_e \in \mathbb{B}$. We consider now the union-closed set system $\mathscr{H}(V)$ of Hamiltonian graphs over $K(V)$ and let the $x_\cdot$ vary over this set system. Then G has a Hamiltonian cycle if and only if

$$\min_{x_\cdot \in \mathscr{H}(V)} \sum_{e \in K(V)} \lambda_e x_e = 0 \tag{12}$$

which is an optimization of a ($\mathbb{B}$-)linear functional over the $\mathbb{B}$-polytope generaetd by Hamiltonian cycles!

Hence, we have formulated the problem of deciding whether a given graph has a Hamiltonian cycle as a $\mathbb{B}$-analogue of the travelling salesman problem! This observation may open the door towards a potential application of methods from combinatorial optimization to decision problems like Hamiltonicity of graphs. In particular, one might go and try to find ($\mathbb{B}$-)linear programming relaxations of the optimization problem (12) together with efficient separation methods for their "facets" in order to create a full-fledged $\mathbb{B}$-analogue of branch-and-cut-type algorithms [7]. Of course it is completely unclear how well an implementation of this idea would work in practice. It may strongly depend on the particular relaxation employed and the separation method used.

Relation to the hypergraph transversal problem. The material in this paragraph is essentially also contained in [8].

What the present version of the algorithm with its simple redundancy check does is that it generates all the *prime implicants* of the union-closed set system. Here, a prime implicant is an inequality

$$y_i \leq \sum_{j \in R} y_j$$

which is required to be valid have the property that no other inequality with the same left-hand side but strictly smaller right-hand side is also valid. The set of all prime implicants now is intimately related to so-called *hypergraph transversals.* Given a hypergraph $\Gamma = (V, E)$, with $E \subseteq \mathcal{P}(V)$, a Γ-*transversal* is defined to be a set of vertices $T \subseteq V$ such that every edge has at least one vertex in common with T, i.e.

$$T \cap e \neq \emptyset \quad \forall e \in E\,.$$

In order to get back to union-closed set systems now, it turns out to be more instructive to first look at the case of *upward closed set systems*, i.e. set systems which have the property that whenever some set $\subseteq [n]$ is contained in the system, then so is any bigger set $\subseteq [n]$. Obviously, every upward closed set system is automatically union-closed. Now if we regard some upward closed set system on $[n]$ as a hypergraph $\Gamma = ([n], E)$, then we claim that the elements of the set system $x \in E$ can be characterized as follows:

$$x \in E \iff x \cap T \neq \emptyset \quad \forall \text{ transversals } T \text{ of } \Gamma\,. \tag{13}$$

Well, clearly if T is a transversal, then $T \cap x \neq \emptyset$ for every $x \in E$ by the definition of a transversal. On the other hand, if $x \notin E$, then we claim that there is a transversal T on Γ which detects this in the sense that $T \cap x = \emptyset$. Clearly if such a T is to exist, then we may as well enlarge it by all elements not contained in x and obtain $T = [n] \setminus x$, while retaining $T \cap x = \emptyset$. It remains to check that this is a transversal. If there were some $x' \in E$ such that $([n] \setminus x) \cap x' = \emptyset$, this would mean that $x' \subseteq x$. However this we can exclude since $x' \in E$, $x \notin E$ and $x' \subseteq x$ contradict the assumption that E is upward closed. Morally, one may view the statement that the transversals separate E from its complement as a simple kind of $\mathbb{B}$-valued separation theorem of Hahn-Banach type.

An important class of transversals are *minimal transversals*, which are taken to be those transversals that are minimal with respect to inclusion. Clearly it is enough to check the condition on the right-hand side of (13) only for the minimal transversals. Therefore, calculating all minimal transversals is an important problem for hypergraphs. We also notice the following analogy to polytopes: viewing sets $x_1, \ldots, x_k \subseteq [n]$ as generators of an upward closed set system $E \subseteq \mathcal{P}([n])$, the minimal transversals are the "functionals" which characterize E and play the role of facets of the "polytope".

We now finally return to study union-closed set systems $E \subseteq [n]$ which are not necessarily upward closed. We call an inequality $y_i \leq \sum_{j\in R} y_j$, which is assumed to be valid for E, *minimal* whenever the right-hand side R cannot be decreased while retaining validity. Then the statement is the following:

Proposition 2. An inequality $y_i \leq \sum_{j\in R} y_j$ is valid for E if and only if the index set $R \subseteq [n] \setminus i$ is a transversal of the upward-closed set system generated by the hypergraph $\Gamma_i \equiv ([n] \setminus i, E_i)$ where $E_i = \{e \setminus \{i\} \mid e \in E, i \in e\}$ contains all those edges which contain i. The inequality is minimal if and only if the transversal is minimal.

Proof. If the inequality is valid for E, then it is in particular valid for every $e \in E$ with $i \in e$. Therefore, every such e needs to have non-empty intersection with R. Hence, R is a transversal of Γ_i. The converse implication works in essentially the same way.

Hence we have an order-preserving bijection between transversals of Γ_i and minimal inequalities with y_i as the left-hand side. The minimality statement is then clear since in both cases, we have defined minimality with respect to inclusion. □

In conclusion, the present Fourier-Motzkin elimination algorithm with its simple redundancy elimination does nothing else than calculate the minimal transversals of the hypergraphs Γ_i. It is likely that this can be done more efficiently using standard algorithms for the generation of minimal hypergraph transversals. On the other hand, once a more powerful redundancy elimination for the Fourier-Motzkin algorithm will have been found and implemented, then this reduction to the hypergraph transversal problem will cease to work: the set of inequalities then calculated should be significantly smaller than the collection of minimal hypergraph transversals for all the Γ_i's.

Fraenkl's union-closed set conjecture. An important open problem in combinatorics is the union-closed set conjecture due to Péter Frankl [6]. It asks the following:

> **Open problem:** Given a set system $X \subseteq \mathcal{P}([n])$ which is closed under union, is there always an $i \in [n]$ which is contained in at least half of the sets in the family X?

Since our "possibilistic Bell polytopes" are union-closed set systems, it should be interesting to see whether the union-closed set conjecture holds for these. A very interesting feature of these is that they have a high degree of symmetry: clearly for any two indices $i, j \in [n]$, there is a permutation of $[n]$, induced from a permutation from the measurement settings and the measurement outcomes, which maps i to j and preserves the union-closed set system. In particular, if the union-closed set conjecture holds for some $i \in [n]$, then it automatically holds for all $i \in [n]$. Hence, it is sufficient to check the following:

> **Question:** Given a fixed Bell scenario, is $\Pi(1, 1|1, 1)$ possible in at least half of the local realistic models?

Given that no counterexample to the union-closed set conjecture has been found since its introduction in 1979, one should expect the answer to be "yes". In any case, the possibilistic Bell polytopes might be an interesting family of test cases for this open problem.

References

[1] Samson Abramsky, Relational Hidden Variables and Non-Locality, `arXiv:1007.2754`

[2] Lucien Hardy. Nonlocality for two particles without inequalities for almost all entangled states. Phys. Rev. Lett., 71(11):16651668, Sep 1993.

[3] D. Boschi, S. Branca, F. De Martini, and L. Hardy. Ladder proof of nonlocality without inequalities: Theoretical and experimental results. *Phys. Rev. Lett.*, 79(15):27552758, Oct 1997.

[4] Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series 298, Cambridge University Press (2004).

[5] William F. Dowling, Jean H. Gallier, *Linear-time algorithms for testing the satisfiability of propositional horn formulae*, *J. Logic Programming* 1 (1984), no. 3, 267284.

[6] Péter Frankl, *Extremal set systems.* Handbook of combinatorics, Vols. 1, 2, 1293-1329, Elsevier, Amsterdam, 1995.

[7] John Mitchell, *Branch-and-cut algorithms for integer programming*, *Encyclopedia of Optimization*, Volume II, pages 519-525, Kluwer Academic Publishers, August 2001.

[8] Dimitris Kavvadias, Christos Papadimitriou, Martha Sideri, *On Horn envelopes and hypergraph transversals* (extended abstract). *Algorithms and computation* (Hong Kong, 1993), 399405, Lecture Notes in Comput. Sci., 762, Springer, Berlin, 1993.