Spherical Bernstein theorems for codimension 1 and 2

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In 1984, B. Solomon [16] has proven that if the Gauss map of a compact minimal hypersurface \( M^k \) of the sphere \( S^{k+1} \), with \( H^1(M) = 0 \), omits an neighborhood of a \( S^{k-1} \), then the Gauss map is constant and \( M \) totally geodesic in \( S^{k+1} \). We present an easier proof for his theorem using the methods developed in a previous work of the authors [2]. The intention of our proof is to exploit the geometry behind the condition \( H^1(M) = 0 \) on the level of universal covers. Later, we provide a similar result as the one of Solomon in the case of codimension 2.

**Keywords:** Bernstein theorem, minimal graph, harmonic map, Grassmannian, Gauss map, maximum principle

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1 Introduction

A cornerstone of the theory of minimal surfaces is Bernstein’s theorem, stating that the only entire minimal graphs in Euclidean 3-space are planes. In other words, if $f(x, y)$ is a smooth function defined on all of $\mathbb{R}^2$ whose graph in $\mathbb{R}^3$, $(x, y, f(x, y))$, is a minimal surface, then $f$ is a linear function and its graph is a plane.

Profound methods in analysis and geometric measure theory were developed to generalize Bernstein’s theorem to higher dimensions, culminating in the theorem of J. Simons [15] stating that an entire minimal graph has to be planar for dimension $d \leq 7$. This dimension constraint is optimal, as Bombieri, de Giorgi, and Giusti [3] constructed a counter-example to such an assertion in dimension 8 and higher. This reveals the subtlety and difficulty of the problem. Under the additional assumption that the slope of the graph is uniformly bounded, Moser [13] proved a Bernstein-type result in arbitrary dimension.

All the preceding results consider minimal hypersurfaces, that is, minimal graphs in Euclidean space of codimension 1. For higher codimension, the situation is more complicated. On one hand, Lawson-Osserman [12] have given explicit counterexamples to Bernstein-type results in higher codimension. Namely, the cone over a Hopf map is an entire Lipschitz solution to the minimal surface system. Since the slope of the graph of such a cone is bounded, even a Moser-type result for codimension higher than one cannot hold. In our previous work [2], we have proved instead that Moser’s Bernstein theorem holds true though for the case of codimension two, giving like this a sharp result on where the codimension issue starts being problematic (that is, bigger than 2).

S. Chern has introduced the Spherical Bernstein theorems. They concern compact $(n - 1)$-dimensional minimal submanifolds of the sphere $S^{n+m-1}$, and analogously to the previous case, the aim is to prove that they are totally geodesic (i.e. equatorial) subspheres when their normal planes do not change their directions too much.

Spherical Bernstein theorems are interesting on their own, but they find very strong applications in geometric analysis. Let us now recall the relation between spherical and the Euclidean Bernstein problem, and then state our results for the latter. Fleming’s [5] idea was that by re-scaling a nontrivial minimal graph in Euclidean space, one obtains a non flat minimal cone, and the intersection of that cone with the unit sphere then
is a compact minimal submanifold of the latter. Therefore, conditions ruling out the latter can be translated into conditions ruling out the former, that is, spherical Bernstein theorems can prove Euclidean Bernstein theorems. Because of the non compact nature of minimal graphs in Euclidean space, as an important technical ingredient, one needs to invoke Allard’s regularity theory [1].

We thank Mario Micallef for calling our attention to B. Solomon’s paper. The first author is in debt with Slava Matveev for very helpful discussions.

2 Preliminaries

Let \((M, g)\) and \((N, h)\) be Riemannian manifolds without boundaries. By Nash’s Theorem we have an isometric embedding \(N \hookrightarrow \mathbb{R}^L\).

**Definition 2.0.1.** A map \(\phi \in W^{1,2}(M, N)\) is called harmonic iff it is a critical point of the energy functional

\[
E(\phi) := \frac{1}{2} \int_M \|d\phi\|^2 dvol_g,
\]

where \(\|\cdot\|^2 = \langle \cdot, \cdot \rangle\) is the metric over the bundle \(T^*M \otimes \phi^{-1}TN\) induced by \(g\) and \(h\).

Recall that the Sobolev space \(W^{1,2}(M, N)\) is defined as:

\[
W^{1,2}(M, N) := \left\{ v : M \rightarrow \mathbb{R}^L ; \|v\|^2_{W^{1,2}(M)} = \int_M (|v|^2 + \|dv\|^2) \, dv_g < +\infty \text{ and } (2.2) \right\}.
\]

The Euler-Lagrange equations for the energy functional are:

\[
\tau(\phi) = 0, \quad (2.4)
\]

and \(\tau\) is called the tension field of the map \(\phi\).

In local coordinates

\[
e(\phi) = \|d\phi\|^2 = g^{ij} \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} h_{\beta\gamma}, \quad (2.5)
\]

\[
\tau(\phi) = \left( \Delta_g \phi^\alpha + g^{ij} \Gamma^\alpha_{\beta\gamma} \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} h_{\beta\gamma} \right) \frac{\partial}{\partial \phi^\alpha}, \quad (2.6)
\]

where \(\Gamma^\alpha_{\beta\gamma}\) denote the Christoffel symbols of \(N\).
Definition 2.0.2 (Gauss map). Let $M^p \hookrightarrow \mathbb{R}^n$ be a $p$-dimensional oriented submanifold in Euclidean space. For any $x \in M$, by parallel translation in $\mathbb{R}^n$, the tangent space $T_xM$ can be moved to the origin, obtaining a $p$-subspace of $\mathbb{R}^n$, i.e., a point in the oriented Grassmannian manifold $G^+_{p,n}$. This defines a map $\gamma : M \rightarrow G^+_{p,n}$ called the Gauss map of the embedding $M \hookrightarrow \mathbb{R}^n$.

Theorem 2.0.3 (Ruh-Vilms). Let $M$ be a submanifold in $\mathbb{R}^n$ and let $\gamma : M \rightarrow G^+_{p,n}$ be its Gauss map. Then $\gamma$ is harmonic if and only if $M$ has parallel mean curvature.

Harmonic maps have interesting geometric properties. By using Ruh-Vilms Theorem, one can try to find subsets $A \subset G^+_{p,n}$ for which there can be no non-constant harmonic map $\phi$ defined on some compact manifold $M$ with $\phi(M) \subset A$. In that regard, it is often useful to use the composition formula for $\phi : M \rightarrow N$, $\psi : N \rightarrow P$ where $(P,i)$ is another Riemannian manifold,

$$
\tau(\psi \circ \phi) = d\psi \circ \tau(\phi) + \text{tr} \nabla d\psi (d\phi, d\phi). \tag{2.7}
$$

When $\phi$ is harmonic, i.e. $\tau(\phi) = 0$, the formula is particularly useful. In particular if $P = \mathbb{R}$, and $\psi$ is a (strictly) convex function, then $\tau(\psi \circ \phi) \geq 0$ ($> 0$). That is, $\psi \circ \phi : M \rightarrow \mathbb{R}$ is a (strictly) subharmonic function on $M$. The maximum principle then implies the following proposition:

Proposition 2.0.4. Let $M$ be a compact manifold without boundary, $\phi : M \rightarrow N$ a harmonic map with $\phi(M) \subset V \subset N$. Assume that there exists a strictly convex function on $V$. Then $\phi$ is a constant map.

In our setting, since we deal with compact minimal submanifolds of spheres $M^k \subset S^n \subset \mathbb{R}^{n+1}$, we can still pursue the Gauss map

$$
g : M \rightarrow G^+_{k,n} \tag{2.8}
$$

and in the case of hypersurfaces of the sphere, the target becomes simply $S^{k+1}$

Then to obtain such a subset $A \subset G^+_{p,n}$, one tries to find a strictly convex function $f : A \rightarrow \mathbb{R}$. This strategy was used by Hildebrandt-Jost-Widman, Jost-Xin, Jost-Xin-Yang and others [6], [8], [9].

We follow our previous work [2] and instead of using strong analytical arguments to obtain a subset that admits a strictly convex function, we want to explore the geometry of regions that can contain the image of a non-constant harmonic map plus the topological consequences of the Gauss map omitting a set of codimension 2.
2.1 Non-existence of harmonic maps into subsets of spheres

Let us recall the following theorem.

**Theorem 2.1.1 (SMP).** Let \( \phi : (M, g) \rightarrow (N, h) \) be a non-constant harmonic map, where \( M \) is a compact Riemannian manifold, \( N \) is a complete Riemannian manifold, and \( S \subset N \) is a hypersurface with definite second fundamental form at a point \( y = \phi(x) \). Then no neighborhood of \( x \in M \) is mapped entirely to the concave side of \( S \).

A proof can be found in [14] and [4].

**Remark 2.1.2.** Take a geodesic ball \( B(p, r) \) in a complete manifold \( N \) such that \( r \) is smaller than the convexity radius of \( N \) at \( p \). Then \( \partial B(p, r) \) is a hypersurface of \( N \) with definite second fundamental form for every point \( q \in \partial B(p, r) \).

The main technical result of our previous work [2] can be seen as a corollary of Sampson’s maximum principle.

**Theorem 2.1.3.** Let \( (N, h) \) be a complete Riemannian manifold and \( \Gamma : [a, b] \rightarrow N \) a smooth embedded curve. Consider a smooth function \( r : [a, b] \rightarrow \mathbb{R}_+ \) and a region

\[
\mathcal{R} := \bigcup_{t \in [a, b]} B(\Gamma(t), r(t)),
\]

where \( B(\cdot, \cdot) \) is the geodesic ball and \( r(t) \) is smaller than the convexity radius of \( N \) for any \( t \). If, for each \( t_0 \in (a, b) \), the set \( \mathcal{R} \setminus B(\Gamma(t_0), r(t_0)) \) is the union of two disjoint connected sets, namely the connected component of \( \Gamma(a) \) and the one of \( \Gamma(b) \), then there exists no compact manifold \( (M, g) \) and non-constant harmonic map \( \phi : M \rightarrow N \) such that \( \phi(M) \subset \mathcal{R} \).

**Proof.** [2].

We have the following immediate consequences of Theorem 2.1.3.

**Corollary 2.1.4.** Let \( \mathcal{R} \) be a region on a complete Riemannian manifold \( (N, h) \) that admits a sweep-out \( \{S_t\} \) by convex hypersurfaces \( S_t \) with the following property:

(i) For each leave \( S_t \) of the sweep-out, \( \mathcal{R} \setminus S_t \) is the union of two disjoint connected sets.

Then, there is no closed embedded minimal hypersurface \( M \subset \mathcal{R} \).
These results have very interesting applications. As an illustration, let us revisit an example in [2] and in [10], but showing something slightly more general: In those papers, it was shown, with different techniques, that in $S^2 \setminus (S^1 / \sim)_{\epsilon > 0}$, where $\sim$ is the antipodal identification, there are no non-constant harmonic maps defined on a closed manifold.

**Example 2.1.5.** Let $p$ be a point in $(S^2, \dot{g})$, $\varphi(p)$ its antipodal point, and $\gamma : [0, 1] \to S^2$ a connected curve such that $\gamma(0) = p$, and $\gamma(1) = \varphi(p)$ (it is not necessary, but one can suppose that $\gamma([0, 1])$ is contained in the south hemisphere with respect to $p$ and $\varphi(p)$). For a given $\epsilon > 0$, define

$$
\mathcal{R} := \bigcup_{t \in S^1} (\partial B(\Gamma(t), \frac{\pi}{2} - \frac{\epsilon}{2}) / \sim),
$$

where $\Gamma(t)$ is the great circle such that $\langle \Gamma(t), p \rangle = 0$, for every $t \in S^1$. Although $\mathcal{R}$ is by definition sweep-out by convex hypersurfaces, $\mathcal{R}$ obviously does not satisfy (i) in Corollary 2.1.4. On the other hand, denoting by

$$
(\gamma([0, 1]))_{\epsilon > 0} := \{ x \in S^2 \mid d_g(x, \gamma([0, 1])) < \epsilon \},
$$

it is clear that $\mathcal{R} \setminus (\gamma([0, 1]))_{\epsilon} = S^2 \setminus (\gamma([0, 1]))_{\epsilon}$ does satisfy (i), and therefore there are no non-constant harmonic maps defined on closed manifolds with image in $S^2 \setminus (\gamma([0, 1]))_{\epsilon}$. See Figure 1. This obviously implies that there are no closed geodesics in that region as well.

The main idea of the proof of Theorem 2.1.3 is that $\partial \mathcal{R}$ is a barrier to the existence of non-constant harmonic maps. With the help of the maximum principle and the definition of $\mathcal{R}$ as a union of convex balls, we push the image of the harmonic map to this barrier. That is exactly what we do in the above example.

Note that the classical method used to prove non-existence of non-constant harmonic maps, that is, the method of looking for a strictly convex function $f : \mathcal{R} \to \mathbb{R}$ (in the
geodesic sense), is not very flexible. Once one changes the boundary of $\mathcal{R}$ slightly, one can no longer guarantee that there exists a strictly convex function $\tilde{f} : \tilde{\mathcal{R}} \to \mathbb{R}$.

**Example 2.1.6.** The argument in example 2.1.5 can be adapted to the case of $S^{k+1} \setminus (S^k(x_1)/\sim)_{\epsilon>0}$, where $x_1 \in S^{k+1}$ and $S^k(x_1) := \{p \in S^{k+1} | \langle p, x_1 \rangle = 0\}$ is a totally geodesic equatorial $S^k$. Using the region

$$\mathcal{R}^{k+2} := \bigcup_{t \in S^1} \left( \partial B \left( \Gamma^{k+2}(t), \frac{\pi}{2} - \frac{\epsilon}{2} \right) / \sim \right)$$

(2.11)
determined by the great circle $\Gamma^{k+2}(t)$ defined by $\Gamma^{k+2}(t) := \cos(t)x_1 + \sin(t)x_{k+2}$, where $\{x_1, x_2, \ldots, x_{k+2}\}$ is an orthonormal basis for $\mathbb{R}^{k+2}$, we have that $\mathcal{R}^{k+2} = S^{k+1} \setminus (S^{k-1}(x_1, x_{k+2}))_{\epsilon>0}$ (here, $S^{k-1}(x_1, x_{k+2})$ denotes a totally geodesic sphere orthogonal to $x_1$ and $x_{k+2}$). Therefore, $(S^k(x_1)/\sim)$ is clearly a barrier for the existence of non-constant harmonic maps.

Finding more flexible barriers like $\gamma([0, 1])_{\epsilon>0}$ in the case of higher dimensions can also be done with this method, although some extra care on the choice of the barrier is needed. Namely, we must understand the antipodal points $p, \varphi(p)$ in example 2.1.5 as a $S^0$ sphere, that is, a codimension two totally geodesic subset. More precisely, we have the following.

**Theorem 2.1.7.** Let $C^{k+1}$ be an open connected subset of $S^{k+1}$ such that there exists a totally geodesic embedding $: S^{k-1} \to S^{k+1}$, such that $\subset C$. In addition, suppose that $S^{k-1}$ is homotopic (in $C$) to a point $x \in C$. Then there is no closed Riemannian manifold $M$ and non-constant harmonic map $\phi : M \to S^{k+1}$ such that $\phi(M) \subset S^{k+1} \setminus C$.

**Remark 2.1.8** (A quasi-counterexample: The Clifford torus). The Clifford torus in $S^3$ decomposes the sphere into two regions that admit mean convex sweep-outs, for instance the equidistant one. By Hopf’s maximum principle, there cannot exist other minimal hypersurfaces in one of these two regions. In fact, it is known more generally that every two minimal hypersurfaces intersect on manifolds with positive Ricci curvature. On the other hand, these two regions have several closed geodesics; or in other words, many codimension two totally geodesic spheres. Those geodesic $S^1$ are not homotopic to a point inside the region they are contained.

### 3 Solomon’s Bernstein theorem

#### 3.1 Codimension 1

B. Solomon [16] has proven the following theorem.
Theorem 3.1.1 (Solomon). Let $M^k \subset S^{k+1}$ be a smooth, compact minimal hypersurface. If $H^1(M) = 0$ and $g(M) \subset S^{k+1} \setminus (S^{k-1})_\epsilon$, embedded as an equator, then $M$ is a totally geodesic hypersurface in $S^{k+1}$.

We present a simple proof for this using Theorem 2.1.3.

Proof. Since $M$ is minimal, its Gauss map $g : M \to S^{k+1}$ is a harmonic map. Let $\tilde{M}$ be the universal cover of $M$ and denote by $(S^{k+1} \setminus S^{k-1})$ the universal cover of $S^{k+1} \setminus S^{k-1}$. Let $\psi_1 : \tilde{M} \to M$ and $\psi_2 : (S^{k+1} \setminus S^{k-1}) \to (S^{k+1} \setminus S^{k-1})$ be the isometries given by the respective covering maps. Since $S^{k-1}$ has codimension two in $S^{k+1}$, its complement is not simple connected; in fact, $(S^{k+1} \setminus S^{k-1})$ is an infinite strip in $R^{k+1}$ with metric $\psi_2^*(\hat{\gamma})$, that is, the pull-back of the round metric $\hat{\gamma}$ by the covering map.

Since $H^1(M) = 0$, we have that $\Pi_1(g \circ \psi_1) = 0$, where $g \circ \psi_1 : \tilde{M} \to S^{k+1}$. This implies that the Gauss map $g$ lifts to the universal cover as a harmonic map $\tilde{\gamma} : \tilde{M} \to (S^{k+1} \setminus S^{k-1})$, and $\tilde{g}(\tilde{M})$ is a compact subset of $(S^{k+1} \setminus S^{k-1})$. But the latter admits a sweep-out by convex hypersurfaces, given by the lifting of the region in $S^{k-1} \setminus (S^{k-1})_\epsilon$ given by equation (2.11). The theorem now follows from Corollary 2.1.4.

3.2 Codimension 2

To study the case of codimension 2, we need a careful study of Grassmannian manifolds. We follow D. Hoffman and R. Osserman [7], S. Kozlov[11], and the work of the second author with Y. Xin [8].

The oriented Grassmanian $G^+_{2,k+2}$ (which is isometric to $G^+_{k,k+2}$) has a natural orientation induced from a complex structure that can be defined as follows: given an oriented 2-plane $P$ in $R^{k+2}$, let $v, w$ be orthonormal vectors spanning $P$ (with the correct orientation). Define

$$z = v + iw$$

and note that this complex vector assigns a point of $C^{k+2}$ to $P$. If one rotates $v$ and $w$ in $P$ by an angle $\theta$, we assign the complex vector $\exp i\theta$ to $P$. Therefore, each oriented 2-plane $P$ is assigned to a unique point in the complex projective space $C P^{k+1}$. From the fact $v$ and $w$ are orthonormal, we have that

$$\sum_{j=1}^{k+2} z_j^2 = 0 \quad (3.1)$$

where $z_j = v_j + iw_j$ for every $j \in \{1, ..., k + 2\}$. The above equation defines a quadric $Q_k \subset C P^{k+1}$.

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Moreover, the Fubini-Study metric on $\mathbb{C}P^{k+1}$ given by
\[
d s^2 = 2 \sum_{j<l} |z_jdz_l - z_ldz_j|^2 \\
\sum_{j=1}^{k+2} |z_j|^2
\]
gives an isometry between $(Q_k, ds^2|_{Q_k})$ and $G^+_2, k+2$ with the metric of homogenous space.

**Theorem 3.2.1** (Hoffman-Osserman). Let $H$ be the hyperplane in $\mathbb{C}P^{k+1}$ given by $H : z_1 - iz_2 = 0$. Then $Q_k^* := Q_k \setminus H$ there exist a biholomorphic map $\varphi : Q_k^* \longrightarrow \mathbb{C}^k$ given by
\[
(z_1, ..., z_{k+2}) = \left(\frac{z_1 - iz_2}{2} \left(1 - \sum_{j=1}^{k} \xi_j^2, i \left(1 + \sum_{j=1}^{k} \xi_j^2\right), 2\xi_1, ..., 2\xi_k\right) \right)
\]
where
\[
\xi_1 = \frac{z_3}{z_1 - iz_2}, ..., \xi_k = \frac{z_k + 2}{z_1 - iz_2}
\]
A proof can be found in D. Hoffman and R. Osserman [7].

Consider another hyperplane $H' : z_1 + iz_2 = 0$ in $\mathbb{C}P^{k+1}$. Obviously, $Q_k \cap (H \cup H') = \{(z_1, ..., z_{k+2}) \in Q_k | z_1^2 + z_2^2 = 0\}$ has codimension two, and therefore
\[
Q_k \setminus (H' \cup H) = \mathbb{C}^k \setminus \varphi(H')
\]
is not simply connected.

For the case $k = 2$, we have $Q_2 = G^+_2, 4$ and
\[
Q_k \setminus (H' \cup H) = S^2 \times S^2 \setminus [(S^0 \times S^2) \cup (S^2 \times S^0)].
\]
Defining
\[
\mathcal{R} := \bigcup_{t,s \in S^1} \partial B(\Gamma(t), \frac{\pi}{2} - \frac{\epsilon}{2}) \times \partial B(\Gamma(s), \frac{\pi}{2} - \frac{\epsilon}{2}),
\]
where $\partial B(\Gamma(t), \frac{\pi}{2} - \frac{\epsilon}{2})$ is given in example 2.1.5, we get the following.

Let $M^2 \subset S^4$ is a codimension two compact minimal $Q_2$ submanifold, $H^1(M) = 0$ and the gauss map $g : M \longrightarrow Q_2$ omits two hyperplanes like above, then $g$ is constant and $M$ a totally geodesic 2-sphere in $S^4$.

Since we assume $H^1(M) = 0$, we are basically assuming that $M$ is topologically $S^2$ to start with. But the harmonic gauss map $g : S^2 \longrightarrow S^2 \times S^2$ must omit a considerably large set in $S^2 \times S^2$. Therefore the question of when we can have that a minimal immersion of $S^2$ into $S^4$ is totally geodesic is resumed to the question that if we can find in the image of
the gauss map two antipodal points in each of the $S^2$ components of the Grassmannian $G_{2,k+2}^+$. To find convex sets in the general quadric $Q_k$, we need the following theorems by S. Kozlov and Y. Xin (together with the second author).

Let $(w, X) \in TG_{p,n}^+$ be an element of the tangent bundle and $\{\eta_{i\alpha}\}_{i=1,\ldots,p}$ a basis for $T_w G_{p,n}^+$ like above. There exist $m_i \in V_w^\perp$ such that

$$X = m_1 \wedge e_2 \wedge \ldots \wedge e_p + \ldots + e_1 \wedge \ldots \wedge e_{p-1} \wedge m_p. \quad (3.5)$$

These $m_i$ are not necessarily pairwise orthogonal.

**Theorem 3.2.2** (Kozlov). Let $w \in G_{p,n}^+$ and $X \in T_w G_{p,n}^+$, $X \not= 0$. Then there exists an orthonormal basis $\{e_i\}_{i=1}^p$ in $V_w$ and a system $\{m_i\}_{i=1}^r$, with $1 \leq r \leq \min\{p, q\}$, of non-zero pairwise orthogonal vectors in $V_w^\perp$, such that

$$w = e_1 \wedge \ldots \wedge e_p, \quad (3.6)$$

$$X = (m_1 \wedge e_2 \wedge \ldots \wedge e_r + \ldots + e_1 \wedge \ldots \wedge e_{r-1} \wedge m_r) \wedge (e_{r+1} \wedge \ldots \wedge e_p). \quad (3.7)$$

Let $w \in G_{p,n}^+$ and $X \in T_w G_{p,n}^+$ a unit tangent vector. We know that there exists an orthonormal basis $\{e_i, n_\alpha\}_{i=1,\ldots,p}^{\alpha=1,\ldots,q}$ of $\mathbb{R}^n$ and a number $r \leq \min(p, q)$ such that $w = \text{span}\{e_i\}$, $n = p + q$ and

$$X = (\lambda^1 n_1 \wedge e_2 \wedge \ldots \wedge e_r + \ldots + X e_1 \wedge \ldots \wedge e_{r-1} \wedge n_r) \wedge X_0, \quad (3.8)$$

and $|X| = (\sum_{n=1}^r |\lambda\alpha|^2)^{\frac{1}{2}} = 1$.

**Definition 3.2.3.** Let $X \in T_w G_{p,n}^+$ be given by Equation (3.8). Define

$$t_X := \frac{\pi}{2(|\lambda_\alpha| + |\lambda_\beta|)}, \quad (3.9)$$

where $|\lambda_\alpha| := \max\{\lambda^\alpha\}$ and $|\lambda_\beta| := \max\{\lambda^\alpha; \lambda^\alpha \neq \lambda_\alpha\}$.

**Definition 3.2.4.** Let $w \in G_{p,n}^+$ and $X \in T_w G_{p,n}^+$ a unit tangent vector like in Equation (3.8). Define

$$B_G(w) := \{w_X(t) \in G_{p,n}^+; 0 \leq t \leq t_X\}.$$ 

**Theorem 3.2.5** (Jost-Xin). The set $B_G(w)$ is a (geodesically) convex set and contains the largest geodesic ball centered at $w$. 

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Setting \( n = k + 2 \) in the above, we can define

\[
R := \left( \bigcup_{t \in S^1} \partial B_G(w_{X_1}(t)) \right).
\]  

(3.10)

where \( w_{X_1} \) is the geodesic with tangent vector given by \( X_1 = n_1 \wedge e_2 \wedge e_3 \wedge \ldots \wedge e_p \).

This region clearly gives a sweep-out by convex hypersurfaces of a set that contains none of the points \( w_{X_2}(\pm t_{X_2}) \), where \( X_2 = \frac{1}{\sqrt{2}}(n_1 \wedge e_2 + e_1 \wedge n_2) \wedge e_3 \wedge \ldots \wedge e_p \) (or the rotation of any other two tangent vector into normal ones). If we take the biholomorphism \( \varphi \) in theorem 3.2.1, we have that for an appropriate basis and two different vectors \( X_2 \) and \( \tilde{X}_2 \) (that is, two different rotations of basis vectors), \( \varphi^{-1}(w_{X_2}) = H \) and \( \varphi^{-1}(w_{X_2}) = H' \).

This construction gives us the following theorem.

**Theorem 3.2.6.** Let \( M^k \) be a codimension two compact minimal submanifold of \( S^{k+2} \) with \( H^1(M) = 0 \). Suppose that its gauss image is contained in the region \( R \) given by equation (3.10). Then \( g \) is constant and \( M \) a totally geodesic submanifold of \( S^{k+2} \).

**Proof.** By the above argument, \( R \) is contained in a region that is not simply connected in \( Q_k \). As in the proof of Theorem 3.1.1, the gauss map lifts to a map \( \tilde{g} : \tilde{M} \to Q_k \setminus (H' \cup H) \), and the image \( \tilde{g}(\tilde{M}) \) is compact in \( \tilde{R} \subset Q_k \setminus (H' \cup H) \). Now since we can lifting the convex sweep-out of \( R \) to \( \tilde{R} \), we get that \( (\tilde{g}) \) is constant. Therefore \( g \) is constant and \( M \) is a totally geodesic submanifold of \( S^{k+2} \). \( \square \)
References


[2] Renan Assimos and Jürgen Jost, *The geometry of maximum principles and a Bernstein theorem in codimension 2*. ↑1, 2, 4, 5, 6


