**Week 5 - Characters**

**Proposition 1.** Let \( \rho: G \to \text{GL}_n \) be a representation. Then the dual representation of \( \rho \) is isomorphic to the representation \( \psi: G \to \text{GL}_n, g \mapsto \rho(g)^{-1} \).

**Proof.** The dual representation of \( \rho \) is the homomorphism \( \rho^*: G \to \text{GL}(\mathbb{C}^n), g \mapsto (f \mapsto f \circ \rho(g)^{-1}) \). Let \( x_1, \ldots, x_n \) be the dual basis of the basis \( e_1, \ldots, e_n \) of \( \mathbb{C}^n \). Then the linear map

\[
L: \mathbb{C}^n \to (\mathbb{C}^n)^* \quad \quad (a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n
\]

is a linear isomorphism. We claim that this is also a \( G \)-linear map from the representation \( \psi \) to the representation \( \rho^* \). We know that for all \( i \in \{1, \ldots, n\} \) and \( g \in G \), there are \( b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{C} \) such that

\[
\psi(g)(e_i) = b_1e_1 + \ldots + b_ne_n, \quad \rho^*(g)(x_i) = c_1x_1 + \ldots + c_nx_n.
\]

To prove that the map \( L \) is \( G \)-linear, we need to prove that in this case \( b_1 = c_1, \ldots, b_n = c_n \). Write \( A = \rho(g)^{-1} \in \text{GL}_n \). Then

\[
\psi(g)(e_i) = A^T e_i = A_{i1}e_1 + \ldots + A_{in}e_n
\]

and

\[
\rho^*(g)(x_i) = (e_j \mapsto x_i(Ae_j)) = (e_j \mapsto x_i(A_{i1}, \ldots, A_{in})) = (e_j \mapsto A_{i1}x_1 + \ldots + A_{in}x_n)
\]

and so the coefficients are indeed the same. Hence \( L \) is \( G \)-linear and so an isomorphism of representations. \( \square \)

**Definition 2.** Let \( A \in \mathbb{C}^{n \times m}, B \in \mathbb{C}^{k \times \ell} \) be matrices. The **Kronecker product** of \( A, B \) is the matrices \( A \otimes B \in \mathbb{C}^{nk \times m\ell} \) defined by

\[
A \otimes B := \begin{pmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nm}B \end{pmatrix}
\]

**Lemma 3.**

1. We have \( I_n \otimes I_m = I_{nm} \).
2. Let \( A, A' \in \mathbb{C}^{n \times n}, B, B' \in \mathbb{C}^{m \times m} \) be matrices. Then \( (AA') \otimes (BB') = (A \otimes B)(A' \otimes B') \).

**Proposition 4.** Let \( \rho: G \to \text{GL}_n \) and \( \psi: H \to \text{GL}_m \) be representations. Then their tensor product is isomorphic to

\[
\varphi: G \times H \to \text{GL}_{nm} \quad (g, h) \mapsto \rho(g) \otimes \psi(h)
\]

**Proof.** Recall that the tensor product of \( \rho \) and \( \psi \) is the homomorphism

\[
\rho \otimes \psi: G \times H \to \text{GL}(\mathbb{C}^n \otimes \mathbb{C}^m)
\]

\[
(g, h) \mapsto \left( \sum_{i=1}^k \lambda_i \cdot v_i \otimes w_i \mapsto \sum_{i=1}^k \lambda_i \rho(g)(v_i) \otimes \psi(h)(w_i) \right)
\]

and that the linear map

\[
L: \mathbb{C}^n \otimes \mathbb{C}^m \to \mathbb{C}^{nm}
\]

\[
\sum_{i=1}^n \sum_{j=1}^m a_{ij} \cdot e_i \otimes e_j \mapsto (a_{11}, a_{12}, \ldots, a_{1m}, \ldots, a_{n1}, \ldots, a_{nm})
\]

is a linear isomorphism. This isomorphism is an fact \((G \times H)\)-linear and hence an isomorphism of representations. \( \square \)

**Proposition 5.**

1. Let \( V \) be a representation of \( G \). Then we have \( \chi_V(g) = \overline{\chi_V(g)} \) for all \( g \in G \).
2. Let \( V \) be a representation of \( G \) and \( W \) a representation of \( H \). Then we have \( \chi_{V \otimes W}(g, h) = \chi_V(g)\chi_W(h) \) for all \( g \in G \) and \( h \in H \).
3. Let \( V, W \) be representations of \( G \). Then we have \( \chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g) \) for all \( g \in G \).
Proof. (1) We may assume that $V = \mathbb{C}^n$ and that $V$ is unitary with respect to the standard inner product, so that the corresponding homomorphism is of the form $\rho: G \to U_n$. We know that the dual of $V$ is isomorphic to $\psi: G \to \text{GL}_n, g \mapsto \rho(g)^{-T}$. Since $\rho(g) \in U_n$, we have $\rho(g)^{-T} = \overline{\rho(g)}$. Hence $\chi_V(g) = \text{tr}(\rho(g)) = \text{tr}(\rho(g)) = \chi_V(g)$.

(2) We know that $V \otimes W$ is isomorphic to the representation $\rho \otimes \psi: G \times H \to \text{GL}_{nm}, (g,h) \mapsto \rho(g) \otimes \rho(h)$, where $\rho: G \to \text{GL}_n, \psi: H \to \text{GL}_m$ are isomorphic to $V,W$. Write $A = \rho(g), B = \psi(h)$. Then $\chi_{V \otimes W}(g,h) = \text{tr}(A \otimes B) = \text{tr}(A_{11}B) + \ldots + \text{tr}(A_{nn}B) = A_{11} \text{tr}(B) + \ldots + A_{nn} \text{tr}(B) = \text{tr}(A) \text{tr}(B) = \chi_V(g) \chi_W(h)$.

(3) Now assume that $G = H$. Then $V \otimes W$ is a representation of $G$ by setting $g \cdot T := (g,g) \cdot T$ for all $g \in G$ and $T \in V \otimes W$. We get $\chi_{V \otimes W}(g) = \chi_{V \otimes W}(g,g) = \chi_V(g) \chi_W(g)$. □