This week, we cover Section 3.1 from the book together with the following.

**Definition 1.** Let $V$ be a representation of a group $G$. A subrepresentation of $V$ is a $G$-invariant subspace $W$ of $V$. A subrepresentation is itself a representation, where the map $G \times W \to W$ is the restriction of the map $G \times V \to V$ to $G \times W$. ♦

**Definition 2.** Let $V$ be a subrepresentation of a representation $V$ of $G$. Then the map
\[
G \times V/W \to V/W \\
(g, v + W) \mapsto g \cdot v + W
\]
turns $V/W$ into a representation of $G$. This representation is called the quotient of $V$ by $W$. ♦

**Definition 3.** Let $V, W$ be representations of a group $G$. A $G$-linear map $f : V \to W$ that is
- linear, so $\varphi(\lambda v + \mu v') = \lambda \varphi(v) + \mu \varphi(v')$ for all $\lambda, \mu \in \mathbb{C}$ and $v, v' \in V$.
- $G$-equivariant, this means that $\varphi(g \cdot v) = g \cdot \varphi(v)$ for all $g \in G$ and $v \in V$.

**Proposition 4.** Let $\varphi : V \to W$ be a $G$-linear map. Then $\ker(\varphi)$ is a subrepresentation of $V$ and $\text{im}(\varphi)$ is a subrepresentation of $W$.

**Proof.** The set $\ker(\varphi)$ is a subspace of $V$. For all $g \in G$ and $v \in \ker(\varphi)$, we have $\varphi(g \cdot v) = g \cdot \varphi(v) = g \cdot 0 = 0$ and so $g \cdot v \in \ker(\varphi)$. So $\ker(\varphi)$ is also $G$-invariant. So it is a subrepresentation of $V$. The set $\text{im}(\varphi)$ is a subspace of $W$. For all $g \in G$ and $w \in \text{im}(\varphi)$, we have $g \cdot w = g \cdot \varphi(v) = \varphi(g \cdot v) \in \text{im}(\varphi)$ for some $v \in V$. So $\text{im}(\varphi)$ is also $G$-invariant. So it is a subrepresentation of $W$. □

**Example 5.** Let $V_{\text{triv}}$ be the representation of $S_n$ on $\mathbb{C}$ defined by
\[
\sigma \cdot x = x
\]
for all $\sigma \in S_n$ and $x \in \mathbb{C}$. Let $V_{\text{alt}}$ be the representation of $S_n$ on $\mathbb{C}$ defined by
\[
\sigma \cdot x = \text{sign}(\sigma) x
\]
for all $\sigma \in S_n$ and $x \in \mathbb{C}$. Let $V_{\text{std}}$ be the representation of $S_n$ on $\mathbb{C}^n$ defined by
\[
\sigma \cdot (v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})
\]
for all $\sigma \in S_n$ and $(v_1, \ldots, v_n) \in \mathbb{C}^n$.
- (1) The only $G$-linear map $V_{\text{triv}} \to V_{\text{alt}}$ is the map $0$.
- (2) The map $V_{\text{triv}} \to V_{\text{std}}, x \mapsto (x, \ldots, x)$ is $G$-linear.
- (3) The map $V_{\text{std}} \to V_{\text{triv}}, (v_1, \ldots, v_n) \mapsto v_1 + \ldots + v_n$ is $G$-linear. ♠

**Definition 6.** Let $f : H \to G$ be a homomorphism and let $V$ be a representation of $G$. Then the map
\[
H \times V \to V \\
(h, v) \mapsto f(h) \cdot v
\]
turns $V$ into a representation of $H$. ♦

**Definition 7.** Let $V$ be a representation of $G$ and $W$ a representation of $H$. Then the map
\[
(G \times H) \times (V \otimes W) \to V \otimes W \\
(g, h, \sum_{i=1}^{k} \lambda_i \cdot (v_i \otimes w_i)) \mapsto \sum_{i=1}^{k} \lambda_i \cdot (g \cdot v_i \otimes h \cdot w_i)
\]
turns $V \otimes W$ into a representation of $G \times H$. When $G = H$, we can use the homomorphism
\[
G \to G \times G \\
g \mapsto (g, g)
\]
to get a representation of $G$. ♦
The goal of this course is to get the same results for any finite group $G$ in a unique way. Namely $e \cdot v = v$ for all $v \in V$. Any subspace of $V$ is a subrepresentation and any linear map is also $G$-linear. So we see that a representation is irreducible if and only if it has dimension 1. Hence, every irreducible representation is isomorphic to $\mathbb{C}$. We have the following fundamental result:

1. Every finite-dimensional vector space is isomorphic to $\mathbb{C}^n$ for some unique $n \in \mathbb{Z}_{\geq 0}$.
2. Every linear map $\mathbb{C}^n \to \mathbb{C}^m$ is of the form
   \[ v \mapsto Av \]
   for some matrix $A \in \mathbb{C}^{m \times n}$. Conversely, for all matrices $A \in \mathbb{C}^{n \times m}$, the map
   \[ \mathbb{C}^n \to \mathbb{C}^m \]
   \[ v \mapsto Av \]
   is linear.
3. Let $v_1, \ldots, v_n \in V$ be a basis of $V$. Then the map
   \[ \mathbb{C}^n \to V \]
   \[ (x_1, \ldots, x_n) \mapsto x_1v_1 + \ldots + x_nv_n \]
   is an isomorphism.

Example 8. Let us study the case where $G = \{e\}$. In this case, every vector space $V$ is a representation of $G$ in a unique way. Namely $e \cdot v = v$ for all $v \in V$. Any subspace of $V$ is a subrepresentation and any linear map is also $G$-linear. So we see that a representation is irreducible if and only if it has dimension 1. Hence, every irreducible representation is isomorphic to $\mathbb{C}$. We have the following fundamental result:

1. Every finite-dimensional vector space is isomorphic to $\mathbb{C}^n$ for some unique $n \in \mathbb{Z}_{\geq 0}$.
2. Every linear map $\mathbb{C}^n \to \mathbb{C}^m$ is of the form
   \[ v \mapsto A v \]
   for some matrix $A \in \mathbb{C}^{m \times n}$. Conversely, for all matrices $A \in \mathbb{C}^{n \times m}$, the map
   \[ \mathbb{C}^n \to \mathbb{C}^m \]
   \[ v \mapsto A v \]
   is linear.
3. Let $v_1, \ldots, v_n \in V$ be a basis of $V$. Then the map
   \[ \mathbb{C}^n \to V \]
   \[ (x_1, \ldots, x_n) \mapsto x_1v_1 + \ldots + x_nv_n \]
   is an isomorphism.

The goal of this course is to get the same results for any finite group $G$.

1. We want to find representations $V_1, \ldots, V_k$ such that every representation of $G$ is isomorphic to
   \[ V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \]
   for some unique $a_1, \ldots, a_k \geq 0$.
2. We want to understand the $G$-linear maps between representations of this form.
3. Given a representation $V$, we want to be able to determine $a_1, \ldots, a_k$ such that
   \[ V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \]
   (and maybe even write down the isomorphism).

Let’s start with (2).

Lemma 9 (Schur’s Lemma).

1. Let $V, W$ be irreducible representations of a group $G$ and let $\varphi: V \to W$ be a $G$-linear map. Then either $\varphi = 0$ or $\varphi$ is an isomorphism.
2. Let $V$ be a finite-dimensional irreducible representation of a group $G$. Then every $G$-linear map $V \to V$ is of the form $v \mapsto \lambda v$ for some $\lambda \in \mathbb{C}$ (and every map of this form is $G$-linear).

Proof. (1) Let $\varphi: V \to W$ be a $G$-linear map. Then $\ker(\varphi)$ is a subrepresentation of $V$. Since $V$ is irreducible, either $\ker(\varphi) = V$ or $\ker(\varphi) = 0$. The first case, we have $\varphi = 0$ and we are done. Assume the second case. Next note that $\im(\varphi)$ is a subrepresentation of $W$. Since $W$ is irreducible, either $\im(\varphi) = 0$ or $\im(\varphi) = W$. In the first case, we have $\varphi = 0$ and we are done. Assume that second case. Now $\varphi$ is both injective and surjective. So it is an isomorphism.

(2) Let $\varphi: V \to V$ be a $G$-linear map. Then $\varphi$ has an eigenvalue $\lambda \in V$ with corresponding eigenvector $w \in V \setminus \{0\}$. Define $\psi: V \to V$ by $\psi(v) := \varphi(v) - \lambda v$ for all $v \in V$. One can check that $\psi$ is a $G$-linear map. So by part (1), we know that $\psi = 0$ or $\psi$ is an isomorphism. Note that $\psi(w) = \varphi(w) - \lambda w = 0$. So $\psi$ cannot be an isomorphism. So $\psi = 0$. So $\varphi(v) = \lambda v$ for all $v \in V$. \qed

We get the following statement:

Proposition 10. Let $G$ be a group and let $V_1, \ldots, V_k$ be irreducible representations of $G$ such that $V_i \not\cong V_j$ for all $i \neq j$. Let $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Z}_{\geq 0}$. Then every $G$-linear map
   \[ \varphi: V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \to V_1^{\oplus b_1} \oplus \cdots \oplus V_k^{\oplus b_k} \]
   is of the form
   \[ (v^{(1)}, \ldots, v^{(k)}) \mapsto (A^{(1)}v^{(1)}, \ldots, A^{(k)}v^{(k)}) \]
(and any map of this form is $G$-linear). Here $A^{(i)} \in \mathbb{C}^{b_i \times a_i}$, $v^{(i)} \in V_i^{\oplus a_i}$ and

$$A^{(i)}v^{(i)} := (A^{(i)}_{11}v_{1}^{(i)} + \ldots + A^{(i)}_{1a_i}v_{a_i}^{(i)}, \ldots, A^{(i)}_{a_1i}v_{1}^{(i)} + \ldots + A^{(i)}_{a_ia_i}v_{a_i}^{(i)})$$

Proof. To prove this, we need to show that there exist $\lambda_1, \ldots, \lambda_{b_i} \in \mathbb{C}$ such that

$$\varphi(0, \ldots, 0, v, \ldots, 0, \ldots, 0) = (0, \ldots, 0, \lambda_1v, \ldots, \lambda_{b_i}v, \ldots, 0)$$

for all $i, j$ and $v \in V_i$. Here $(0, \ldots, v, \ldots, 0)$ is at position $i$ and contains $v$ at position $j$. Note that the map

$$V_i \rightarrow V_1^{\oplus b_1} \oplus \ldots \oplus V_k^{\oplus b_k}$$

$$v \mapsto \varphi(0, \ldots, 0, v, \ldots, 0, \ldots, 0)$$

is $G$-linear. The projection on a $V_k$ is also $G$-linear. So the composition $V_i \rightarrow V_k$ is as well. If $k \neq i$, then this composition is 0. If $k = i$, this composition is $V_i \rightarrow V_i, v \mapsto \lambda v$ for some $\lambda \in \mathbb{C}$. This gives us what we want. □

Corollary 11. Let

$$\varphi : V_1^{\oplus a_1} \oplus \ldots \oplus V_k^{\oplus a_k} \rightarrow V_1^{\oplus b_1} \oplus \ldots \oplus V_k^{\oplus b_k}$$

be a $G$-linear map and let $A_1, \ldots, A_k$ be the associated matrices. Then the following hold:

1. $\ker(\varphi) \cong V_1^{\oplus \dim(\ker A_1)} \oplus \ldots \oplus V_k^{\oplus \dim(\ker A_k)}$
2. $\im(\varphi) \cong V_1^{\oplus \dim(\im A_1)} \oplus \ldots \oplus V_k^{\oplus \dim(\im A_k)}$
3. The map $\varphi$ is an isomorphism if and only if $A_1, \ldots, A_k$ are all invertible.