Exercises 3

1. Let $V_1, \ldots, V_k$ be finite-dimensional representations of $G$. Let $a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_k \in \mathbb{Z}_{\geq 0}$. Consider the following $G$-linear maps

$$\varphi: V_1^\oplus a_1 \oplus \cdots \oplus V_k^\oplus a_k \to V_1^\oplus b_1 \oplus \cdots \oplus V_k^\oplus b_k, \quad \psi: V_1^\oplus b_1 \oplus \cdots \oplus V_k^\oplus b_k \to V_1^\oplus c_1 \oplus \cdots \oplus V_k^\oplus c_k$$

$$(v_1, \ldots, v_k) \mapsto (A_1 v_1, \ldots, A_k v_k) \quad (w_1, \ldots, w_k) \mapsto (B_1 w_1, \ldots, B_k w_k)$$

where $A_i \in \mathbb{C}_{b_i \times a_i}$ and $B_i \in \mathbb{C}_{c_i \times a_i}$. Prove the following:

(a) The composition $\psi \circ \varphi: V_1^\oplus a_1 \oplus \cdots \oplus V_k^\oplus a_k \to V_1^\oplus c_1 \oplus \cdots \oplus V_k^\oplus c_k$ is given by

$$(v_1, \ldots, v_k) \mapsto ((B_1 A_1) v_1, \ldots, (B_k A_k) v_k)$$

Hint: First consider the case where $V_i = \mathbb{C}$ is the trivial representation for all $i$. The proof of the general case is the same.

(b) The map $\varphi$ is the identity map if and only if $a_i = b_i$ and $A_i = I_{a_i}$ for all $i$.

(c) The map $\varphi$ is invertible if and only if $a_i = b_i$ and $\det(A_i) \neq 0$ for all $i$.

(d) The kernel of $\varphi$ is isomorphic to $V_1^\oplus \dim \ker(A_1) \oplus \cdots \oplus V_k^\oplus \dim \ker(A_k)$.

(e) The image of $\varphi$ is isomorphic to $V_1^\oplus \text{rk}(A_1) \oplus \cdots \oplus V_k^\oplus \text{rk}(A_k)$.

2. Exercise 3.5 from the book.

3. Consider the representations $\mathbb{C}^n$ of $\text{GL}_n$ and $\mathbb{C}^m$ of $\text{GL}_m$ both defined by $P \cdot v := Pv$. Prove that the representations $\mathbb{C}^n \otimes \mathbb{C}^m$ of $\text{GL}_n \times \text{GL}_m$ is isomorphic to the representation $\mathbb{C}^{n \times m}$ of $\text{GL}_n \times \text{GL}_m$ defined by $(P, Q) \cdot A := PAQ^\top$.

4. Let $V, W$ be isomorphic finite-dimensional representations of $G$. Prove the following:

(a) If $V$ is irreducible, then so is $W$.

(b) If $V$ is decomposable, then so is $W$.

(c) If $V$ is completely reducible, then so is $W$.

5. Let $V$ be a finite-dimensional representation of $G$. We define a new representation $V'$ of $G$ as follows: as a vector space we take $V' := V$ and action of $G$ on $V'$ is defined by $g \ast v := g^{-\top} \cdot v$. In other words, the new representation is defined by the homomorphism $(g \mapsto g \ast v) = (g \mapsto (v \mapsto g \cdot v)) \circ (g \mapsto g^{-\top})$. Prove the following:

(a) $V'$ is a representation of $G$.

(b) We have $(V')' = V$.

(c) If $V$ is irreducible/decomposable/completely decomposable, then $V'$ is as well.

(d) Find an example where $V \not\cong V'$. 

1