Exercises 1

1. Let $G$ be a finite group. Then $\mathbb{C}[G]$ is a $\mathbb{C}$-algebra we define as follows:
   - For every $g \in G$, we let $[g]$ be a new symbol.
   - We define $\mathbb{C}[G]$ as the vector space with basis $\{[g] \mid g \in G\}$.
   - We define the multiplication operation:
     $$ - \cdot - : \mathbb{C}[G] \times \mathbb{C}[G] \rightarrow \mathbb{C}[G] $$
     $$ \left( \sum_{g \in G} c_g \cdot [g], \sum_{h \in G} b_h \cdot [h] \right) \mapsto \sum_{g,h \in G} c_g b_h \cdot [gh] $$
   - Warning: When $G$ is abelian this means that $[x] + [y] \neq [x + y] = [x] \cdot [y]$.

   That $\mathbb{C}[G]$ is a $\mathbb{C}$-algebra means that the following axioms are satisfied:
   1. The map $- \cdot -$ is bilinear.
   2. We have the element $[e]$ with $[e] \cdot x = x$ for all $x \in \mathbb{C}[G]$.
   3. We have $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \mathbb{C}[G]$.

   A $\mathbb{C}[G]$-module is a vector space $V$ together with a map
   $$ \mathbb{C}[G] \times V \rightarrow V $$
   $$(x, v) \mapsto x \cdot v $$
   such that the following hold:
   - We have $x \cdot (\lambda v + \mu w) = \lambda (x \cdot v) + \mu (x \cdot w)$ for all $x \in \mathbb{C}[G]$, $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$.
   - We have $(x + y) \cdot v = (x \cdot v) + (y \cdot v)$ for all $x, y \in \mathbb{C}[G]$ and $v \in V$.
   - We have $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ for all $x, y \in \mathbb{C}[G]$ and $v \in V$.
   - We have $[e] \cdot v = v$ for all $v \in V$.

   Prove the following:
   (a) Let $V$ be a representation of $G$ (so that $g \cdot v$ is defined for all $g \in G$ and $v \in V$). Then $V$ together with the map
       $$ \mathbb{C}[G] \times V \rightarrow V $$
       $$ \left( \sum_{g \in G} c_g \cdot [g], v \right) \mapsto \sum_{g \in G} c_g (g \cdot v) $$
   is a $\mathbb{C}[G]$-module.
   (b) Let $V$ be a module of $\mathbb{C}[G]$ (so that $x \cdot v$ is defined for all $x \in \mathbb{C}[G]$ and $v \in V$). Then $V$ together with the map
       $$ G \times V \rightarrow V $$
       $$ (g, v) \mapsto [g] \cdot v $$
   is a representation of $G$.

2. Let $V, W$ be finite dimensional vector spaces. Prove that the map
   $$ V^* \otimes W \rightarrow \text{Hom}(V, W) $$
   $$ \sum_{i=1}^{k} \lambda_i \psi_i \otimes w_i \mapsto \left( v \mapsto \sum_{i=1}^{k} \lambda_i \psi_i(v) w_i \right) $$
   is a linear isomorphism.