Week 6 - Fourier analysis on finite abelian groups

This week, we discuss Fourier analysis on finite abelian groups. Before we do this, we prove the following result.

Proposition 1. *Let G, H be finite groups. Then every irreducible representation of $G \times H$ is isomorphic to $V \otimes W$ for some irreducible representations V of G and W of H.*

For the we need the following lemmas.

Lemma 2. *Every conjugacy class of $G \times H$ is of the form $C \times C'$ where C is a conjugacy class of G and C' is a conjugacy class of H. In particular, the number of conjugacy classes of $G \times H$ equals the product of the numbers of conjugacy classes of G and H.*

Proof. Let $(x, y) \in G \times H$, let C be the conjugacy class of x and C' the conjugacy class of y. Then

$$(g,h)(x,y)(g,h)^{-1} = (g,h)(x,y)(g^{-1},h^{-1}) = (gxg^{-1}, hyh^{-1}) \in C \times C'$$

for all $(g, h) \in G \times H$. So the conjugacy class of (x, y) is contained in $C \times C'$. Conversely, every element of $C \times C'$ is of the form (gxg^{-1}, hyh^{-1}) with $g \in G$ and $h \in H$ and such an element is conjugate to (x, y). So $C \times C'$ is the conjugacy class of (x, y). □

Proposition 3. *Let V, V' be irreducible representations of G and W, W' irreducible representations of H.*

(1) *The representation $V \otimes W$ of $G \times H$ is irreducible.*
(2) *If $V \otimes W \cong V' \otimes W'$, then $V \cong V'$ and $W \cong W'$.*

Proof. (1) We know that

$$\frac{1}{|H|}\sum_{h\in H}\chi_W(h)\overline{\chi_W(h)} = \langle \chi_W, \chi_W\rangle = 1 = \langle \chi_V, \chi_V\rangle = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_V(g)}$$

and $\chi_{V\otimes W}(g, h) = \chi_V(g)\chi_W(h)$. So

$$\begin{aligned}
\langle \chi_{V\otimes W}, \chi_{V\otimes W}\rangle &= \frac{1}{|G\times H|}\sum_{(g,h)\in G\times H}\chi_{V\otimes W}(g,h)\overline{\chi_{V\otimes W}(g,h)} \\
&= \frac{1}{|G|\cdot|H|}\sum_{g\in G}\sum_{h\in H}\chi_V(g)\chi_W(h)\overline{\chi_V(g)}\,\overline{\chi_W(h)} \\
&= \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_V(g)}\cdot\frac{1}{|H|}\sum_{h\in H}\chi_W(h)\overline{\chi_W(h)} \\
&= 1\cdot 1 = 1.
\end{aligned}$$

Hence $V \otimes W$ is irreducible.
(2) Suppose that $V \otimes W \cong V' \otimes W'$. Then

$$\chi_V(g)\chi_W(h) = \chi_{V\otimes W}(g,h) = \chi_{V'\otimes W'}(g,h) = \chi_{V'}(g)\chi_{W'}(h)$$

for all $g \in G$ and $h \in H$. Choose $h = e$, then we get

$$\chi_V(g) = \chi_V(g)\cdot 1 = \chi_V(g)\chi_W(e) = \chi_{V'}(g)\chi_{W'}(e) = \chi_{V'}(g)\cdot 1 = \chi_{V'}(g)$$

for all $g \in G$. Similarly, we get $\chi_W(h) = \chi_{W'}(h)$ for all $h \in H$ using $g = e$. Hence $\chi_V = \chi_{V'}$ and $\chi_W = \chi_{W'}$. So $V \cong V'$ and $W \cong W'$. □

Proof of Proposition 1. Let $V_1, \ldots, V_s$ be the irreducible representations of G and $W_1, \ldots, W_r$ the irreducible representations of H. Then the $V_i \otimes W_j$ for $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$ are irreducible non-isomorphic representations of $G \times H$. We have sr of them, which equals the number of conjugacy classes of $G \times H$. Hence these are all irreducible representations of $G \times H$. □