Week 5 - Characters

Proposition 1. *Let $\rho\colon G \to \mathrm{GL}_n$ be a representation. Then the dual representation of ρ is isomorphic to the representation $\psi\colon G \to \mathrm{GL}_n, g \mapsto \rho(g)^{-\top}$.*

Proof. The dual representation of ρ is the homomorphism $\rho^*\colon G \to \mathrm{GL}((\mathbb{C}^n)^*), g \mapsto (f \mapsto f \circ \rho(g)^{-1})$. Let $x_1, \ldots, x_n$ be the dual basis of the basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$. Then the linear map

$$\begin{array}{rcl} L\colon \mathbb{C}^n & \to & (\mathbb{C}^n)^* \\ (a_1, \ldots, a_n) & \mapsto & a_1x_1 + \ldots + a_nx_n \end{array}$$

is a linear isomorphism. We claim that this is also a G-linear map from the representation ψ to the representation ρ^*. We know that for all $i \in \{1, \ldots, n\}$ and $g \in G$, there are $b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{C}$ such that

$$\psi(g)(e_i) = b_1e_1 + \ldots + b_ne_n, \quad \rho^*(g)(x_i) = c_1x_1 + \ldots + c_nx_n.$$

To prove that the map L is G-linear, we need to prove that in this case $b_1 = c_1, \ldots, b_n = c_n$. Write $A = \rho(g)^{-1} \in \mathrm{GL}_n$. Then

$$\psi(g)(e_i) = A^\top e_i = A_{i1}e_1 + \ldots + A_{in}e_n$$

and

$$\rho^*(g)(x_i) = (e_j \mapsto x_i(Ae_j)) = (e_j \mapsto x_i(A_{1j}, \ldots, A_{nj})) = (e_j \mapsto A_{ij}) = A_{i1}x_1 + \ldots + A_{in}x_n$$

and so the coefficients are indeed the same. Hence L is G-linear and so an isomorphism of representations. ☐

Definition 2. Let $A \in \mathbb{C}^{n\times m}, B \in \mathbb{C}^{k\times \ell}$ be matrices. The *Kronecker product* of A, B is the matrices $A \otimes B \in \mathbb{C}^{nk\times m\ell}$ defined by

$$A \otimes B := \begin{pmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nm}B \end{pmatrix}$$

◆

Lemma 3.

1. *We have $I_n \otimes I_m = I_{nm}$.*
2. *Let $A, A' \in \mathbb{C}^{n\times n}, B, B' \in \mathbb{C}^{m\times m}$ be matrices. Then $(AA') \otimes (BB') = (A \otimes B)(A' \otimes B')$.*

Proposition 4. *Let $\rho\colon G \to \mathrm{GL}_n$ and $\psi\colon H \to \mathrm{GL}_m$ be representations. Then their tensor product is isomorphic to*

$$\begin{array}{rcl} \varphi\colon G \times H & \to & \mathrm{GL}_{nm} \\ (g, h) & \mapsto & \rho(g) \otimes \psi(h) \end{array}$$

Proof. Recall that the tensor product of ρ and ψ is the homomorphism

$$\begin{array}{rcl} \rho \otimes \psi\colon G \times H & \to & \mathrm{GL}(\mathbb{C}^n \otimes \mathbb{C}^m) \\ (g, h) & \mapsto & \left(\displaystyle\sum_{i=1}^k \lambda_i \cdot v_i \otimes w_i \mapsto \sum_{i=1}^k \lambda_i\rho(g)(v_i) \otimes \psi(h)(w_i)\right) \end{array}$$

and that the linear map

$$\begin{array}{rcl} L\colon \mathbb{C}^n \otimes \mathbb{C}^m & \to & \mathbb{C}^{nm} \\ \displaystyle\sum_{i=1}^n\sum_{j=1}^m a_{ij} \cdot e_i \otimes e_j & \mapsto & (a_{11}, a_{12}, \ldots, a_{1m}, \ldots, a_{n1}, \ldots, a_{nm}) \end{array}$$

is a linear isomorphism. This isomorphism is an fact $(G \times H)$-linear and hence an isomorphism of representations. ☐

Proposition 5.

1. *Let V be a representation of G. Then we have $\chi_{V^*}(g) = \overline{\chi_V(g)}$ for all $g \in G$.*
2. *Let V be a representation of G and W a representation of H. Then we have $\chi_{V\otimes W}(g, h) = \chi_V(g)\chi_W(h)$ for all $g \in G$ and $H \in H$.*
3. *Let V, W be representations of G. Then we have $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$ for all $g \in G$.*

Proof. (1) We may assume that $V = \mathbb{C}^n$ and that V is unitairy with respect to the standard inner product, so that the correpsonding homomorphism is of the form $\rho\colon G \to U_n$. We know that the dual of V is isomorphic to $\psi\colon G \to \mathrm{GL}_n, g \mapsto \rho(g)^{-T}$. Since $\rho(g) \in U_n$, we have $\rho(g)^{-\top} = \overline{\rho(g)}$. Hence $\chi_{V^*}(g) = \mathrm{tr}(\overline{\rho(g)}) = \overline{\mathrm{tr}(\rho(g))} = \overline{\chi_V(g)}$.
(2) We know that $V \otimes W$ is isomorphic to the representation $\rho \otimes \psi\colon G \times H \to \mathrm{GL}_{nm}, (g, h) \mapsto \rho(g) \otimes \rho(h)$, where $\rho\colon G \to \mathrm{GL}_n, \psi\colon H \to \mathrm{GL}_m$ are isomorphic to V, W. Write $A = \rho(g), B = \psi(h)$. Then

$$\chi_{V\otimes W}(g, h) = \mathrm{tr}(A \otimes B) = \mathrm{tr}(A_{11}B) + \ldots + \mathrm{tr}(A_{nn}B) = A_{11}\,\mathrm{tr}(B) + \ldots + A_{nn}\,\mathrm{tr}(B) = \mathrm{tr}(A)\,\mathrm{tr}(B) = \chi_V(g)\chi_W(h).$$

(3) Now assume that $G = H$. Then $V \otimes W$ is a representation of G by setting $g \cdot T := (g, g) \cdot T$ for all $g \in G$ and $T \in V \otimes W$. We get $\chi_{V\otimes W}(g) = \chi_{V\otimes W}(g, g) = \chi_V(g)\chi_W(g)$. $\square$