Week 3 - Complete reducibility

This week, we cover the remainder of Chapter 3 from the book together with the following.

Let again see how we can break G-linear between direct sums of irreducible representations into pieces. Recall Schur's Lemma.

Lemma 1 (Schur's Lemma)**.**

(1) *Let V, W be irreducible representations of a group G and let $\varphi\colon V \to W$ be a G-linear map. Then either $\varphi = 0$ are φ is an isomorphism.*

(2) *Let V be a finite-dimensional irreducible representation of a group G. Then every G-linear map $V \to V$ is of the form $v \mapsto \lambda v$ for some $\lambda \in \mathbb{C}$ (and every map of this form is G-linear).*

The following lemma states how to break G-linear maps into pieces.

Lemma 2. *Let V, V', W, W' be representations of G.*

(1) *A map $\psi\colon V \oplus V' \to W$ is G-linear if and only if it is of the form $(v, v') \mapsto \varphi(v) + \varphi'(v')$ where $\varphi\colon V \to W$ and $\varphi'\colon V' \to W$ are G-linear maps.*

(2) *A map $\psi\colon V \to W \oplus W'$ is G-linear if and only if it is of the form $v \mapsto (\varphi(v), \varphi'(v))$ where $\varphi\colon V \to W$ and $\varphi'\colon V \to W'$ are G-linear maps.*

Proof.

(1) Let ψ be a G-linear map. Then $V \to W, v \mapsto \psi(v, 0)$ and $V' \to W, v' \mapsto \psi(0, v')$ are also G-linear. Conversely, when φ, φ' are G-linear, then $V \oplus V' \to W, (v, v') \mapsto \varphi(v) + \varphi'(v')$ is as well.

(2) Let ψ be a G-linear map. Then we can write $\psi(v) = (\varphi(v), \varphi'(v))$ for maps $\varphi\colon V \to W, \varphi'\colon V \to W'$. These maps are G-linear. Conversely, when φ, φ' are G-linear, then $V \to W \oplus W', v \mapsto (\varphi(v), \varphi'(v))$ is as well. □

We will work our way up in steps.

Lemma 3. *Let V, W be finite-dimensional irreducible representations of G. Assume that $V \ncong W$. Let $a, b \in \mathbb{Z}_{\geq 0}$.*

(1) *Every G-linear map $V^{\oplus a} \to V^{\oplus b}$ is of the form*

$$v = (v_1, \dots, v_a) \mapsto (\lambda_{11} v_1 + \ldots + \lambda_{1a} v_a, \dots, \lambda_{b1} v_1 + \ldots + \lambda_{ba} v_a) =: Av$$

for some $\lambda_{11}, \dots, \lambda_{ba} \in \mathbb{C}$, where $A = (\lambda_{ij})_{i,j=1}^{b,a} \in \mathbb{C}^{b \times a}$.

(2) *The only G-linear map $V^{\oplus a} \to W^{\oplus b}$ is $(v_1, \dots, v_a) \mapsto (0, \dots, 0)$.*

Proof.

(1) By the previous lemma, every G-linear map $V^{\oplus a} \to V^{\oplus b}$ is of the form

$$v = (v_1, \dots, v_a) \mapsto (\varphi_{11}(v_1) + \ldots + \varphi_{1a}(v_a), \dots, \varphi_{b1}(v_1) + \ldots + \varphi_{ba}(v_a))$$

for G-linear maps $\varphi_{11}, \dots, \varphi_{ba}\colon V \to V$. By Schur's Lemma, we have $\varphi_{ij}(v) = \lambda_{ij} v$ for some $\lambda_{ij} \in \mathbb{C}$.

(2) By the previous lemma, every G-linear map $V^{\oplus a} \to W^{\oplus b}$ is of the form

$$v = (v_1, \dots, v_a) \mapsto (\varphi_{11}(v_1) + \ldots + \varphi_{1a}(v_a), \dots, \varphi_{b1}(v_1) + \ldots + \varphi_{ba}(v_a))$$

for G-linear maps $\varphi_{11}, \dots, \varphi_{ba}\colon V \to W$. By Schur's Lemma, we have $\varphi_{ij}(v) = 0$. □

Proposition 4. *Let $V_1, \dots, V_k$ be finite-dimensional representations of G. Suppose that $V_i \ncong V_j$ for all $i \neq j$. Let $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{Z}_{\geq 0}$. Then every G-linear map*

$$V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \to V_1^{\oplus b_1} \oplus \cdots \oplus V_k^{\oplus b_k}$$

is of the form $(v_1, \dots, v_k) \mapsto (A_1 v_1, \dots, A_k v_k)$ where $v_i \in V_i^{\oplus a_i}$ and $A_i \in \mathbb{C}^{b_i \times a_i}$.

Proof. By Lemma 2, every such G-linear map is of the form

$$(v_1, \dots, v_k) \mapsto (\varphi_{11}(v_1) + \ldots + \varphi_{k1}(v_k), \dots, \varphi_{1k}(v_1) + \ldots + \varphi_{kk}(v_k))$$

where $\varphi_{ij}\colon V_i^{\oplus a_i} \to V_j^{\oplus b_j}$ are G-linear maps. When $i \neq j$, then $\varphi_{ij} = 0$. We have $\varphi_{ii}(v) = A_i v$ for some $A_i \in \mathbb{C}^{b_i \times a_i}$. □

Proposition 5. *Let U, V, W be finite-dimensional vector spaces and let $B\colon V \times W \to U$ be a bilinear form. Then there exists a unique linear map $\varphi\colon V \otimes W \to U$ such that $\varphi(v \otimes w) = B(v, w)$ for all $v \in V$ and $w \in W$. Conversely, for all linear maps $\varphi\colon V \otimes W \to U$ the map $B\colon V \times W \to U$ defined by $B(v, w) = \varphi(v \otimes w)$ is bilinear.*

Proof. Let $B\colon V \times W \to U$ be a bilinear map. Define φ by

$$\varphi\left(\sum_{i=1}^{k} \lambda_i v_i \otimes w_i\right) := \sum_{i=1}^{k} \lambda_i B(v_i, w_i)$$

We need to show that φ is well-defined. Afterwards, it is easy to see that φ is linear. For well-definedness, we notice that

$$\lambda B(v, w) = B(\lambda v, w) = B(v, \lambda w) \quad \forall \lambda \in \mathbb{C}, v \in V, w \in W \tag{1}$$
$$B(v + v', w) = B(v, w) + B(v', w) \quad \forall v, v' \in V, w \in W \tag{2}$$
$$B(v, w + w') = B(v, w) + B(v, w') \quad \forall v \in V, w, w' \in W \tag{3}$$

We need to show that if $T, T' \in V \otimes W$ and you can get from T to T' by the following steps, then $\varphi(T) = \varphi(T')$.

$$\lambda v \otimes w \leftrightarrow (\lambda v) \otimes w \leftrightarrow v \otimes (\lambda w) \tag{4}$$
$$(v + v') \otimes w \leftrightarrow v \otimes w + v' \otimes w \tag{5}$$
$$v \otimes (w + w') \leftrightarrow v \otimes w + v \otimes w' \tag{6}$$

Now, (1) shows that exchanges of the form (4) are okay, (2) shows (5) are okay and (3) shows that (6) is okay. So φ is well-defined.

Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ a basis of W. Then the condition $\varphi(v \otimes w) = B(v, w)$ implies that $\varphi(v_i \otimes w_j) = B(v_i, w_j)$ for all i, j. Hence the condition determines the values of φ at a basis of $V \otimes W$. So φ is unique.

Now, let $\varphi\colon V \otimes W \to U$ be a linear map and defined $B(v, w) := \varphi(v \otimes w)$. We check that B is a bilinear map. We have

$$B(\lambda v + \mu v', w) = \varphi((\lambda v + \mu v') \otimes w) = \varphi(\lambda(v \otimes w) + \mu(v' \otimes w)) = \lambda\varphi(v \otimes w) + \mu\varphi(v' \otimes w) = \lambda B(v, w) + \mu B(v', w)$$

and similarly $B(v, \lambda w + \mu w') = \lambda B(v, w) + \mu B(v, w')$. So B is indeed bilinear. □

Example 6. The map

$$\begin{array}{ccc} \mathbb{C}^{n \times m} & \to & \mathbb{C}^n \otimes \mathbb{C}^m \\ A & \mapsto & \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} e_i \otimes e_j \end{array}$$

is an isomorphism. To see this, we note that the map $\mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^{n \times m}$ sending $(v, w) \mapsto vw^\top$ is bilinear. The map

$$\begin{array}{ccc} \mathbb{C}^n \otimes \mathbb{C}^m & \to & \mathbb{C}^{n \times m} \\ \displaystyle\sum_{i=1}^{k} \lambda_i v_i \otimes w_i & \mapsto & \displaystyle\sum_{i=1}^{k} \lambda_i v_i w_i^\top \end{array}$$

is the inverse map. ♠

Example 7. The map

$$\begin{array}{ccc} V \otimes W & \to & W \otimes V \\ \displaystyle\sum_{i=1}^{k} \lambda_i v_i \otimes w_i & \mapsto & \displaystyle\sum_{i=1}^{k} \lambda_i w_i \otimes v_i \end{array}$$

is an isomorphism. The map is well-defined since $(v, w) \mapsto w \otimes v$ is bilinear. Its inverse is well-defined since $(w, v) \mapsto v \otimes w$ is bilinear. ♠

Example 8. Let $\varphi\colon V \to V'$ and $\psi\colon W \to W'$ be linear maps. Then

$$\begin{aligned} \varphi \otimes \psi \colon V \otimes W &\to V' \otimes W' \\ \sum_{i=1}^{k} \lambda_i v_i \otimes w_i &\mapsto \sum_{i=1}^{k} \lambda_i \varphi(v_i) \otimes \psi(w_i) \end{aligned}$$

is a linear map, since $(v, w) \mapsto \varphi(v) \otimes \psi(w)$ is a bilinear map. ♠

Example 9. The linear map

$$\begin{aligned} (U \oplus V) \otimes W &\to (U \otimes W) \oplus (V \otimes W) \\ \sum_{i=1}^{k} \lambda_i (u_i, v_i) \otimes w_i &\mapsto \left(\sum_{i=1}^{k} \lambda_i u_i \otimes w_i, \sum_{i=1}^{k} \lambda_i v_i \otimes w_i \right) \end{aligned}$$

is an isomorphism. Here we use that $(u, v) \mapsto u, (u, v) \mapsto v, u \mapsto (u, 0), v \mapsto (0, v), w \mapsto w$ are linear maps. ♠