Week 2 - G-linear maps

This week, we cover Section 3.1 from the book together with the following.

Definition 1. Let V be a representation of a group G. A *subrepresentation* of V is a G-invariant subspace W of V. A subrepresentation is itself a representation, where the map $G \times W \to W$ is the restriction of the map $G \times V \to V$ to $G \times W$. ♦

Definition 2. Let V be a subrepresentation of a representation V of G. Then the map

$$\begin{array}{rcl} G \times V/W & \to & V/W \\ (g, v + W) & \mapsto & g \cdot v + W \end{array}$$

turns V/W into a representation of G. This representation is called the *quotient* of V by W. ♦

Definition 3. Let V, W be representations of a group G. A *G-linear map* from V to W is a map $\varphi \colon V \to W$ that is

- linear, so $\varphi(\lambda v + \mu v') = \lambda\varphi(v) + \mu\varphi(v')$ for all $\lambda, \mu \in \mathbb{C}$ and $v, v' \in V$.
- *G-equivariant*, this means that $\varphi(g \cdot v) = g \cdot \varphi(v)$ for all $g \in G$ and $v \in V$. ♦

Proposition 4. *Let $\varphi \colon V \to W$ be a G-linear map. Then* $\ker(\varphi)$ *is a subrepresentation of V and* $\operatorname{im}(\varphi)$ *is a subrepresentation of W.*

Proof. The set $\ker(\varphi)$ is a subspace of V. For all $g \in G$ and $v \in \ker(\varphi)$, we have $\varphi(g \cdot v) = g \cdot \varphi(v) = g \cdot 0 = 0$ and so $g \cdot v \in \ker(\varphi)$. So $\ker(\varphi)$ is also G-invariant. So it is a subrepresentation of V. The set $\operatorname{im}(\varphi)$ is a subspace of W. For all $g \in G$ and $w \in \operatorname{im}(\varphi)$, we have $g \cdot w = g \cdot \varphi(v) = \varphi(g \cdot v) \in \operatorname{im}(\varphi)$ for some $v \in V$. So $\operatorname{im}(\varphi)$ is also G-invariant. So it is a subrepresentation of W. □

Example 5. Let V_{triv} be the representation of S_n on $\mathbb{C}$ defined by

$$\sigma \cdot x = x$$

for all $\sigma \in S_n$ and $x \in \mathbb{C}$. Let V_{alt} be the representation of S_n on $\mathbb{C}$ defined by

$$\sigma \cdot x = \operatorname{sign}(\sigma)x$$

for all $\sigma \in S_n$ and $x \in \mathbb{C}$. Let V_{std} be the representation of S_n on $\mathbb{C}^n$ defined by

$$\sigma \cdot (v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$$

for all $\sigma \in S_n$ and $(v_1, \ldots, v_n) \in \mathbb{C}^n$.

1. The only G-linear map $V_{\text{triv}} \to V_{\text{alt}}$ is the map 0.
2. The map $V_{\text{triv}} \to V_{\text{std}}, x \mapsto (x, \ldots, x)$ is G-linear.
3. The map $V_{\text{std}} \to V_{\text{triv}}, (v_1, \ldots, v_n) \mapsto v_1 + \ldots + v_n$ is G-linear. ♠

Definition 6. Let $f \colon H \to G$ be a homomorphism and let V be a representation of G. Then the map

$$\begin{array}{rcl} H \times V & \to & V \\ (h, v) & \mapsto & f(h) \cdot v \end{array}$$

turns V into a representation of H. ♦

Definition 7. Let V be a representation of G and W a representation of H. Then the map

$$\begin{array}{rcl} (G \times H) \times (V \otimes W) & \to & V \otimes W \\ \left((g,h), \sum_{i=1}^{k} \lambda_i \cdot (v_i \otimes w_i)\right) & \mapsto & \sum_{i=1}^{k} \lambda_i \cdot (g \cdot v_i \otimes h \cdot w_i) \end{array}$$

turns $V \otimes W$ into a representation of $G \times H$. When $G = H$, we can use the homomorphism

$$\begin{array}{rcl} G & \to & G \times G \\ g & \mapsto & (g, g) \end{array}$$

to get a representation of G. ♦

Example 8. Let us study the case where $G = \{e\}$. In this case, every vector space V is a representation of G in a unique way. Namely $e \cdot v = v$ for all $v \in V$. Any subspace of V is a subrepresentation and any linear map is also G-linear. So we see that a representation is irreducible if and only if it has dimension 1. Hence, every irreducible representation is isomorphic to $\mathbb{C}$. We have the following fundamental result:

(1) Every finite-dimensional vector space is isomorphic to $\mathbb{C}^{\oplus n}$ for some unique $n \in \mathbb{Z}_{\geq 0}$.
(2) Every linear map $\mathbb{C}^{\oplus n} \to \mathbb{C}^{\oplus m}$ is of the form
$$v \mapsto Av$$
for some matrix $A \in \mathbb{C}^{m \times n}$. Conversely, for all matrices $A \in \mathbb{C}^{n \times m}$, the map
$$\begin{array}{rcl} \mathbb{C}^{\oplus n} & \to & \mathbb{C}^{\otimes m} \\ v & \mapsto & Av \end{array}$$
is linear.
(3) Let $v_1, \dots, v_n \in V$ be a basis of V. Then the map
$$\begin{array}{rcl} \mathbb{C}^{\oplus n} & \to & V \\ (x_1, \dots, x_n) & \mapsto & x_1 v_1 + \dots + x_n v_n \end{array}$$
is an isomorphism.

♠

The goal of this course is to get the same results for any finite group G.

(1) We want to find representations $V_1, \dots, V_k$ such that every representation of G is isomorphic to
$$V_1^{\oplus a_1} \oplus \cdots \oplus C_k^{\oplus a_k}$$
for some unique $a_1, \dots, a_k \geq 0$.
(2) We want to understand the G-linear maps between representations of this form.
(3) Given a representation V, we want to be able to determine $a_1, \dots, a_k$ such that
$$V \cong V_1^{\oplus a_1} \oplus \cdots \oplus C_k^{\oplus a_k}$$
(and maybe even write down the isomorphism).

Let's start with (2).

Lemma 9 (Schur's Lemma)**.**

(1) *Let V, W be irreducible representations of a group G and let $\varphi \colon V \to W$ be a G-linear map. Then either $\varphi = 0$ are φ is an isomorphism.*
(2) *Let V be a finite-dimensional irreducible representation of a group G. Then every G-linear map $V \to V$ is of the form $v \mapsto \lambda v$ for some $\lambda \in \mathbb{C}$ (and every map of this form is G-linear).*

Proof. (1) Let $\varphi \colon V \to W$ be a G-linear map. Then $\ker(\varphi)$ is a subrepresentation of V. Since V is irreducible, either $\ker(\varphi) = V$ or $\ker(\varphi) = 0$. The the first case, we have $\varphi = 0$ and we are done. Assume the second case. Next note that $\operatorname{im}(\varphi)$ is a subrepresentation of W. Since W is irreducible, either $\operatorname{im}(\varphi) = 0$ or $\operatorname{im}(\varphi) = W$. In the first case, we have $\varphi = 0$ and we are done. Assume that second case. Now φ is both injective and surjective. So it is an isomorphism.
(2) Let $\varphi \colon V \to V$ be a G-linear map. Then φ has an eigenvalue $\lambda \in V$ with corresponding eigenvector $w \in V \setminus \{0\}$. Define $\psi \colon V \to V$ by $\psi(v) := \varphi(v) - \lambda v$ for all $v \in V$. One can check that ψ is a G-linear map. So by part (1), we know that $\psi = 0$ or ψ is an isomorphism. Note that $\psi(w) = \varphi(w) - \lambda w = 0$. So ψ cannot be an isomorphism. So $\psi =$. So $\varphi(v) = \lambda v$ for all $v \in V$. □

We get the following statement:

Proposition 10. *Let G be a group and let $V_1, \dots, V_k$ be irreducible representations of G such that $V_i \not\cong V_j$ for all $i \neq j$. Let $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{Z}_{\geq 0}$. Then every G-linear map*
$$\varphi \colon V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \to V_1^{\oplus b_1} \oplus \cdots \oplus V_k^{\oplus b_k}$$
is of the form
$$(v^{(1)}, \dots, v^{(k)}) \mapsto (A^{(1)} v^{(1)}, \dots, A^{(k)} v^{(k)})$$

(and any map of this form is G-linear). Here $A^{(i)} \in \mathbb{C}^{b_i \times a_i}$, $v^{(i)} \in V_i^{\oplus a_i}$ and

$$A^{(i)}v^{(i)} := (A_{11}^{(i)}v_1^{(i)} + \ldots + A_{1a_i}^{(i)}v_{a_i}^{(i)}, \ldots, A_{a_i1}^{(i)}v_1^{(i)} + \ldots + A_{a_ia_i}^{(i)}v_{a_i}^{(i)})$$

Proof. To prove this, we need to show that there exist $\lambda_1, \ldots, \lambda_{b_i} \in \mathbb{C}$ such that

$$\varphi(0, \ldots, (0, \ldots, v, \ldots, 0), \ldots, 0) = (0, \ldots, (\lambda_1 v, \ldots, \lambda_{b_i} v), \ldots, 0)$$

for all i, j and $v \in V_i$. Here $(0, \ldots, v, \ldots, 0)$ is at position i and contains v at position j. Note that the map

$$\begin{array}{rcl} V_i & \to & V_1^{\oplus b_1} \oplus \cdots \oplus V_k^{\oplus b_k} \\ v & \mapsto & \varphi(0, \ldots, (0, \ldots, v, \ldots, 0), \ldots, 0) \end{array}$$

is G-linear. The projection on a V_k is also G-linear. So the composition $V_i \to V_k$ is as well. If $k \neq i$, then this composition is 0. If $k = i$, this composition is $V_i \to V_i, v \mapsto \lambda v$ for some $\lambda \in \mathbb{C}$. This gives us what we want. □

Corollary 11. *Let*

$$\varphi \colon V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \to V_1^{\oplus b_1} \oplus \cdots \oplus V_k^{\oplus b_k}$$

be a G-linear map and let $A_1, \ldots, A_k$ be the associated matrices. Then the following hold:

(1) $\ker(\varphi) \cong V_1^{\oplus \dim(\ker A_1))} \oplus \cdots \oplus V_k^{\oplus \dim(\ker A_k))}$
(2) $\mathrm{im}(\varphi) \cong V_1^{\oplus \dim(\mathrm{im}\, A_1))} \oplus \cdots \oplus V_k^{\oplus \dim(\mathrm{im}\, A_k))}$
(3) *The map φ is an isomorphism if and only if $A_1, \ldots, A_k$ are all invertible.*