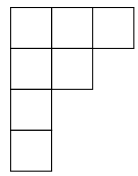
Week 15 - More properties of Sprecht representations

Definition 1. Let λ be a Young diagram. Then the *hook length* of a box B in λ is the number of boxes in λ that are either below or to the right of B (including B itself). ♦

Example 2. Let λ be the following Young diagram



Then the boxes of λ have hook lengths

$$\begin{array}{lll} 6 & 3 & 1 \\ 4 & 1 & \\ 2 & & \\ 1 & & \end{array}$$

♠

Theorem 3 (Hook length formula)**.** *Let λ be a Young diagram with n boxes. Then the dimension of the representation S^λ equals the quotient of $n!$ by the product of all hook lengths of boxes in λ.*

Example 4. Let λ be the Young diagram from the previous example. Then

$$\dim S^\lambda = \frac{7!}{6 \cdot 3 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1} = 35$$

♠

The hook length formula is a special case of the following theorem.

Theorem 5. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a Young diagram with n boxes and let $\sigma \in S_s$ be a permutation with cycle type $\mu = (\mu_1, \ldots, \mu_m)$. Then we have*

$$\chi_{S^\lambda}(\sigma) = \mathrm{coeff}_{x_1^{\lambda_1+k-1} x_2^{\lambda_2+k-2} \cdots x_k^{\lambda_k}} \left(\prod_{1 \le i < j \le k} (x_i - x_j) \prod_{i=1}^{m} (x_1^{\mu_i} + \ldots + x_k^{\mu_i}) \right)$$

Example 6. Let $\lambda = (3,2)$ and take $\sigma = (12) \in S_5$ with cycle type $\mu = (2,1,1,1)$. Then $\chi_{S^\lambda}(\sigma)$ is the coefficient of the monomial $x_1^4 x_2^2$ in the polynomial

$$(x_1 - x_2)(x_1^2 + x_2^2)(x_1 + x_2)^3 = x_1^6 + 2x_1^5 x_2 + x_1^4 x_2^2 - x_1^2 x_2^4 - 2x_1 x_2^5 - x_2^6,$$

so we find $\chi_{S^\lambda}(\sigma) = 1$. ♠