Week 14 - Alternate definition of Sprecht representations

Let λ be a Young diagram with n boxes and let t be the Young tableaux of shape λ we get by filling the ith row of λ with the numbers $\lambda_1 + \ldots + \lambda_{i-1} + 1, \ldots, \lambda_1 + \ldots + \lambda_{i-1} + \lambda_i$. Let

$$C_t := \{\sigma \in S_n \mid i, \sigma(i) \text{ in the same column for all } i \in \{1, \ldots, n\}\}$$

and

$$R_t := \{\sigma \in S_n \mid i, \sigma(i) \text{ in the same row for all } i \in \{1, \ldots, n\}\}$$

be the column and row stabilizer of t, respectively. Take

$$a_t := \sum_{\sigma \in C_t} \text{sign}(\sigma)\sigma \in \mathbb{C}[S_n] \quad \text{and} \quad b_t := \sum_{\sigma \in R_t} \sigma \in \mathbb{C}[S_n]$$

with product $c_t := a_t b_t \in \mathbb{C}[S_n]$. Then the set

$$V_\lambda := \mathbb{C}[S_n]c_t = \{xc_t \mid x \in \mathbb{C}[S_n]\} \subseteq \mathbb{C}[S_n]$$

is a subrepresentation of $\mathbb{C}[S_n]$. We can this representation the Sprecht representation associated to the Young diagram λ.

Example 1. Take $n = 3$. Then $(3), (2, 1), (1, 1, 1)$ are all the Young diagrams with n boxes.

(a) For $\lambda = (3)$, we get $C_t = \{(1)\}$ and $R_t = S_3$. So

$$c_t = a_t b_t = (1) \cdot \sum_{\sigma \in S_n} \sigma = \sum_{\sigma \in S_n} \sigma \neq 0.$$

The representation V_λ is spanned by $\{\pi c_t \mid \pi \in S_3\}$ and we have

$$\pi c_t = \pi \sum_{\sigma \in S_3} \sigma = \sum_{\sigma \in S_3} \pi\sigma = c_t$$

Hence V_λ has basis c_t and is isomorphic to the 1-dimensional trivial representation of S_3.

(b) For $\lambda = (1, 1, 1)$, we get $C_t = S_3$ and $R_t = \{(1)\}$. So

$$c_t = a_t b_t = \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma \cdot (1) = \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma \neq 0.$$

The representation V_λ is spanned by $\{\pi c_t \mid \pi \in S_3\}$ and we have

$$\pi c_t = \pi \sum_{\sigma \in S_3} \text{sign}(\sigma)\sigma = \text{sign}(\pi) \sum_{\sigma \in S_3} \text{sign}(\pi\sigma)\pi\sigma = \text{sign}(\pi)c_t$$

Hence V_λ has basis c_t and is isomorphic to the alternating representation of S_3.

(c) For $\lambda = (2, 1)$, we get $C_t = \{(1), (13)\}$ and $R_t = \{(1), (12)\}$. So

$$c_t = a_t b_t = ((1) - (13)) \cdot (((1) + (12)) = (1) + (12) - (13) - (13)(12) = (1) + (12) - (13) - (123) \neq 0.$$

Write

$$\begin{aligned} v_1 &:= (1) + (12) - (13) - (123) = (1) \cdot c_t \\ v_2 &:= (1) + (12) - (23) - (132) = (12) \cdot c_t \end{aligned}$$

Then $v_1, v_2 \in V_\lambda$ are linearly independent and we have

$$(12) \cdot v_1 = v_2 \qquad (13) \cdot v_1 = -v_1$$
$$(12) \cdot v_2 = v_1 \qquad (13) \cdot v_2 = -v_1 + v_2$$

Since $(12), (13)$ generate S_3, we conclude from this that V_λ has basis v_1, v_2. Note that

$$\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}$$

has the basis $w_1 := (1, 0, -1), w_2 := (0, 1, -1)$ which satisfies

$$(12) \cdot w_1 = w_2 \qquad (13) \cdot w_1 = -w_1$$
$$(12) \cdot w_2 = w_1 \qquad (13) \cdot w_2 = -w_1 + w_2$$

and so the linear map $V_\lambda \to \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}$ sending $v_i \mapsto w_i$ is an isomorphism of representations. ♠