Week 1 - Introduction

Organisational.

- The website for the course: `https://personal-homepages.mis.mpg.de/arbik/reptheory22.html`
- My email address: `arthur.bik@mis.mpg.de`
- The course will be taught in-person only.
- Exceptions are some weeks where I am away. Then the lecture is on Zoom. See schedule.
- The course will be taught in English.
  Room: SG 3-10
  Time: Thursdays 7:30-9:00 and Fridays 9:15-10:45
- We follow the book *Representation Theory of Finite Groups* by Benjamin Steinberg.
  When something is not from this book, I will put notes online.
- Every week I put some exercises online. These are optional, but of course recommended.
- Thursdays 7:30-8:15 will be an exercise class starting from week 3.
- No lecture on April 15th and May 26th.
- There will be a final written exam at the end of the semester.
- Proposed date of the written exam: Friday July 22th

The definitions of a representation

Definition 1. A *representation* of a group G can be defined in the following 3 ways:

(1) It is a complex vector space V together with a (left) group action

$$\begin{aligned} G \times V &\to V \\ (g, v) &\mapsto g \cdot v \end{aligned}$$

which is linear. This means that

$$g \cdot (\lambda v + \mu w) = \lambda (g \cdot v) + \mu (g \cdot w)$$

for all $g \in G$, $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$.

(2) It is a homomorphism of groups

$$\varphi \colon G \to \mathrm{GL}(V)$$

where V is a complex vector space.

(3) It is a $\mathbb{C}[G]$-module. ♦

Groups

Definition 2. A *group* is a triple $(G, *, e)$ where G is a set, $e \in G$ is an element and

$$- * - \colon G \times G \to G$$

is a binary operation such that the following conditions hold:

- Associativity: we have $f * (g * h) = (f * g) * h$ for all $f, g, h \in G$.
- Neutral element: we have $e * g = g * e = g$ for all $g \in G$.
- Inverse element: for each $g \in G$, there exists an $h \in G$ with $g * h = h * g = e$. ♦

Definition 3. A *homomorphism* between two groups G, H is a map

$$\varphi \colon G \to H$$

such that $\varphi(e) = e$ and $\varphi(g * h) = \varphi(g) * \varphi(h)$ for all $g, h \in G$. ♦

In general, we often use the symbol $\cdot$ instead of $*$ and denote the inverse element of g by g^{-1}.

Definition 4. A group G is *abelian* when $g * h = h * g$ for all $g, h \in G$. ♦

When G is abelian, we often use the symbol $+$ instead of $*$, denote the neutral element by 0 and the inverse element of g by $-g$.

Example 5. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}^n$ are abelian groups. ♠

Example 6. The sets $\{\pm 1\}$, $\mathbb{Q} \setminus \{0\}$, $\mathbb{R} \setminus \{0\}$, $\mathbb{C} \setminus \{0\}$, $\{z \in \mathbb{C} \mid |z| = 1\}$ are groups. ♠

Example 7. The set S_n of bijections $[n] \to [n] := \{1, \dots, n\}$ is a group where $\sigma \cdot \tau := \sigma \circ \tau$. ♠

Definition 8. A *cycle* in S_n is written as $(k_1\ k_2\ \cdots\ k_\ell)$ where $k_1, k_2, \dots, k_\ell \in \{1, \dots, n\}$ are distinct elements. It is the bijection that sends $k_i \mapsto k_{i+1}$ for $i \in \{1, \dots \ell - 1\}$, sends $k_\ell \mapsto k_1$ and $x \mapsto x$ for all $x \in \{1, \dots, n\} \setminus \{k_1, \dots, k_\ell\}$. ♦

The same cycle can be written in different ways. For example $(123) = (231)$. Every element of S_n can be written (basically uniquely) as a product of cycle with distinct entries. For example:

$$(123)(36)(245)(16) = (1)(24536) = (24536)(1) = (53624)(1)$$

in S_6.

Example 9. The set $\mathrm{GL}(V)$ of linear isomorphisms $V \to V$ is a group where $f \cdot g := f \circ g$. ♠

We write $\mathrm{GL}_n := \mathrm{GL}(\mathbb{C}^n)$.

Example 10. The set $\mathbb{Z}/n\mathbb{Z}$ of classes of integers modulo n is an abelian group. The elements are of the form $(a \bmod n)$ where $a \in \mathbb{Z}$ and the following are equivalent:

- $(a \bmod n) = (b \bmod n)$
- $a \equiv b \bmod n$
- $n \mid a - b$

We have $(a \bmod n) + (b \bmod n) = (a + b \bmod n)$. ♠

Example 11. Let G, H be groups. Then the set $G \times H$ is a group with $(g, h) \cdot (g', h') := (gg', hh')$. ♠

Example 12. The following maps are homomorphisms of groups:

- $G \to H, g \mapsto e$
- $G \to G, g \to g$
- $G \to G \times G, g \mapsto (g, g)$
- $\mathbb{Z} \to \mathbb{C}, n \mapsto n$
- $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, k \mapsto (k \bmod n)$
- $\mathbb{Z}/n\mathbb{Z} \to \mathbb{C} \setminus \{0\}, (k \bmod n) \mapsto e^{k \cdot 2\pi i/n}$
- $S_n \to \mathrm{GL}_n, \sigma \mapsto P_\sigma$ where $P_\sigma e_i = e_{\sigma(i)}$ for $i \in \{1, \dots, n\}$ with $e_1, ..., e_n$ the standard basis of $\mathbb{C}^n$. So for example:

$$P_{(123)(45)} = \begin{pmatrix} & & 1 & & \\ 1 & & & & \\ & 1 & & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix}$$

♠

Definition 13. We say that element $a, b \in G$ are *conjugate* when there is an $g \in G$ with $a = gbg^{-1}$. Being conjugate is an equivalence relation. The *conjugacy class* of $a \in G$ is $C_a := \{gag^{-1} \mid g \in G\}$. The following are equivalent:

- a, b are conjugate
- $a \in C_b$
- $b \in C_a$
- $C_a = C_b$

The group G is a disjoint union of its conjugacy classes. ♦

Example 14. Let A be an abelian group and let $a, b \in A$ be conjugate. Then there exists an $g \in A$ with

$$a = g + b + (-g) = g + (-g) + b = b$$

and so $C_a = \{a\}$ for all $a \in A$. ♠

Example 15. For two matrices $A, B \in \mathrm{GL}_n$, the following are equivalent:

- A, B are conjugate.
- A, B representation the same linear maps for different choices of basis.
- A, B have the same Jordan Normal Form.

The matrix A is diagonalizable if and only if C_A contains a diagonal matrix. ♠

Definition 16. A partition of $n \in \mathbb{N}$ is a tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$ such that $\lambda_1 + \dots + \lambda_k = n$. We write $\lambda \vdash n$. ♦

Example 17. Every element $\sigma \in S_n$ can be written as

$$\sigma = (a_{1,1} \ \cdots \ a_{1,\lambda_1}) \cdots (a_{k,1} \ \cdots \ a_{k,\lambda_k})$$

with $\{a_{1,1}, \dots, a_{k,\lambda_k}\} = \{1, \dots, n\}$ for some unique partition $(\lambda_1, \dots, \lambda_k) \vdash n$. This partition is called the *cycle type* of σ. Two permutations $\sigma, \tau \in S_n$ are conjugate if and only if they have the same cycle type. In fact, we have

$$\pi\sigma\pi^{-1} = (\pi(a_{1,1}) \ \cdots \ \pi(a_{1,\lambda_1})) \cdots (\pi(a_{k,1}) \ \cdots \ \pi(a_{k,\lambda_k}))$$

for all $\pi \in S_n$. ♠

Definition 18. A *(left) group action* of a group G on a set X is a map

$$\begin{array}{rcl} f\colon G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

such that $e \cdot x = x$ for all $x \in X$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$. ♦

Example 19. Take $G = D_4$ the dihedral group of order 8. The consists of the rotations in $\mathbb{R}^2$ around the origin with $0, \pi/2, \pi, 3\pi/2$ degrees together with reflections in the lines $y = 0$, $x = y$, $y = 0$ and $y = -x$. Take

$$X = \{(x, y) \in \mathbb{R}^2, |x|, |y| \leq 1\}$$

the square in $\mathbb{R}^2$ with center $(0, 0)$ and edge-lengths 2. Let $g \cdot v \in X$ denote the result of applying the rotation/reflection g to the point $v \in X$. Then

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

is an action of D_4 on X. ♠

Let G be any group and V be any (complex) vector space.

Example 20. The map

$$\begin{array}{rcl} G \times V & \to & V \\ (g, v) & \mapsto & v \end{array}$$

is an action. This turns V into a representation of G called a *trivial* representation. ♠

Example 21. The map

$$\begin{array}{rcl} \mathrm{GL}(V) \times V & \to & V \\ (\ell, v) & \mapsto & g(v) \end{array}$$

is an action. This turns V into a representation of $\mathrm{GL}(V)$. ♠

Example 22. The map

$$\begin{array}{rcl} S_n \times \mathbb{C}^n & \to & \mathbb{C}^n \\ (\sigma, (v_1, \dots, v_n)) & \mapsto & (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)}) \end{array}$$

is an action. This turns $\mathbb{C}^n$ into a representation of S_n called the *standard* representation of S_n. ♠

Example 23. The map

$$\begin{array}{rcl} \mathrm{GL}_n \times \mathbb{C} & \to & \mathbb{C} \\ (A, x) & \mapsto & \det(A)x \end{array}$$

is an action. This turns $\mathbb{C}$ into a representation of GL_n. ♠

Example 24. The map

$$\begin{array}{rcl} S_n \times \mathbb{C} & \to & \mathbb{C} \\ (\sigma, x) & \mapsto & \operatorname{sign}(\sigma) x \end{array}$$

is an action. This turns $\mathbb{C}$ into a representation of S_n. ♠

We now show how to go between Definitions (1) and (2) of a representation.

(1)⇒(2) Let a vector space V and a linear action $G \times V \to V, (g, v) \mapsto g \cdot v$ be given. Then for all $g \in G$, the map $V \to V, v \mapsto g \cdot v$ is a linear map. The map

$$\begin{array}{rcl} \varphi\colon G & \to & \mathrm{GL}(V) \\ g & \mapsto & (v \mapsto g \cdot v) \end{array}$$

is a homomorphism.

(2)⇒(1) Let a vector space V and a homomorphism $\varphi\colon G \to \mathrm{GL}(V)$ be given. Then

$$\begin{array}{rcl} G \times V & \to & V \\ (g, v) & \mapsto & \varphi(g)(v) \end{array}$$

is a linear action of G on V.

We now give the same examples again using Definition (2).

Example 25. The map

$$\begin{array}{rcl} G & \to & \mathrm{GL}(V) \\ g & \mapsto & \mathrm{id}_V \end{array}$$

is a representation of G. ♠

Example 26. The map

$$\begin{array}{rcl} \mathrm{GL}(V) & \to & \mathrm{GL}(V) \\ \ell & \mapsto & \ell \end{array}$$

is a representation of $\mathrm{GL}(V)$. ♠

Example 27. The map

$$\begin{array}{rcl} S_n & \to & \mathrm{GL}_n \\ \sigma & \mapsto & P_\sigma \end{array}$$

is a representation of S_n ♠

Example 28. The map

$$\begin{array}{rcl} \mathrm{GL}_n & \to & \mathrm{GL}_1 = \mathbb{C} \setminus \{0\} \\ A & \mapsto & \det(A) \end{array}$$

is representation of GL_n. ♠

Example 29. The map

$$\begin{array}{rcl} S_n & \to & \mathrm{GL}_1 = \mathbb{C} \setminus \{0\} \\ \sigma & \mapsto & \operatorname{sign}(\sigma) \end{array}$$

is a representation of S_n. ♠

1. Linear algebra

Definition 30. A *(complex) vector space* is an abelian group V together with a map

$$\begin{aligned} \mathbb{C} \times V &\to V \\ (\lambda, v) &\mapsto \lambda \cdot v \end{aligned}$$

such that $\lambda \cdot (\mu \cdot v) = (\lambda\mu) \cdot v$, $1 \cdot v = v$, $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$, $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$ for all $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$. ♦

Definition 31. A *subspace* W of V is a subset $W \subseteq V$ such that $W \neq \emptyset$, $v + w, \lambda v \in W$ for all $\lambda \in \mathbb{C}$ and $v, w \in W$. The *quotient* V/W is a vector space. As a set, it is $V/W = \{v + W \mid v \in V\}$ where $(v+W) = (v'+W)$ if and only if $v-v' \in W$. We have $\lambda(v+W) := \lambda v+W$ and $(v+W)+(v'+W) = (v+v')+W$ for all $\lambda \in \mathbb{C}$ and $v, v' \in V$. ♦

Definition 32. A map $\ell \colon V \to W$ is *linear* when $\ell(\lambda v + \mu w) = \lambda\ell(v) + \mu\ell(w)$ for all $v, w \in V$ and $\lambda, \mu \in V$. The kernel $\ker(\ell) := \{v \in V \mid \ell(v) = 0\}$ and the image $\operatorname{im}(\ell) := \{\ell(v) \mid v \in V\}$ of a linear map ℓ are subspaces of V, W, respectively. ♦

Let V, W be complex vector spaces.

Definition 33. The *dual* of V is the vector space $V^* := \operatorname{Hom}(V, \mathbb{C})$. Here $\operatorname{Hom}(V, W)$ is the set of linear maps $V \to W$. This is a vector space with $(\lambda f + \mu g)(v) := \lambda f(v) + \mu g(v)$ for all $f, g \in \operatorname{Hom}(V, W)$ and $\lambda, \mu \in \mathbb{C}$. ♦

We have a linear map

$$\begin{aligned} V &\to V^{**} \\ v &\mapsto (v \mapsto f(v) \end{aligned}$$

which is an isomorphism when $\dim V < \infty$.

Definition 34. The *direct sum* of V, W is the vector space $V \oplus W$. As a set, it is $V \times W$. We have

$$\lambda(v, w) := (\lambda v, \lambda w), \quad (v, w) + (v', w') = (v + v', w + w')$$

for all $v, v' \in V$, $w, w' \in W$ and $\lambda \in \mathbb{C}$. ♦

Let V_1, V_2 be subspaces of V, we say that V is the *direct sum* of V_1, V_2 (and write $V = V_1 \oplus V_2$) when every element of V can be written *uniquely* as $v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$.

Definition 35. An *(hermitian) inner product* on V is a map

$$\langle -, - \rangle \colon V \times V \to \mathbb{C}$$

such that the following hold:

- We have $\langle \lambda v + \mu v', w\rangle = \lambda\langle v, w\rangle + \mu\langle v', w\rangle$ for all $v, v', w \in V$ and $\lambda, \mu \in \mathbb{C}$.
- We have $\langle w, v\rangle = \overline{\langle v, w\rangle}$ for all $v, w \in V$.
- We have $\langle v, v\rangle \geq 0$ for all $v \in V$, with $\langle v, v\rangle = 0$ if and only if $v = 0$.

For a subspace $W \subseteq V$, we define its *orthogonal complement* as the subspace

$$W^\perp := \{v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in W\}.$$

We then have $V = W \oplus W^\perp$. ♦

Example 36. The map

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\to \mathbb{C} \\ ((v_1, \ldots, v_n), (w_1, \ldots, w_n)) &\mapsto \sum_{i=1}^n v_i\overline{w_i} \end{aligned}$$

is an (hermitian) inner product on $\mathbb{C}^n$. ♠

Definition 37. The *tensor product* of V, W is a vector space $V \otimes W$. Every element of $V \otimes W$ can written as in a non-unique way as

$$\sum_{i=1}^{k} \lambda_i \cdot (v_i \otimes w_i)$$

where $k \in \mathbb{Z}_{\geq 0}$, $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$, $v_1, \ldots, v_k \in V$ and $w_1, \ldots, w_k \in W$. Conversely, this expression is always an element of $V \otimes W$. The addition and scalar multiplication are defined by

$$\sum_{i=1}^{k} \lambda_i \cdot (v_i \otimes w_i) + \sum_{j=k+1}^{k+\ell} \lambda_j \cdot (v_j \otimes w_j) := \sum_{i=1}^{k+\ell} \lambda_i \cdot (v_i \otimes w_i)$$

and

$$\mu \cdot \sum_{i=1}^{k} \lambda_i \cdot (v_i \otimes w_i) := \sum_{i=1}^{k} (\mu\lambda_i) \cdot (v_i \otimes w_i)$$

for all tensors $\sum_{i=1}^{k} \lambda_i \cdot (v_i \otimes w_i), \sum_{j=k+1}^{k+\ell} \lambda_j \cdot (v_j \otimes w_j) \in V \otimes W$ and $\mu \in \mathbb{C}$.

In $V \otimes W$, the following equalities hold:

- We have $\lambda \cdot (v \otimes w) = (\lambda v) \otimes w$ and $\lambda \cdot (v \otimes w) = v \otimes (\lambda w)$ for all $\lambda \in \mathbb{C}$, $v \in V$ and $w \in W$.
- We have $(v + v') \otimes w = (v \otimes w) + (v' \otimes w)$ for all $v, v' \in V$ and $w \in W$.
- We have $v \otimes (w + w') = (v \otimes w) + (v \otimes w')$ for all $v \in V$ and $w, w' \in W$.

Conversely, if two tensors $T_1, T_2 \in V \otimes W$ are equal, then they can be shown to be equal using these rules. ♦

Example 38. We have

$$0 \otimes w = 0 \otimes w + 0 \otimes w - 0 \otimes w = (0 + 0) \otimes w - 0 \otimes w = 0$$

for all $w \in W$. ♠

Example 39. We have

$$\begin{aligned}
(v_1 + v_2) \otimes (w_1 + w_2) - v_1 \otimes w_2 - v_2 \otimes w_1 &= v_1 \otimes (w_1 + w_2) + v_2 \otimes (w_1 + w_2) - v_1 \otimes w_2 - v_2 \otimes w_1 \\
&= v_1 \otimes w_1 + v_1 \otimes w_2 + v_2 \otimes w_1 + v_2 \otimes w_2 - v_1 \otimes w_2 - v_2 \otimes w_1 \\
&= v_1 \otimes w_1 + v_2 \otimes w_2
\end{aligned}$$

for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$. ♠

Lemma 40. *Let $\varphi\colon V \to V'$ and $\psi\colon W \to W'$ be linear maps. Then*

$$\begin{aligned}
\varphi \otimes \psi\colon V \otimes W &\to V' \otimes W' \\
\sum_{i=1}^{k} \lambda_i \cdot (v_i \otimes w_i) &\mapsto \sum_{i=1}^{k} \lambda_i \varphi(v_i) \psi(w_i)
\end{aligned}$$

is a linear map.

Proof. Proving that $\varphi \otimes \psi$ is linear is not hard, but we need to prove that the map is well-defined! This means

$$\sum_{i=1}^{k} \lambda_i \cdot (v_i \otimes w_i) = \sum_{j=1}^{\ell} \mu_j \cdot (x_j \otimes y_j) \Rightarrow \sum_{i=1}^{k} \lambda_i \varphi(v_i) \psi(w_i) = \sum_{j=1}^{\ell} \mu_j \varphi(x_j) \psi(y_j).$$

For this we check the following:

- We have $\lambda \varphi(v) \psi(w) = \varphi(\lambda v) \psi(w)$ and $\lambda \cdot \varphi(v) \psi(w) = \varphi(v) \psi(\lambda w)$ for all $\lambda \in \mathbb{C}$, $v \in V$ and $w \in W$.
- We have $\varphi(v + v') \psi(w) = \varphi(v) \psi(w) + \varphi(v') \psi(w)$ for all $v, v' \in V$ and $w \in W$.
- We have $\varphi(v) \psi(w + w') = \varphi(v) \psi(w) + \varphi(v) \psi(w')$ for all $v \in V$ and $w, w' \in W$.

When two elements in $V \otimes W$ are equal, we can get from one to the other in steps. We have shown that in each step, the value of $\varphi \otimes \psi$ does not change. So $\varphi \otimes \psi$ is well-defined. □

Proposition 41. *If $\{v_1, \ldots, v_n\}$ is a basis of V and $\{w_1, \ldots, w_m\}$ is a basis of W, then*

$$\{v_i \otimes w_j \mid i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$$

is a basis of $V \otimes W$.

Proof. It is easy to see that this set spans $V \otimes W$: we can always write

$$\lambda \cdot (v \otimes w) = \lambda \cdot ((a_1 v_1 + \ldots + a_n v_n) \otimes (b_1 w_1 + \ldots + b_m w_m)) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda a_i b_j \cdot (v_i \otimes w_j)$$

for some $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathbb{C}$. So $\lambda \cdot (v \otimes w)$ is in the span of the set. So sums of such terms are as well.

Suppose that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \cdot (v_i \otimes w_j) = 0$$

for some $a_{ij} \in \mathbb{C}$. We need to show that $a_{ij} = 0$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Fix one such pair (i, j). Let $\varphi\colon V \to \mathbb{C}$ be a linear map sending $v_i \mapsto 1$ and $v_k \mapsto 0$ for $k \neq i$. Let $\psi\colon W \to \mathbb{C}$ be a linear map sending $w_j \mapsto 1$ and $w_k \mapsto 0$ for $k \neq j$. We claim that

$$\begin{aligned} \varphi \otimes \psi\colon V \otimes W &\to \mathbb{C} \\ \sum_{k=1}^{\ell} \lambda_i (v_k \otimes w_k) &\mapsto \sum_{k=1}^{\ell} \lambda_i \varphi(v_k) \psi(w_k) \end{aligned}$$

is a linear map. Since $\varphi \otimes \psi$ is a linear map, we have

$$a_{ij} = (\varphi \otimes \psi) \left(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \cdot (v_i \otimes w_j) \right) = (\varphi \otimes \psi)(0) = 0,$$

which is what we want to prove. □

Example 42. The map

$$\begin{aligned} V &\to V \otimes \mathbb{C} \\ v &\mapsto v \otimes 1 \end{aligned}$$

is an isomorphism. ♠

Example 43. The map

$$\begin{aligned} \mathbb{C}^{n \times m} &\to \mathbb{C}^n \otimes \mathbb{C}^m \\ A &\mapsto \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} \cdot e_i \otimes e_j \end{aligned}$$

is an isomorphism. ♠