Representation theory of finite groups

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MAX-PLANCK-GESELLSCHAFT

Exercise 8.11

Let G be a group and H a subgroup. Let $\rho\colon H \to \mathrm{GL}(V)$ be a representation. Let $\mathrm{Hom}_H(G,V)$ be the vector space of all maps $f\colon G \to V$ such that

$$f(gh) = \rho(h)^{-1}(f(g))$$

for all $g \in G$ and $h \in H$. Define

$$g \cdot f := (x \mapsto f(g^{-1}x))$$

for all $g \in G$ and $f \in \mathrm{Hom}_H(G,V)$. Prove that $\mathrm{Hom}_H(G,V)$ is a representation of G that is isomorphic to $\mathrm{Ind}_H^G V$.

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First, check that $(x \mapsto f(g^{-1}x)) \in \mathrm{Hom}_H(G, V)$ for all $g \in G$ and $f \in \mathrm{Hom}_H(G, V)$. Then check the axioms for a representation.

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For $g, g' \in G$, $h \in H$, $f \in \mathrm{Hom}_H(G, V)$ and $\hat{f} = (x \mapsto f(g^{-1}x))$, we have

$$\hat{f}(g'h) = f(g^{-1}g'h) = \rho(h)^{-1}(f(g^{-1}g')) = \rho(h)^{-1}(\hat{f}(g))$$

and so indeed $\hat{f} \in \mathrm{Hom}_H(G, V)$.

Checking the axioms in not hard.

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Checking the axioms in not hard.

Let $v_1, \ldots, v_n$ be a basis of V and $t_1, \ldots, t_m$ a complete set of representatives of the cosets of H in G. Then

$$\chi_{\mathrm{Ind}_H^G V}(g) = \sum_{i=1}^{m} \dot{\chi}_V(t_i^{-1} g t_i)$$

We want to prove that $\chi_{\mathrm{Hom}_H(G,V)}$ is the same function.

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Let $v_1, \ldots, v_n$ be a basis of V and $t_1, \ldots, t_m$ a complete set of representatives of the cosets of H in G.

Take $f_{ij} \in \mathrm{Hom}_H(G, V)$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ as

$$f_{ij}(x) = \begin{cases} \rho(t_i^{-1}x)^{-1}(v_j) & \text{if } x \in t_iH, \\ 0 & \text{otherwise} \end{cases}$$

Then the f_{ij}'s form a basis of $\mathrm{Hom}_H(G, V)$.

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Then the f_{ij}'s form a basis of $\mathrm{Hom}_H(G, V)$.

Write $f = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} f_{ij}$. Then

$$\mathrm{coeff}_k(f(t_\ell)) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \, \mathrm{coeff}_k(f_{ij}(t_\ell)) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \, \mathrm{coeff}_k(\delta_{i\ell} v_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \delta_{i\ell} \delta_{jk} = c_{\ell k} \qquad \text{where } \mathrm{coeff}_k(\lambda_1 v_1 + \ldots + \lambda_n v_n) = \lambda_k$$

Exercise 8.11

Now, we have

$$\begin{aligned}
\chi_{\mathrm{Hom}_H(G,V)}(g) &= \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{coeff}_j((g\cdot f_{ij})(t_i)) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{coeff}_j(f_{ij}(g^{-1}t_i)) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{coeff}_j\left(\left\{\begin{array}{cl} \rho(t_i^{-1}g^{-1}t_i)^{-1}(v_j) & \text{if } g^{-1}t_i \in t_iH \\ 0 & \text{otherwise} \end{array}\right\}\right) \\
&= \sum_{i=1}^{m} \left\{\begin{array}{cl} \sum_{j=1}^{n} \mathrm{coeff}_j(\rho(t_i^{-1}gt_i)(v_j)) & \text{if } t_i^{-1}gt_i \in H \\ 0 & \text{otherwise} \end{array}\right\} \\
&= \sum_{i=1}^{m} \left\{\begin{array}{cl} \chi_V(t_i^{-1}gt_i) & \text{if } t_i^{-1}gt_i \in H \\ 0 & \text{otherwise} \end{array}\right\} = \sum_{i=1}^{m} \dot{\chi}_V(t_i^{-1}gt_i)
\end{aligned}$$

The group ring $\mathbb{C}[G]$

Let G be a finite group.

Definition

The *group ring* $\mathbb{C}[G]$ is a $\mathbb{C}$-*algebra* we define as follows:

- For every $g \in G$, we let $[g]$ be a new symbol.
- We define $\mathbb{C}[G]$ as the vector space with basis $\{[g] \mid g \in G\}$. So the element of $\mathbb{C}[G]$ can all be written uniquely as:

$$\sum_{g \in G} c_g \cdot [g]$$

- We define the multiplication operation:

$$\begin{aligned} - \cdot - \colon \mathbb{C}[G] \times \mathbb{C}[G] &\to \mathbb{C}[G] \\ \left(\sum_{g \in G} c_g \cdot [g], \sum_{h \in G} b_h \cdot [h] \right) &\mapsto \sum_{g,h \in G} c_g b_h \cdot [gh] \end{aligned}$$

The group ring $\mathbb{C}[G]$

That $\mathbb{C}[G]$ is a $\mathbb{C}$-algebra means that the following axioms are satisfied:

1. The map $- \cdot -$ is bilinear.
2. We have the element $[e]$ with $[e] \cdot x = x$ for all $x \in \mathbb{C}[G]$.
3. We have $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \mathbb{C}[G]$.

(left) $\mathbb{C}[G]$-modules

Definition

A (left) $\mathbb{C}[G]$-module is a vector space V together with a map

$$\begin{array}{rcl} \mathbb{C}[G] \times V & \to & V \\ (x, v) & \mapsto & x \cdot v \end{array}$$

such that the following hold:

- We have $x \cdot (\lambda v + \mu w) = \lambda(x \cdot v) + \mu(x \cdot w)$ for all $x \in \mathbb{C}[G]$, $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$.
- We have $(\lambda x + \mu y) \cdot v = \lambda(x \cdot v) + \mu(y \cdot v)$ for all $x, y \in \mathbb{C}[G]$, $\lambda, \mu \in \mathbb{C}$ and $v \in V$.
- We have $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ for all $x, y \in \mathbb{C}[G]$ and $v \in V$.
- We have $[e] \cdot v = v$ for all $v \in V$.

(left) $\mathbb{C}[G]$-modules

Example

The vector space $\mathbb{C}[G]$ is a (left) $\mathbb{C}[G]$-module with the map

$$\begin{array}{rcl} \mathbb{C}[G] \times \mathbb{C}[G] & \to & \mathbb{C}[G] \\ (x, v) & \mapsto & x \cdot v \end{array}$$

Example

Let V be a representation of G. Then V is a (left) $\mathbb{C}[G]$-module with the map

$$\begin{array}{rcl} \mathbb{C}[G] \times V & \to & V \\ \left(\sum_{g \in G} c_g [g], v \right) & \mapsto & \sum_{g \in G} c_g (g \cdot v) \end{array}$$

(left) $\mathbb{C}[G]$-modules

It also goes the other way around:

When V is a (left) $\mathbb{C}[G]$-module, then V is also a representation of G via

$$g \cdot v := [g] \cdot v$$

right $\mathbb{C}[G]$-modules

Definition

A right $\mathbb{C}[G]$-module is a vector space V together with a map

$$\begin{array}{rcl} V \times \mathbb{C}[G] & \to & V \\ (v, x) & \mapsto & v \cdot x \end{array}$$

such that the following hold:

- We have $(\lambda v + \mu w) \cdot x = \lambda(v \cdot x) + \mu(w \cdot x)$ for all $x \in \mathbb{C}[G]$, $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$.
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- We have $(v \cdot x) \cdot y = v \cdot (x \cdot y)$ for all $x, y \in \mathbb{C}[G]$ and $v \in V$.
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right $\mathbb{C}[G]$-modules

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Example

Let V be a representation of G. Then V is a right $\mathbb{C}[G]$-module with the map

$$\begin{array}{rcl} V \times \mathbb{C}[G] & \to & V \\ \left(v, \sum_{g \in G} c_g [g]\right) & \mapsto & \sum_{g \in G} c_g (g^{-1} \cdot v) \end{array}$$

Tensor product over $\mathbb{C}[G]$

Definition

Let V be a right $\mathbb{C}[G]$-module and W a left $\mathbb{C}[G]$-module. Then $V \otimes_{\mathbb{C}[G]} W$ is the vector space spanned by elements $v \otimes w$ where we have the relations

$$\begin{aligned} (\lambda v_1 + \mu v_2) \otimes w &= \lambda \cdot v_1 \otimes w + \mu \cdot v_2 \otimes w \\ v \otimes (\lambda w_1 + \mu w_2) &= \lambda v \otimes w_1 + \mu \cdot v \otimes w_2 \\ (v \cdot x) \otimes w &= v \otimes (x \cdot w) \end{aligned}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $\lambda, \mu \in \mathbb{C}$ and $x \in \mathbb{C}[G]$.

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(v \cdot x) \otimes w &= v \otimes (x \cdot w)
\end{aligned}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $\lambda, \mu \in \mathbb{C}$ and $x \in \mathbb{C}[G]$.

Why all this stuff with left and right?

$$(v \cdot (xy)) \otimes w = ((v \cdot x) \cdot y) \otimes w = (v \cdot x) \otimes (y \cdot w)$$

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Another construction of $\mathrm{Ind}_H^G V$

Let H be a subgroup of G and V a representation of H. Then V is also a (left) $\mathbb{C}[H]$-module via

$$\sum_{h \in H} c_h [h] \cdot v := \sum_{h \in H} c_h (h \cdot v)$$

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Also, the vector space $\mathbb{C}[G]$ is a right $\mathbb{C}[H]$-module by restricting the usual multiplication map $\mathbb{C}[G]\times\mathbb{C}[G]\to\mathbb{C}[G]$ to a map $\mathbb{C}[G]\times\mathbb{C}[H]\to\mathbb{C}[G]$. Hence, we get a vector space

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spanned by elements of the form $[x]\otimes v$ with $x\in G$ and $v\in V$.

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spanned by elements of the form $[x]\otimes v$ with $x\in G$ and $v\in V$.

We now turn this into a representation of G via:

$$g\cdot([x]\otimes v) = [gx]\otimes v$$

Another construction of $\operatorname{Ind}_H^G V$

Now, why does $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \cong \operatorname{Ind}_H^G V$ hold?

Claim: Let $t_1, \dots, t_m$ be a complete set of representatives of the cosets of H in G and let $v_1, \dots, v_n$ be a basis of V. Then

$$\{[t_i] \otimes v_j \mid i \in \{1, \dots, m\}, j \in \{1, \dots, n\}\}$$

is a basis of $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

Now, compute the character or find an explicit isomorphism.

Another construction of $\mathrm{Ind}_H^G V$

Example

Take $H = \{1\}$. Then $\mathbb{C}[H] \cong \mathbb{C}$. Let $V = \mathbb{C}$ be the trivial representation. Then

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \mathbb{C}[G] \otimes \mathbb{C} \cong \mathbb{C}[G]$$

is the regular representation.

When is $\mathrm{Ind}_H^G V$ irreducible?

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We have $\langle \operatorname{Ind}_H^G \chi_V, \operatorname{Ind}_H^G \chi_V \rangle = \langle \chi_V, \operatorname{Res}_H^G \operatorname{Ind}_H^G V \rangle$.

Today, we focus on the map

$$\begin{array}{rcl} \operatorname{Class}(K) & \to & \operatorname{Class}(H) \\ f & \mapsto & \operatorname{Res}_H^G \operatorname{Ind}_K^G f \end{array}$$

where H, K are subgroups of G.

Double cosets

Let H, K be subgroups of G

Definition

For $g \in G$, we define the *double coset* of g to be

$$HgK = \{hgk \mid h \in H, k \in K\}$$

For $g, g' \in G$, we have $HgK = Hg'K$ or $HgK \cap Hg'K = \emptyset$ and so

$$G = \bigcup_{g \in G} HgK$$

is a disjoint union.

Remark

When $H = K$ is normal, then $HgK = HgH = gHH = gH$ is just the coset of g.

Mackey's Theorem

Theorem (Mackey)

Let H, K be subgroups of G and let S be a complete set of double coset representatives. Then we have

$$\operatorname{Res}^G_H \operatorname{Ind}^G_K f = \sum_{s\in S} \operatorname{Ind}^H_{H\cap sKs^{-1}} \operatorname{Res}^{sKs^{-1}}_{H\cap sKs^{-1}} f^s$$

for all $f \in \operatorname{Class}(K)$, where $f^s \in \operatorname{Class}(sKs^{-1})$ is defined by

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Goal: Compute "good" complete set of representatives of the cosets of K in G.

Proof

Choose, for each $s \in S$, a complete set V_s of representatives of the cosets of $H \cap sKs^{-1}$ in H. Then

$$H = \bigcup_{v \in V_s} v(H \cap sKs^{-1}) \qquad \text{(disjoint union)}$$

and so

$$HsK = \bigcup_{v \in V_s} v(H \cap sKs^{-1})(sKs^{-1})s = \bigcup_{v \in V_s} v(sKs^{-1})s = \bigcup_{v \in V_s} vsK$$

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Claim: This union is disjoint.

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Claim: This union is disjoint.
Assume that $vsK = v'sK$ with $v, v' \in V_s$.
Then $v^{-1}v' \in H \cap sKs^{-1}$ and so $v = v'$.

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Assume that $vsK = v'sK$ with $v, v' \in V_s$.
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We get disjoint union $G = \bigcup_{s \in S} HsK = \bigcup_{s \in S} \bigcup_{v \in V_s} vsK$.

Mackey's Theorem

Now, we have

$$
\begin{aligned}
\operatorname{Res}_H^G \operatorname{Ind}_K^G f(h) &= \sum_{s\in S}\sum_{v\in V_s} \dot{f}((vs)^{-1}h(vs)) \\
&= \sum_{s\in S}\sum_{v\in V_s} \dot{f}(s^{-1}(v^{-1}hv)s) \\
&= \sum_{s\in S}\sum_{\substack{v\in V_s \\ v^{-1}hv\in sKs^{-1}}} f(s^{-1}(v^{-1}hv)s) \\
&= \sum_{s\in S}\sum_{\substack{v\in V_s \\ v^{-1}hv\in sKs^{-1}}} \operatorname{Res}^K_{H\cap sKs^{-1}} f^s(v^{-1}hv) \\
&= \sum_{s\in S} \operatorname{Ind}^H_{H\cap sKs^{-1}} \operatorname{Res}^K_{H\cap sKs^{-1}} f^s
\end{aligned}
$$

Mackey's irreducibility criterion

Theorem (Mackey's irreducibility criterion)

Let H be a subgroup of G and let $\varphi\colon H \to \mathrm{GL}(V)$ be a representation with character χ. Then $\mathrm{Ind}_H^G \varphi$ is irreducible if and only if φ is irreducible and

$$\langle \mathrm{Res}^H_{H\cap sHs^{-1}} \varphi, \mathrm{Res}^{sHs^{-1}}_{H\cap sHs^{-1}} \varphi^s \rangle = 0$$

for all $s \in G \setminus H$, where $\varphi^s(x) := \varphi(s^{-1}xs)$ for all $x \in sHs^{-1}$.

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for all $s \in G \setminus H$, where $\varphi^s(x) := \varphi(s^{-1}xs)$ for all $x \in sHs^{-1}$.

Proof

Let S be a complete set of representatives of the double cosets of H, H and assume that $1 \in S$. Write $S^\# = S \setminus \{1\}$. Then

$$\begin{aligned} \operatorname{Res}^G_H \operatorname{Ind}^G_H \chi &= \sum_{s\in S} \operatorname{Ind}^H_{H\cap sHs^{-1}} \operatorname{Res}^{sHs^{-1}}_{H\cap sHs^{-1}} \chi^s \\ &= \chi + \sum_{s\in S^\#} \operatorname{Ind}^H_{H\cap sHs^{-1}} \operatorname{Res}^{sHs^{-1}}_{H\cap sHs^{-1}} \chi^s \end{aligned}$$

Mackey's irreducibility criterion

We get

$$\begin{aligned}
\langle \operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi \rangle &= \langle \chi, \operatorname{Res}_H^G \operatorname{Ind}_H^G \chi \rangle \\
&= \langle \chi, \chi \rangle + \sum_{s \in S^\#} \langle \chi, \operatorname{Ind}_{H \cap sHs^{-1}}^H \operatorname{Res}_{H \cap sHs^{-1}}^{sHs^{-1}} \chi^s \rangle \\
&= \langle \chi, \chi \rangle + \sum_{s \in S^\#} \langle \operatorname{Res}_{H \cap sHs^{-1}}^H \chi, \operatorname{Res}_{H \cap sHs^{-1}}^{sHs^{-1}} \chi^s \rangle
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So we see that $\operatorname{Ind}_H^G \chi$ is the character of an irreducible representation if and only if χ is irreducible and

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for all $s \in S^\#$.

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for all $s \in S^\#$.

Now, this holds for one choice of S if and only if it holds for all choices of S. And every element of $G \setminus H$ can be an element of $S^\#$. This gives the result.

Theorem (Mackey's irreducibility criterion)

Let H be a subgroup of G and let $\varphi\colon H \to \mathrm{GL}(V)$ be a representation with character χ. Then $\mathrm{Ind}_H^G \varphi$ is irreducible if and only if φ is irreducible and

$$\langle \mathrm{Res}^{H}_{H\cap sHs^{-1}} \varphi, \mathrm{Res}^{sHs^{-1}}_{H\cap sHs^{-1}} \varphi^s \rangle = 0$$

for all $s \in G \setminus H$, where $\varphi^s(x) := \varphi(s^{-1}xs)$ for all $x \in sHs^{-1}$.

What happens when H is a normal subgroup of G?

Mackey's irreducibility criterion

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Let H be a subgroup of G and let $\varphi\colon H \to \mathrm{GL}(V)$ be a representation with character χ. Then $\mathrm{Ind}_H^G \varphi$ is irreducible if and only if φ is irreducible and

$$\langle \mathrm{Res}^{H}_{H\cap sHs^{-1}} \varphi, \mathrm{Res}^{sHs^{-1}}_{H\cap sHs^{-1}} \varphi^s \rangle = 0$$

for all $s \in G \setminus H$, where $\varphi^s(x) := \varphi(s^{-1}xs)$ for all $x \in sHs^{-1}$.

What happens when H is a normal subgroup of G?

- We have $H = sHs^{-1} = H \cap sHs^{-1}$. (can ignore Res)
- If φ irreducible, then φ^s irreducible.
- So just need to check that $\chi_\varphi \neq \chi_{\varphi^s}$ for $s \in G \setminus H$.
- Actually, is enough to take one s for each coset of H. (except for H itself, we don't check this coset)