Exercises 7

1. Let p be a prime and let G be a group of order p^2. Then G is abelian.

The commutator subgroup $[G, G]$ of a group G is the subgroup generated by all elements of the form $[g, h] := g^{-1}h^{-1}gh$ with $g, h \in G$. It is a normal subgroup, which means that $g[G, G]g^{-1} = [G, G]$ for all $g \in G$. For $g \in G$, we write $g[G, G] := \{gh \mid h \in [G, G]\} \subseteq G$. The set $G/[G, G] := \{g[G, G] \mid g \in G\}$ is an abelian group with $g[G, G] \cdot h[G, G] = gh[G, G]$ for all $g, h \in G$. This quotient group has the following property: every homomorphism from G to an abelian group A can be uniquely written as $f \circ \pi$ where $f\colon G/[G, G] \to A$ is a homomorphism and $\pi\colon G \to G/[G, G], g \mapsto g[G, G]$ is the quotient homomorphism.

2. Prove that the number of non-isomorphic 1-dimensional representations of G equals $|G/[G, G]|$. Use the fact that $|G/[G, G]| = |G|/|[G, G]|$ to show that the number of 1-dimensional representations of G divides $|G|$.

3. Let p, q be primes with $p < q$ and $q \not\equiv 1 \mod p$. Prove that any group G of order pq is abelian.

4. Exercises 6.2, 6.3