Exercises 3

1. Let $V_1, \ldots, V_k$ be finite-dimensional representations of G. Let $a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_k \in \mathbb{Z}_{\geq 0}$. Consider the following G-linear maps

$$\begin{array}{rcl} \varphi\colon V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} & \to & V_1^{\oplus b_1} \oplus \cdots \oplus V_k^{\oplus b_k}, \\ (v_1, \ldots, v_k) & \mapsto & (A_1 v_1, \ldots, A_k v_k) \end{array} \qquad \begin{array}{rcl} \psi\colon V_1^{\oplus b_1} \oplus \cdots \oplus V_k^{\oplus b_k} & \to & V_1^{\oplus c_1} \oplus \cdots \oplus V_k^{\oplus c_k} \\ (w_1, \ldots, w_k) & \mapsto & (B_1 w_1, \ldots, B_k w_k) \end{array}$$

where $A_i \in \mathbb{C}^{b_i \times a_i}$ and $B_i \in \mathbb{C}^{c_i \times a_i}$. Prove the following:

(a) The composition $\psi \circ \varphi\colon V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \to V_1^{\oplus c_1} \oplus \cdots \oplus V_k^{\oplus c_k}$ is given by

$$(v_1, \ldots, v_k) \mapsto ((B_1 A_1) v_1, \ldots, (B_k A_k) v_k)$$

Hint: First consider the case where $V_i = \mathbb{C}$ is the trivial representation for all i. The proof of the general case is the same.

(b) The map φ is the identity map if and only if $a_i = b_i$ and $A_i = I_{a_i}$ for all i.

(c) The map φ is invertible if and only if $a_i = b_i$ and $\det(A_i) \neq 0$ for all i.

(d) The kernel of φ is isomorphic to $V_1^{\oplus \dim \ker(A_1)} \oplus \cdots \oplus V_k^{\oplus \dim \ker(A_k)}$.

(e) The image of φ is isomorphic to $V_1^{\oplus \operatorname{rk}(A_1)} \oplus \cdots \oplus V_k^{\oplus \operatorname{rk}(A_k)}$.

2. Exercise 3.5 from the book.

3. Consider the representations $\mathbb{C}^n$ of GL_n and $\mathbb{C}^m$ of GL_m both defined by $P \cdot v := Pv$. Prove that the representations $\mathbb{C}^n \otimes \mathbb{C}^m$ of $\mathrm{GL}_n \times \mathrm{GL}_m$ is isomorphic to the representation $\mathbb{C}^{n \times m}$ of $\mathrm{GL}_n \times \mathrm{GL}_m$ defined by $(P, Q) \cdot A := PAQ^\top$.

4. Let V, W be isomorphic finite-dimensional representations of G. Prove the following:

(a) If V is irreducible, then so is W.

(b) If V is decomposable, then so is W.

(c) If V is completely reducible, then so is W.

5. Let V be a finite-dimensional representation of G. We define a new representation V' of G as follows: as a vector space we take $V' := V$ and action of G on V' is defined by $g * v := g^{-\top} \cdot v$. In other words, the new representation is defined by the homomorphism $(g \mapsto g * v) = (g \mapsto (v \mapsto g \cdot v)) \circ (g \mapsto g^{-\top})$. Prove the following:

(a) V' is a representation of G.

(b) We have $(V')' = V$.

(c) If V is irreducible/decomposable/completely decomposable, then V' is as well.

(d) Find an example where $V \not\cong V'$.