Exercises 2

1. Let $\varphi\colon V \to W$ and $\psi\colon U \to V$ be G-linear maps. Prove that the composition $\varphi \circ \psi$ is a G-linear map.

2. Let V be a finite-dimensional representation of G.
(a) Prove that the dual space V^* is a representation of G together with the map

$$\begin{array}{rcl} G \times V^* & \to & V^* \\ (g, \varphi) & \mapsto & (v \mapsto \varphi(g^{-1} \cdot v)) \end{array}$$

(b) Prove that V is irreducible if and only V^* is.
Hint: $V \cong V^{**}$

3. Find all S_3-linear maps from the standard representation V_{std} to itself.

4. Find all S_3-linear maps $V_{\text{alt}} \to V_{\text{std}}$.

5. Let V_{triv}, V_{alt} be the trivial and alternating representations of S_n. Prove that $V_{\text{alt}} \otimes V_{\text{alt}} \cong V_{\text{triv}}$.

6. Consider the subrepresentation $V = \{(v_1, v_2, v_3) \in \mathbb{C}^3 \mid v_1 + v_2 + v_3 = 0\}$ of the representation V_{std} of the group S_3. Find an isomorphism between $V \otimes V$ and a direct sum of irreducible representations of S_3.