Exercises 1

1. Let G be a finite group. Then $\mathbb{C}[G]$ is a $\mathbb{C}$-*algebra* we define as follows:

- For every $g \in G$, we let $[g]$ be a new symbol.
- We define $\mathbb{C}[G]$ as the vector space with basis $\{[g] \mid g \in G\}$.
- We define the multiplication operation:

$$\begin{array}{rcl} - \cdot -\colon \mathbb{C}[G] \times \mathbb{C}[G] & \to & \mathbb{C}[G] \\ \left(\sum_{g \in G} c_g \cdot [g], \sum_{h \in G} b_h \cdot [h] \right) & \mapsto & \sum_{g,h \in G} c_g b_h \cdot [gh] \end{array}$$

- Warning: When G is abelian this means that $[x] + [y] \neq [x+y] = [x] \cdot [y]$.

That $\mathbb{C}[G]$ is a $\mathbb{C}$-algebra means that the following axioms are satisfied:

(1) The map $- \cdot -$ is bilinear.
(2) We have the element $[e]$ with $[e] \cdot x = x$ for all $x \in \mathbb{C}[G]$.
(3) We have $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \mathbb{C}[G]$.

A $\mathbb{C}[G]$-module is a vector space V together with a map

$$\begin{array}{rcl} \mathbb{C}[G] \times V & \to & V \\ (x, v) & \mapsto & x \cdot v \end{array}$$

such that the following hold:

- We have $x \cdot (\lambda v + \mu w) = \lambda(x \cdot v) + \mu(x \cdot w)$ for all $x \in \mathbb{C}[G]$, $\lambda, \mu \in \mathbb{C}$ and $v, w \in V$.
- We have $(x + y) \cdot v = (x \cdot v) + (y \cdot v)$ for all $x, y \in \mathbb{C}[G]$ and $v \in V$.
- We have $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ for all $x, y \in \mathbb{C}[G]$ and $v \in V$.
- We have $[e] \cdot v = v$ for all $v \in V$.

Prove the following:
(a) Let V be a representation of G (so that $g \cdot v$ is defined for all $g \in G$ and $v \in V$). Then V together with the map

$$\begin{array}{rcl} \mathbb{C}[G] \times V & \to & V \\ \left(\sum_{g \in G} c_g \cdot [g], v \right) & \mapsto & \sum_{g \in G} c_g (g \cdot v) \end{array}$$

is a $\mathbb{C}[G]$-module.
(b) Let V be a module of $\mathbb{C}[G]$ (so that $x \cdot v$ is defined for all $x \in \mathbb{C}[G]$ and $v \in V$). Then V together with the map

$$\begin{array}{rcl} G \times V & \to & V \\ (g, v) & \mapsto & [g] \cdot v \end{array}$$

is a representation of G.

2. Let V, W be finite dimensional vector spaces. Prove that the map

$$\begin{array}{rcl} V^* \otimes W & \to & \operatorname{Hom}(V, W) \\ \sum_{i=1}^{k} \lambda_i \cdot \psi_i \otimes w_i & \mapsto & \left(v \mapsto \sum_{i=1}^{k} \lambda_i \psi_i(v) w_i \right) \end{array}$$

is a linear isomorphism.