Final exam - Representation theory of finite groups

Leipzig, 9:30-11:30, July 22th, 2022

Notes, books and calculators are allowed during the exam. Please make sure your NAME in on every sheet of paper you hand in and your EMAIL ADDRESS is on the first sheet.

1. Let V be a representation of a group G. Let $V^* := \mathrm{Hom}(V, \mathbb{C})$ be the space of linear maps $V \to \mathbb{C}$.
(a) Show that the map

$$\begin{array}{rcl} G \times V^* & \to & V^* \\ (g, L) & \mapsto & \big(v \mapsto L(g^{-1} \cdot v)\big) \end{array}$$

is well-defined and gives V^* the structure of a representation of G. This is the *dual* representation of V.
(b) Show that the map

$$\begin{array}{rcl} \mathrm{eval}\colon V & \to & (V^*)^* \\ v & \mapsto & (L \mapsto L(v)) \end{array}$$

is an isomorphism of representations of G.
(Hint: To show that the linear map eval is invertible, recall that V has a finite basis.)
(c) Show that V is irreducible if and only if V^* is irreducible.

2. Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group defined by the equations

$$i^2 = j^2 = k^2 = ijk = -1.$$

(So for example $i \cdot j = -ij \cdot -1 = (-ijk)k = k$.)
(a) Determine all 1-dimensional representations of Q.
(b) Determine the dimensions of the representations in a complete list of irreducible representations of Q (and how often each of these dimensions occurs).
(c) Determine the conjugacy classes of Q.
(d) Determine the character table of Q.

Consider the subgroup $H = \{\pm 1, \pm i\}$ of Q. Let $V = \mathbb{C}$ be the representation of H such that $h \cdot v = hv$ for all $h \in H$ and $v \in V$ (the product on the right-hand-side is the usual multiplication of complex number).

(e) Is $\mathrm{Ind}_H^Q V$ an irreducible representation of Q?
(f) Give a complete list of irreducible representations of Q.

3. Let $V_{\text{std}} = \mathbb{C}^4$ be the standard representation of S_4 defined by

$$\sigma \cdot (x_1, x_2, x_3, x_4) = \big(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}, x_{\sigma^{-1}(4)}\big)$$

for all $\sigma \in S_4$ and $x_1, x_2, x_3, x_4 \in \mathbb{C}$.
(a) Describe a complete list of irreducible representations $V_1, \ldots, V_s$ of the group S_4.
(b) Compute the characters of $V_1, \ldots, V_s$.
(c) A representation V is called *self-dual* when $V \cong V^*$. Is V_{std} self-dual?
(d) Find constants $a_1, \ldots, a_s \in \mathbb{Z}_{\geq 0}$ such that

$$V_{\text{std}}^{\otimes 2022} \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_s^{a_s}.$$

(It is enough to to give formulas for these numbers, you don't need compute these formulas.)