Strength of Polynomials

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The strength of polynomials

Let f be a homogeneous polynomial of degree $d \geq 2$ over $\mathbb{C}$.

Definition

The *strength* of f is the minimal number $\operatorname{str}(f) := r \geq 0$ such that

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

with $g_1, h_1, \ldots, g_r, h_r$ homogeneous polynomials of degree $\leq d - 1$.

Question

What is the strength of $f := x^2 + y^2 + z^2$?

- We have $\operatorname{str}(f) \leq 3$ since $f = x \cdot x + y \cdot y + z \cdot z$.
- We have $\operatorname{str}(f) > 0$ since $f \neq 0$.
- We have $\operatorname{str}(f) > 1$ since f is not reducible.
- We have $\operatorname{str}(f) \leq 2$ since $f = (x + iy) \cdot (x - iy) + z \cdot z$.

So $\operatorname{str}(f) = 2$ (but over $\mathbb{R}$ is would be 3).

Why care about strength?

Reason 1 - Data efficiency

A homogeneous polynomial of degree d in $n+1$ variables has

$$\binom{n+d}{d}$$

coefficients.

A polynomial of degree 3 in 10^6 variables has

$$\approx 10^{17}$$

coefficients.

The number of coefficients in a strength decomposition is:

$$\approx \operatorname{str}(f) \cdot 10^{12}$$

So the strength decomposition uses $\approx 10^5 / \operatorname{str}(f)$ times less space.

Why care about strength?

Reason 2 - Universality

Let $f \in \mathbb{C}[x_1, \ldots, x_n]_d$. For $\star, \ldots, \star \in \mathbb{C}$, the polynomial

$$f(\star y_1 + \ldots + \star y_m, \ldots, \star y_1 + \ldots + \star y_m) \in \mathbb{C}[y_1, \ldots, y_m]_d$$

is a coordinate transformation of f.

Let $\mathcal{P}$ be a property of degree-d polynomials such that

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{every coordinate transformation of } f \text{ has } \mathcal{P}$$

Examples

- $\mathcal{P}_{\mathsf{triv}}$: the polynomial equals itself
- $\mathcal{P}_k$: the polynomial has strength $\leq k$
- $\mathcal{P}_{\mathsf{KZ},\ell}$: every partial derivative of the polynomial has strength $\leq \ell$

Theorem (Kazhdan-Ziegler, B-Danelon-Draisma-Eggermont)

Either $\mathcal{P} = \mathcal{P}_{\mathsf{triv}}$ or there exists a $k \geq 0$ such that

$$f \text{ has } \mathcal{P} \Rightarrow \mathrm{str}(f) \leq k$$

Why care about strength?

Reason 3 - It is like the rank of matrices

We have a one-to-one correspondence

$$\begin{array}{rcl} \{A \in \mathbb{C}^{n\times n} \mid A = A^\top\} & \leftrightarrow & \mathbb{C}[x_1,\dots,x_n]_2 \\ A & \mapsto & (x_1,\dots,x_n)A(x_1,\dots,x_n)^\top \\ (a_1,\dots,a_n)^\top(a_1,\dots,a_n) & \mapsto & (a_1x_1+\ldots+a_nx_n)^2 \\ vw^\top + wv^\top & \mapsto & 2\cdot(x_1,\dots,x_n)v\cdot(x_1,\dots,x_n)w \end{array}$$

Write $f = (x_1,\dots,x_n)A(x_1,\dots,x_n)^\top$. Then

$$\begin{array}{rcl} \mathrm{str}(f) \le k & \Leftrightarrow & f \text{ is a sum of } k \text{ reducible polynomials} \\ & \Leftrightarrow & A \text{ is a sum of } k \text{ matrices of rank} \le 2 \\ & \Leftrightarrow & A \text{ has rank} \le 2k \end{array}$$

So $\mathrm{str}(f) = \lceil \mathrm{rk}(A)/2 \rceil$.

Example

$\mathrm{str}(x^2+y^2+z^2) = \lceil \mathrm{rk}(I_3)/2 \rceil = 2.$

Basic properties of strength

How does strength compare to rank of matrices?

We can compute the rank of a matrix.
(determinants of submatrices / column- and rowoperations)
Q: How do you compute the strength of a polynomial?

The limit of a sequence of matrices of rank $\leq k$ has rank $\leq k$.
Q: Is the subset of polynomials of strength $\leq k$ closed?

An $n \times m$ matrix has maximal rank $\min(n, m)$.
Q: What is the maximal strength of a polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$?

A random $n \times m$ matrix has rank $\min(n, m)$.
Q: What is the strength of a random polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$?

Computing the strength of a polynomial

I don't know how to do this... **Exercise** Find an algorithm.

Tricks

1. We have $\mathrm{str}(f+g) \leq \mathrm{str}(f) + \mathrm{str}(g)$.
2. For $f \in \mathbb{C}[x_1, \dots, x_n]_d$, we define the singular locus:

   $$\mathrm{Sing}(f) := \left\{ \frac{\partial f}{\partial x_1} = \ldots = \frac{\partial f}{\partial x_n} = 0 \right\}$$

   When $f = g_1 \cdot h_1 + \ldots + g_k \cdot h_k$, then

   $$\{g_1 = h_1 = \ldots = g_k = h_k = 0\} \subseteq \mathrm{Sing}(f)$$

   and so $\dim \mathrm{Sing}(f) \geq n - 2\,\mathrm{str}(f)$.
3. Every polynomial in $\mathbb{C}[x, y]_d$ is reducible. Hence

   $$f \in \mathbb{C}[x, y]_d \Rightarrow \mathrm{str}(f) \leq 1$$

Computing the strength of a polynomial

Example

Consider $f = x_1^d + \ldots + x_n^d$.

We have

$$f = \begin{cases} (x_1^d + x_2^d) + \ldots + (x_{2k-1}^d + x_{2k}^d) & \text{if } n = 2k \\ (x_1^d + x_2^d) + \ldots + (x_{2k-1}^d + x_{2k}^d) + x_{2k+1}^d & \text{if } n = 2k+1 \end{cases}$$

and so $\operatorname{str}(f) \leq \lceil n/2 \rceil$.

The singular locus

$$\operatorname{Sing}(f) = \{dx_1^{d-1} = \ldots = dx_n^{d-1} = 0\} = \{(0, \ldots, 0)\} \subseteq \mathbb{C}^n$$

has dimenion $0 \geq n - 2\operatorname{str}(f)$. So $\operatorname{str}(f) \geq \lceil n/2 \rceil$.

So $\operatorname{str}(f) = \lceil n/2 \rceil$.

Strength ≤ 3 is not closed

$\mathbf{Q}_{d,k,n}$: Is $\{f \in \mathbb{C}[x_1, \ldots, x_n]_d \mid \mathrm{str}(f) \leq k\}$ closed?

For $k = 1$, yes. (union of images of projective morphisms).

For $k = 2$, I don't know. (Conjecture: yes)

For $d = 2$, yes. (rank of symmetric matrices)

For $d = 3$, yes. (slice rank of polynomials)

Theorem (Ballico-B-Oneto-Ventura)

The set $\{f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \mathrm{str}(f) \leq 3\}$ is not closed for $n \gg 0$.

Consider

$$^1/_t(x^2+tg)(y^2+tf) - {}^1/_t(u^2-tq)(v^2-tp) - {}^1/_t(xy-uv)(xy+uv)$$
$$=$$
$$x^2f + y^2g + u^2p + v^2q + t(fg - pq)$$

It has strength ≤ 3. For $t \to 0$, we get $x^2f + y^2g + u^2p + v^2q$.

Strength ≤ 3 is not closed

Theorem (Ballico-B-Oneto-Ventura)

For $n \gg 0$, there are polynomials $f, g, p, q \in \mathbb{C}[z_1, \dots, z_n]_2$ such that

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, z_1, \dots, z_n]_4$$

has strength 4.

Consider the polynomial

$$h := x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

where x, y, u, v have degree 1 and $\underbrace{f, g, p, q}_{\text{variables}}$ have degree 2.

Proposition

The polynomial h has strength 4.

Strength ≤ 3 is not closed

Definition

The strength of a polynomial $h \in \mathbb{C}[x, y, u, v, f, g, p, q]_d$ is the minimum number $r \geq 0$ (when this exists) such that

$$h = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

with $g_1, h_1, \ldots, g_r, h_r$ homogeneous polynomials of degree $\leq d-1$.

Example

The polynomial

$$f \cdot g + x \cdot (uh + v^3)$$

is irreducible and hence has strength 2.

Example

When the g_i, h_i have degree 1, then

$$g_1 \cdot h_1 + \ldots + g_r \cdot h_r \in \mathbb{C}[x, y, u, v]_2$$

Hence the variable f has infinite strength.

Strength ≤ 3 is not closed

Proposition

The polynomial

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

has strength 4.

1/4 of the proof

We need to show, for example, that

$$x^2 f + y^2 g + u^2 p + v^2 q \neq \ell_1 \cdot h_1 + \ell_2 \cdot h_2 + \ell_3 \cdot h_3$$

for all $\ell_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_1$ and $h_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_3$.

Strength ≤ 3 is not closed

Proposition

The polynomial

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

has strength 4.

1/4 of the proof

We need to show, for example, that

$$x^2 f + y^2 g + u^2 p + v^2 q \neq \ell_1 \cdot h_1 + \ell_2 \cdot h_2 + \ell_3 \cdot h_3$$

for all $\ell_i \in \mathbb{C}[x, y, u, v]_1$ and $h_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_3$.

Think of $R = \mathbb{C}[x, y, u, v]$ as the set of coefficients.
So $\ell_i \in R$ and $h_i \in R[f, g, p, q]$.

The coefficients of f, g, p, q on the right are all in (ℓ_1, ℓ_2, ℓ_3).
The coefficients x^2, y^2, u^2, v^2 on the left are not all (ℓ_1, ℓ_2, ℓ_3). $\square$

Strength ≤ 3 is not closed

Proposition

The polynomial

$$x^2f + y^2g + u^2p + v^2q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

has strength 4.

$\vdots$

$\vdots$

Theorem (Ballico-B-Oneto-Ventura)

For $n \gg 0$, there are polynomials $f, g, p, q \in \mathbb{C}[z_1, \dots, z_n]_2$ such that

$$x^2f + y^2g + u^2p + v^2q \in \mathbb{C}[x, y, u, v, z_1, \dots, z_n]_4$$

has strength 4.

How to bridge the gap?

Polynomial functors

Definition

The polynomial functor $S^d\colon \mathrm{Vec} \to \mathrm{Vec}$ is the functor

$$\begin{array}{rcl} V & \mapsto & S^d(V) \\ (L\colon V \to W) & \mapsto & \left(S^d(L)\colon S^d(V) \to S^d(W)\right) \\ \mathbb{C}\,x_1 \oplus \cdots \oplus \mathbb{C}\,x_n & \mapsto & \mathbb{C}[x_1, \ldots, x_n]_d \\ (x_i \mapsto \sum_j c_{ij} y_j) & \mapsto & (x_i \mapsto \sum_j c_{ij} y_j) \end{array}$$

Definition

A polynomial transformation

$$\alpha\colon S^{d_1} \oplus \cdots \oplus S^{d_k} \to S^{e_1} \oplus \cdots \oplus S^{e_\ell}$$

is of the form

$$(f_1, \ldots, f_k) \mapsto (F_1(f_1, \ldots, f_k), \ldots, F_\ell(f_1, \ldots, f_k))$$

Here $F_j \in \mathbb{C}[X_1, \ldots, X_k]_{e_j}$ are fixed forms with $\deg(X_i) = d_i$.

Polynomial functors

Example

$$(g_1, h_1, g_2, h_2, g_3, h_3) \mapsto g_1 \cdot h_1 + g_2 \cdot h_2 + g_3 \cdot h_3$$

defines a polynomial transformation

$$\alpha\colon (S^{d_1} \oplus S^{4-d_1}) \oplus (S^{d_2} \oplus S^{4-d_2}) \oplus (S^{d_3} \oplus S^{4-d_3}) \to S^4$$

for all fixed $1 \leq d_1 \leq d_2 \leq d_3 \leq 2$.

Definition

We define the inverse limit

$$S^d_\infty := \{\text{degree-}d \text{ series in } x_1, x_2, \dots\} \ni x_1^d + x_2^d + x_3^d + \dots$$

Proposition (B-Draisma-Eggermont-Snowden)

Let $p \in S^d_\infty$ be a series with projections $p_n \in \mathbb{C}[x_1, \dots, x_n]_d$ and $\alpha\colon P \to S^d$ a polynomial transformation. Then

$$p \in \mathrm{im}(\alpha_\infty) \Leftrightarrow p_n \in \mathrm{im}(\alpha_n) \text{ for all } n$$

Take $p = x^2 f + y^2 g + u^2 p + v^2 q$ for series some $f, g, p, q \in S^2_\infty$.

Polynomial functors

Definition

Write $D^d \subseteq S^d_\infty$ for the subspace of finite strength series.
A system of variables consists of a basis of S^d_∞ / D^d for every $d \geq 1$.

Proposition (B-Draisma-Eggermont-Snowden)

Let $\beta\colon S^{e_1} \oplus \cdots \oplus S^{e_k} \to S^d$ and $\alpha\colon P \to S^d$ be polynomial transformations. Let $f_1 \in S^{e_1}_\infty, \ldots, f_k \in S^{e_k}_\infty, p \in P_\infty$ be a series. Assume that $\beta_\infty(f_1, \ldots, f_k) = \alpha_\infty(p)$ and that $(f_1, \ldots, f_k)$ is part of a system of variables. Then there exists a polynomial transformation $\gamma\colon S^{e_1} \oplus \cdots \oplus S^{e_k} \to P$ such that $\beta = \alpha \circ \gamma$.

Example (which closes the gap)

Take

$$\beta(x, y, u, v, f, g, p, q) = x^2 f + y^2 g + u^2 p + v^2 q$$
$$\alpha(g_1, h_1, g_2, h_2, g_3, h_3) = g_1 \cdot h_1 + g_2 \cdot h_2 + g_3 \cdot h_3$$

Strength ≤ 3 is not closed

Proposition

The polynomial

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

has strength 4.

$\Big\Downarrow$ Polynomial functors

Theorem (Ballico-B-Oneto-Ventura)

For $n \gg 0$, there are polynomials $f, g, p, q \in \mathbb{C}[z_1, \ldots, z_n]_2$ such that

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, z_1, \ldots, z_n]_4$$

has strength 4.

Generic and maximal strength

Q: What is the maximal strength of a polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$?

Q: What is the strength of a random polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$?

Definition

The *slice rank* of f is the minimal $\mathrm{slrk}(f) := r \geq 0$ such that

$$f = \ell_1 \cdot h_1 + \ldots + \ell_r \cdot h_r$$

with $\ell_1, \ldots, \ell_r$ and $h_1, \ldots, h_r$ homogeneous of degrees 1 and $d-1$.

Proposition

1. $\mathrm{str}(f) \leq \mathrm{slrk}(f) \leq n-1$
2. $\mathrm{slrk}(f) = \min\{\mathrm{codim}(U) \mid U \subseteq \mathbb{C}^n, f|_U = 0\}$
3. The subset of polynomials of slice rank $\leq k$ closed.

Generic and maximal strength

Theorem (Harris)

A generic homogeneous polynomial of degree d in $n+1$ variables has slice rank

$$\min\left\{ r \in \mathbb{Z}_{\geq (n+1)/2} \;\middle|\; r(n+1-r) \geq \binom{d+n-r}{d} \right\}.$$

Theorem (B-Oneto)

The strength and slice rank of a homogeneous polynomial of degree d are generically equal for $d \leq 7$ and $d = 9$.

Theorem (Ballico-B-Oneto-Ventura)

The strength and slice rank of a homogeneous polynomial of degree d are generically equal for $d \geq 5$.

Generic and maximal strength

We consider

$$\{g_1 \cdot h_1 + \ldots + g_r \cdot h_r \mid \deg(g_i) = a_i, \deg(h_i) = d - a_i\}$$

inside $\mathbb{C}[x_1, \ldots, x_n]_d$.

Goal

Prove for fixed r that dimension is maximal when $a_1, \ldots, a_r = 1$.

Terracini's Lemma

The dimension is $\dim(g_1, h_1, \ldots, g_r, h_r)_d$ for generic generators.

Proposition

The dimension is at most

$$\binom{n+d}{d} - \mathrm{coeff}_d\left(\frac{\prod_{i=1}^r (1-t^{a_i})(1-t^{d-a_i})}{(1-t)^{n+1}}\right) + \binom{\ell_{d/2}}{2}$$

where $\ell_{d/2} := \#\{i \mid a_i = d/2\}$. Equality when $a_1, \ldots, a_r = 1$.

Generic and maximal strength

For fixed d, r, we want $F(a_1, \dots, a_r) :=$

$$\mathrm{coeff}_d \left(\frac{\prod_{i=1}^r (1 - t^{a_i})(1 - t^{d-a_i})}{(1-t)^{n+1}} \right) - \binom{\ell_{d/2}}{2}$$

to be minimal when $a_1, \dots, a_r = 1$.

Proposition

We have

$$F(a_1, \dots, a_r) - F(a_1, \dots, a_{r-1}, a_r - 1) > 0$$

when $a_r = \theta := \max\{a_1, \dots, a_r\} > 2$.

Proof

Write $c_\ell(k_1, \dots, k_n) := \mathrm{coeff}_\ell(P_{k_1} \cdots P_{k_n}) \geq 0$ where

$$P_k = 1 + t + \ldots + t^k$$

for $k \in \{0, 1, 2, \ldots\} \cup \{\infty\}$. Then the difference equals

$$c_{d-\theta+1}(\infty^{n-r}, d - 2\theta, a_1 - 1, \dots, a_{r-1} - 1) - \ell_{\theta-1} - (\ell_\theta - 1)m$$

where $\ell_j = \#\{i \mid a_i = j\}$ and $m = n - \ell_1$.

Generic and maximal strength

Write $c_\ell(k_1, \dots, k_n) := \operatorname{coeff}_\ell(P_{k_1} \cdots P_{k_n}) \geq 0$ where

$$P_k = 1 + t + \ldots + t^k$$

for $k \in \{0, 1, 2, \dots\} \cup \{\infty\}$.

Proposition

We have

- $c_\ell(k_1, \dots, k_n) = c_\ell(k_{\sigma(1)}, \dots, k_{\sigma(n)})$ for all $\sigma \in S_n$
- $c_\ell(k_1, \dots, k_n, 0) = c_\ell(k_1, \dots, k_n)$
- $c_\ell(k, k_2, \dots, k_n) \geq c_\ell(k', k_2, \dots, k_n)$ for all $0 \leq k' \leq k \leq \infty$
- $c_{\ell+1}(k_1, \dots, k_n) \geq c_\ell(k_1, \dots, k_n)$ when $k_1 = \infty$

We get

$$c_{d-\theta+1}(\infty^{n-r}, d - 2\theta, a_1 - 1, \dots, a_{r-1} - 1)$$

$$\geq \operatorname{coeff}_4(P_\infty^{\ell_\theta} P_1^{m-\ell_\theta-1}) - (\ell_\theta - 1)(m - 1)$$

where $\ell_j = \#\{i \mid a_i = j\}$ and $m = n - \ell_1$.

Strength of polynomials

Q: How do you compute the strength of a polynomial?

Q: Is there an algorithm that computes best low-strength approximations of a polynomial?

Q: What is the highest possible strength of a limit of strength $\leq k$ polynomials?

Thanks for your attention!

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