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Polynomial functors as affine spaces

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Categories and functors

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Definition: A category $\mathcal{C}$ has objects $C, D \in \mathcal{C}$ and morphisms $C \to D$. You can compose morphisms and every object has an identity morphism.

Examples:

(0) The category Set consists of sets and maps.

(1) The category Vec consists of finite-dimensional vector spaces and linear maps.

(2) The category Top consists of topological spaces and continious maps.

(3) The category Vec^k consists of $V = (V_1, \ldots, V_k)$ with $V_i \in \mathrm{Vec}$ and $\ell = (\ell_1, \ldots, \ell_k)\colon V \to W$ with $\ell_i\colon V_i \to W_i$ linear maps.

Categories and functors

Let $\mathcal{C}, \mathcal{D}$ be categories.

Definition: A functor $F\colon \mathcal{C} \to \mathcal{D}$ assigns

(1) to every object $C \in \mathcal{C}$ an object $F(C) \in \mathcal{D}$

(2) to every morphism $\ell\colon C \to C'$ a morphism $F(\ell)\colon F(C) \to F(C')$

such that $F(\ell \circ \ell') = F(\ell) \circ F(\ell')$ and $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$.

Examples:

(1) The functor $\mathrm{For}\colon \mathrm{Vec} \to \mathrm{Set}$ with $\mathrm{For}(V) = V$ and $\mathrm{For}(\ell) = \ell$.

(2) The functor $\mathrm{Zar}\colon \mathrm{Vec} \to \mathrm{Top}$ with $\mathrm{Zar}(V) = V$ and $\mathrm{Zar}(\ell) = \ell$.

(3) The functor $\Delta\colon \mathrm{Vec} \to \mathrm{Vec}^k$ with $\Delta(V) = (V, \dots, V)$
and $\Delta(\ell) = (\ell, \dots, \ell)$.

Polynomial functors are functors

Definition: A polynomial functor $P\colon \mathrm{Vec}^k \to \mathrm{Vec}$

(1) assigns to every $V \in \mathrm{Vec}^k$ a vector space $P(V) \in \mathrm{Vec}$

(2) assigns to every pair $(V,W) \in \mathrm{Vec}^k \times \mathrm{Vec}^k$ a polynomial map

$$\begin{aligned} \mathrm{Mor}(V,W) &\to \mathrm{Hom}(P(V),P(W)) \\ (\ell\colon V \to W) &\mapsto (P(\ell)\colon P(V) \to P(W)) \end{aligned}$$

such that $P(\ell \circ \ell') = P(\ell) \circ P(\ell')$ and $P(\mathrm{id}_V) = \mathrm{id}_{P(V)}$.

Remark: For every $V \in \mathrm{Vec}^k$, the map

$$\begin{aligned} \textstyle\prod_{i=1}^k \mathrm{GL}(V_i) &\to \mathrm{GL}(P(V)) \\ g = (g_1,\ldots,g_k) &\mapsto P(g) \end{aligned}$$

gives an action on $P(V)$.

Polynomial functors are like polynomials

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What are polynomial functions $K^k \to K$?

Examples: Constants $v \mapsto c$ for $c \in K$ and variables $x_i \colon K^k \to K, (v_1, \ldots, v_k) \mapsto v_i$.

Operations: Addition $+$ and multiplication $\cdot$.

Answer: Polynomials are everything you can obtain from constants and variables using additions and multiplications.

Remark: Polynomials have a finite degree.

Polynomial functors are like polynomials

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What are polynomial functors $\mathrm{Vec}^k \to \mathrm{Vec}$?

Examples: Constants for $U \in \mathrm{Vec}$ defined by $V \mapsto U$ and $\ell \mapsto \mathrm{id}_U$ and variables T_i defined by $(V_1, \ldots, V_k) \mapsto V_i$ and $(\ell_1, \ldots, \ell_k) \mapsto \ell_i$.

Operations: Direct sum $\oplus$ and tensor product $\otimes$ defined by

$$(Q \oplus P)(V) = Q(V) \oplus P(V) \quad \text{and} \quad (Q \otimes P)(V) = Q(V) \otimes P(V)$$

Subfunctors and quotients: A functor Q is a subfunctor of P when $Q(V) \subseteq P(V)$ for all V and $Q(\ell) = P(\ell)|_{Q(V)}$ for all $\ell \colon V \to W$. In this case, the quotient P/Q is defined by $(P/Q)(V) = P(V)/Q(V)$.

Answer(Friedlander-Suslin, Touzé): Polynomial functors are everything you can obtain from constants and variables using direct sums, tensor products, taking subfunctors and taking quotients.

Remark: Polynomial functors have a degree. We restrict to polynomial functors with finite degree.

Polynomial functors are like polynomials

Examples:

(1) $T_1 \oplus T_2$ – pairs of vectors

(2) $T \oplus T$ – pairs of vectors of the same size

(3) $T_1 \otimes T_2$ – matrices

(4) $T \otimes T$ – square matrices

(5) $S^2 \subseteq T \otimes T$ – symmetric matrices = hom. degree-2 polynomials

(6) $T_1 \otimes \cdots \otimes T_k$ – k-way tensors

(7) $S^d \subseteq T^{\otimes d}$ – symmetric d-way tensors = hom. degree-d polynomials

(8) $T_1 \oplus T_2 \oplus (T_1 \otimes T_2)$ – (vector v, vector w, matrix A) with

$$vw^T, A$$

same size.

Polynomial functors as affine spaces

Definition: A closed subset $X \subseteq P$ assigns a closed subset

$$X(V) \subseteq P(V)$$

to every $V \in \mathrm{Vec}^k$ such that $P(\ell)(X(V)) \subseteq X(W)$ for all $\ell\colon V \to W$.

Example: Let $P : V \mapsto U, \ell \mapsto \mathrm{id}_U$ be a constant functor and $X \subseteq P$ a closed subset.

(1) $X(V)$ is a closed subset of U for all $V \in \mathrm{Vec}^k$.

(2) $X(V) = \mathrm{id}_U(X(V)) = P(0_{V \to W})(X(V)) \subseteq X(W)$ for all V, W.
$\Rightarrow X(V) = X(W)$ for all V, W.

So

$$\begin{array}{ccc} \{\text{closed subsets of } U\} & \to & \{\text{closed subsets of } P\} \\ Y & \mapsto & (V \mapsto Y) \end{array}$$

is a bijection.

Polynomial functors as affine spaces

Example 1: $X = \{\text{linearly dependent tuples of vectors}\} \subseteq T \oplus \cdots \oplus T$.

- $X(V) = \mathrm{pr}_{V^{\oplus n}}\{(v_1, \ldots, v_n, \lambda) \in V^{\oplus n} \times \mathbb{P}^{n-1} \mid \sum_{i=1}^n \lambda_i v_i = 0\}$ is closed for all $V \in \mathrm{Vec}$.
- $v_1, \ldots, v_n$ linearly dependent $\Rightarrow \ell(v_1), \ldots, \ell(v_n)$ linearly dependent.

Example 2: $X = \{\text{matrices of rank } \leq r\} \subseteq T_1 \otimes T_2$.

- $X(V, W) = Z(\det'\mathrm{s})$ is closed for all $(V, W) \in \mathrm{Vec}^2$.
- $\mathrm{rk}(A) \leq r \Rightarrow \mathrm{rk}(PAQ^T) \leq k$ for all matrices P, Q.

Example 3: $X = \overline{\{\text{tensors of rank } \leq r\}} \subseteq T_1 \otimes \cdots \otimes T_k$.

- $X(V)$ is closed for all $V \in \mathrm{Vec}^k$ by construction.
- $(\ell_1 \otimes \cdots \otimes \ell_k)(\sum_{j=1}^r v_{1j} \otimes \cdots \otimes v_{kj}) = \sum_{j=1}^r \ell_1(v_{1j}) \otimes \cdots \otimes \ell_k(v_{kj})$

Morphisms between polynomial functors

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Let P, Q be polynomial functors.

Definition: A polynomial transformation $\alpha\colon Q \to P$ is a family

$$(\alpha_V\colon Q(V) \to P(V))_{V \in \mathrm{Vec}^k}$$

of polynomial maps such that

$$\begin{array}{ccc} Q(V) & \xrightarrow{\alpha_V} & P(V) \\ \big\downarrow{\scriptstyle Q(\ell)} & & \big\downarrow{\scriptstyle P(\ell)} \\ Q(W) & \xrightarrow{\alpha_W} & P(W) \end{array}$$

commutes for all $\ell\colon V \to W$.

Morphisms between polynomial functors

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Example 1: $\alpha\colon K^{(n-1)\times n} \oplus T^{\oplus(n-1)} \to T^{\oplus n}$ defined by

$$\alpha_V(A, v_1, \ldots, v_{n-1}) = (v_1, \ldots, v_{n-1})A =: (w_1, \ldots, w_n)$$

is a polynomial transformation since

$$\alpha_V(A, \ell(v_1), \ldots, \ell(v_{n-1})) = (\ell(v_1), \ldots, \ell(v_{n-1}))A = (\ell(w_1), \ldots, \ell(w_n)).$$

Example 2: $\alpha\colon (T_1 \oplus T_2)^{\oplus r} \to T_1 \otimes T_2$ defined by

$$\alpha_{(V,W)}(v_1, w_1, \ldots, v_r, w_r) = v_1 w_1^T + \cdots + v_r w_r^T$$

is a polynomial transformation since

$$\alpha_{(V,W)}(Pv_1, Qw_1, \ldots, Pv_r, Qw_r) = P(v_1 w_1^T + \cdots + v_r w_r^T)Q^T.$$

Example 3: $\alpha\colon (T_1 \oplus \cdots \oplus T_k)^{\oplus r} \to T_1 \otimes \cdots \otimes T_k$ defined by

$$\alpha_{(V,W)}(v_{11}, \ldots, v_{kr}) = \sum_{j=1}^{r} v_{1j} \otimes \cdots \otimes v_{kj}$$

is a polynomial transformation.

Closed subsets vs polynomial transformations

Example 1: $\dim \operatorname{span}\{v_1, \ldots, v_n\}$ is the minimal r such that $(v_1, \ldots, v_n)$ is in the image of $\alpha\colon K^{r \times n} \oplus T^{\oplus r} \to T^{\oplus n}$ defined by

$$\alpha_V(A, v_1, \ldots, v_r) = (v_1, \ldots, v_r)A.$$

Example 2: $\operatorname{rk}(A)$ is the minimal r such that A is in the image of $\alpha\colon (T_1 \oplus T_2)^{\oplus r} \to T_1 \otimes T_2$ defined by

$$\alpha_{(V,W)}(v_1, w_1, \ldots, v_r, w_r) = v_1 w_1^T + \cdots + v_r w_r^T.$$

Example 3: $\operatorname{rk}(t)$ is the minimal r such that t is in the image of $\alpha\colon (T_1 \oplus \cdots \oplus T_k)^{\oplus r} \to T_1 \otimes \cdots \otimes T_k$ defined by

$$\alpha_{(V,W)}(v_{11}, \ldots, v_{kr}) = \textstyle\sum_{j=1}^{r} v_{1j} \otimes \cdots \otimes v_{kj}.$$

Closed subsets vs polynomial transformations

Let P, Q be polynomial functors.

Write $Q \prec P$ when $Q_{(d)} = P_{(d)}/P'$ for $d = \max\{e > 0 \mid Q_{(e)} \not\cong P_{(e)}\}$.

Examples

(1) $K^{(n-1)\times n} \oplus T^{\oplus(n-1)} \prec T^{\oplus n}$

(2) $(T_1 \oplus T_2)^{\oplus r} \prec T_1 \otimes T_2$

(3) $(T_1 \oplus \cdots \oplus T_k)^{\oplus r} \prec T_1 \otimes \cdots \otimes T_k$

Dichotomy Theorem (B-Draisma-Eggermont-Snowden)

Let $X \subseteq P$ be a closed subset. Then

- $X = P$ or
- there are polynomial functors $Q_1, \ldots, Q_k \prec P$ and $\alpha_i \colon K^{n_i} \oplus Q_i \to P$ such that $X \subseteq \bigcup_i \operatorname{im}(\alpha_i)$.

Applications

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Theorem (Draisma)
Every descending chain $P \supsetneq X_1 \supseteq X_2 \supseteq \ldots$ of closed subsets stabilizes.

Proof using induction on P. Take $Q_i \prec P$ and $\alpha_i \colon K^{n+i} \oplus Q_i \to P$ such that $X_1 \subseteq \bigcup_i \operatorname{im}(\alpha_i)$ and pull back the chain of closed subsets along each α_i. The resulting chains all have to stabilize. □

Theorem (B-Draisma-Eggermont-Snowden)
Let $X \subseteq Q$ be a constructible subset and let $\alpha \colon Q \to P$ be a morphism. Then $\alpha(X)$ is constructible.

More analogues from finite-dimensional affine algebraic geometry?

Thank you for your attention!

References

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