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Strength and polynomial functors

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The rank of infinite-by-infinite matrices

Definition: The rank of an $\mathbb{N} \times \mathbb{N}$ matrix A is

$$\mathrm{rk}(A) := \sup\{\mathrm{rk}(B) \mid \text{finite submatrices } B \text{ of } A\} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$$

Examples:
(1) The rank of the matrix

$$\begin{pmatrix} 1 & * & \dots & \dots & \dots \\ 0 & 1 & * & \dots & \dots \\ \vdots & 0 & 1 & * & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

is ∞.
(2) For linearly independent subsets $\{v_1, \dots, v_k\}, \{w_1, \dots, w_k\} \subseteq \mathbb{C}^{\mathbb{N}}$ the matrix $v_1 w_1^T + \dots + v_k w_k^T$ has rank k.

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Lemma:
$A \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ has rank $\leqslant k < \infty \Leftrightarrow A = \sum_{i=1}^{k} v_i w_i^T$ with $v_i, w_i \in \mathbb{C}^{\mathbb{N}}$

Proof. Assume A has rank k. Then A has a invertible $k \times k$ submatrix. Permute the columns of A so that the first k columns of A are linearly independent. Call these first k columns $v_1, \ldots, v_k$. To show that

$$A = \sum_{i=1}^{k} v_i w_i^T$$

for some $w_1, \ldots, w_k \in \mathbb{C}^{\mathbb{N}}$, we need to show that every column of A is a linear combination of $v_1, \ldots, v_k$. Let v be another column of A. Then every finite submatrix of $(v\ v_1\ \ldots\ v_k)$ has rank $\leqslant k$. Consider the vector space $V_n = \{\lambda \in \mathbb{C}^{k+1} \mid \mathrm{pr}_n(\lambda_0 v - \lambda_1 v_1 + \cdots + \lambda_k v_k) = 0\} \neq 0$. We have $V_{n+1} \subseteq V_n$ for all n. It follows that $V = \bigcap_n V_n \neq 0$. Any nonzero element of V expresses v as a linear combination of $v_1, \ldots, v_k$. $\square$

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Fact: An $n \times m$ matrix A has rank $\min(n, m) \Leftrightarrow \overline{\mathrm{GL}_n \cdot A \cdot \mathrm{GL}_m} = \mathbb{C}^{n \times m}$

Theorem: An $\mathbb{N} \times \mathbb{N}$ matrix A has rank $\infty \Leftrightarrow \overline{\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty} = \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$

Proof. ($\Leftarrow$) If the matrix A has rank $k < \infty$, then $\overline{\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty}$ is contained in $\{$matrices in $\mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ of rank $\leqslant k\} \subsetneq \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$.

($\Rightarrow$) Suppose $\overline{\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty} \subsetneq \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$. Then there is a nonzero equation on $\mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ that is zero on $\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty$. This equation uses only finitely many entries. So the rank of a particular finite submatrix has to be non-maximal for every element in $\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty$. In particular, this is true for a permutations of A. So the rank of A must be finite. $\square$

Corollary: Let A be an $\mathbb{N} \times \mathbb{N}$ matrix. Then either $\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty$ is dense in $\mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ or $A = \sum_{i=1}^{k} v_i w_i^T$ with $v_1, \ldots, v_k, w_1, \ldots, w_k \in \mathbb{C}^{\mathbb{N}}$.

Similar theorems

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Definition: The rank of a tuple of $\mathbb{N} \times \mathbb{N}$ matrices $(A_1, \dots, A_k)$ is

$$\mathrm{rk}(A_1, \dots, A_k) := \inf\{\mathrm{rk}(\lambda_1 A_1 + \cdots + \lambda_k A_k) \mid (\lambda_1 : \cdots : \lambda_k) \in \mathbb{P}^{k-1}\}$$

Theorem (Draisma-Eggermont)
$\mathrm{rk}(A_1, \dots, A_k) = \infty \Leftrightarrow \overline{\mathrm{GL}_\infty \cdot (A_1, \dots, A_k) \cdot \mathrm{GL}_\infty} = (\mathbb{C}^{\mathbb{N} \times \mathbb{N}})^k$

Definition: The q-rank of a series

$$f = a_{111} x_1^3 + a_{112} x_1^2 x_2 + \cdots + a_{ijk} x_i x_j x_k + \dots$$

is the minimal $k \leqslant \infty$ such that $f = \ell_1 q_1 + \cdots + \ell_k q_k$ with $\deg(\ell_i) = 1$.

Theorem (Derksen-Eggermont-Snowden)
$\mathrm{qrk}(f) = \infty \Leftrightarrow \overline{\mathrm{GL}_\infty \cdot f} = \{$all polynomial series of degree 3$\}$

Similar theorems

Take $d \geqslant 2$.

Definition (Ananyan-Hochster)
The strength of a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]_{(d)}$ is the minimal k such that

$$f = g_1 h_1 + \cdots + g_k h_k$$

with $g_1, \ldots, g_k, h_1, \ldots, h_k \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous of degree $< d$.

Theorem (B-Draisma-Eggermont)
For every n, let $X_n \subseteq \mathbb{C}[x_1, \ldots, x_n]_{(d)}$ be a closed subset such that:

$(*)$ We have $f \circ \ell \in X_m$ for all $f \in X_n$ and all linear maps $\ell \colon \mathbb{C}^m \to \mathbb{C}^n$.

Then either $X_n = \mathbb{C}[x_1, \ldots, x_n]_{(d)}$ for all $n \geqslant 0$ or there is a $k < \infty$ such that $\operatorname{str}(f) \leqslant k$ for all $f \in X_n$ and $n \geqslant 0$.

The semiring of functors $P\colon \mathrm{Vec} \to \mathrm{Vec}$

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Definition: A functor $P\colon \mathrm{Vec} \to \mathrm{Vec}$ sends

$$\begin{array}{rcl} V & \mapsto & P(V) \\ (\ell\colon V \to W) & \mapsto & (P(\ell)\colon P(V) \to P(W)) \end{array}$$

such that $P(\mathrm{id}_V) = \mathrm{id}_{P(V)}$ and $P(\varphi \circ \psi) = P(\varphi) \circ P(\psi)$.

Examples: Take $U \in \mathrm{Vec}$ fixed.

- $C_U\colon V \mapsto U, \ell \mapsto \mathrm{id}_U$
- $T\colon V \mapsto V, \ell \mapsto \ell$

You can add and multiply two functors $P, Q\colon \mathrm{Vec} \to \mathrm{Vec}$.

$$(P \oplus Q)(V) = P(V) \oplus Q(V), \quad (P \otimes Q)(V) = P(V) \otimes Q(V)$$

Definition: The functor Q is a subfunctor of P when $Q(V) \subseteq P(V)$ and $Q(\ell) = P(\ell)|_{Q(\ell)}$. In his case, we have the functor $V \mapsto P(V)/Q(V)$.

Polynomial functors as polynomials

Definition: The class of polynomial functors is the minimal class of functors $\mathrm{Vec} \to \mathrm{Vec}$ containing T and all C_U that is closed under addition, multiplication and taking subfunctors and quotients.

Examples

- Constants: $V \mapsto U$ for $U \in \mathrm{Vec}$ fixed.
- Linear functors: $V \mapsto U \otimes V$ for $U \in \mathrm{Vec}$ fixed.
- Matrices: $V \mapsto V \otimes V$
- Tensors: $V \mapsto V \otimes \cdots \otimes V$
- Polynomials: $V \mapsto S^d V$

Remark: The semiring of polynomial functors is graded.

Polynomial functors as topological spaces

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Definition: Let P, Q be polynomial functors. A morphism $\alpha\colon Q \to P$ is a family $(\alpha_V\colon Q(V) \to P(V))_{V \in \mathrm{Vec}}$ of polynomial maps such that

$$\begin{array}{ccc} Q(V) & \xrightarrow{\alpha_V} & P(V) \\ \downarrow{\scriptstyle Q(\ell)} & & \downarrow{\scriptstyle P(\ell)} \\ Q(W) & \xrightarrow{\alpha_W} & P(W) \end{array}$$

commutes for all linear maps $\ell\colon V \to W$.

Definition: A closed subset $X \subseteq P$ sends

$$V \mapsto \text{closed subset } X(V) \subseteq P(V)$$

such that $P(\ell)(X(V)) \subseteq X(W)$ for all linear maps $\ell\colon V \to W$.

The dichotomy

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Let P, Q be polynomial functors. Write $Q \prec P$ when $Q_{(d)}$ is a quotient of $P_{(d)}$ where d is maximal with $Q_{(d)} \not\cong P_{(d)}$.

Theorem (B-Draisma-Eggermont-Snowden)
Let $X \subseteq P$ be a closed subset. Then $X = P$ or there are polynomial functors $Q_1, \ldots, Q_k \prec P$ and $\alpha_i \colon Q_i \to P$ such that $X \subseteq \bigcup_i \mathrm{im}(\alpha_i)$.

Examples

- $\{$matrices of rank $\leqslant k\} = \{v_1 w_1^T + \cdots + v_k w_k^T \mid v_i, w_i$ vectors$\}$
- $\{$degree-d polynomials that are zero on a codim-k subspace$\} = \{\ell_1 g_1 + \cdots + \ell_k g_k \mid \deg(\ell_i) = 1, \deg(g_i) = d-1\}$

Applications

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The dichotomy can be used to prove all the previous theorems.

Theorem (Draisma)
Every descending chain $P \supsetneq X_1 \supseteq X_2 \supseteq$ ... of closed subsets stabilizes.

Proof. Using induction on P: take $Q_1, \ldots, Q_k \prec P$ and $\alpha_i \colon Q_i \to P$ such that $X_1 \subseteq \bigcup_i \mathrm{im}(\alpha_i)$ and pull back the chain of closed subsets along each α_i. The resulting chains all have to stabilize. □

Theorem (B-Draisma-Eggermont-Snowden)
Let $X \subseteq Q$ be a constructible subset and let $\alpha \colon Q \to P$ be a morphism. Then $\alpha(X)$ is constructible.

More analogues of results from finite-dimensional algebraic geometry?

Thank you for your attention!

References

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