Graph Polynomials from Laplacians

November 11, 2022

Kirchhoff polynomials

Given a graph G, we associate a parameter a_j. The *Kirchhoff polynomial* of G is defined by

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In physics parameters a_j are only associated to internal edges. We set

$$\mathcal{K}_{int}(a_1, \ldots, a_{n_{int}}) = \mathcal{K}(G_{int}) = \sum_{T \in \mathcal{T}_1} \prod_{e_j \in (T \cap E)} a_j.$$

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There is a simple relation between $\mathcal{K}_{int}$ and $\mathcal{U}$:

$$\mathcal{U}(a_1, \ldots, a_{n_{int}}) = a_1 \ldots a_{n_{int}} \mathcal{K}_{int}\left(\frac{1}{a_1}, \ldots, \frac{1}{a_{n_{int}}}\right).$$

Laplacian of a graph

Definition

Let G be a graph with n edges and r vertices. To each edge e_j one associates a parameter a_j. The *Laplacian* of the graph G is a $r \times r$-matrix L, whose entries are given by

$$L_{ij} = \begin{cases} \sum a_k & \text{if } i = j \text{ and edge } e_k \text{ is attached to } v_i \text{ and is not a self-loop,} \\ -\sum a_k & \text{if } i \neq j \text{ and edge } e_k \text{ connects } v_i \text{ and } v_j. \end{cases}$$

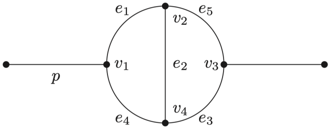
Laplacian with respect to internal vertices and edges

In Feynman graphs one distinguishes between external and internal edges. This motivates the following definition.

Definition

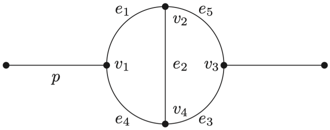
Denote by G_{int} the internal graph of G. The Laplacian of G w.r.t. internal vertices and edges is defined as

$$L_{int}(G) = L(G_{int}).$$



$$\left(\begin{matrix} a_1 + a_4 \\ \\ \\ \end{matrix} \right.$$

Example



$$\left(\begin{array}{cccc} a_1 + a_4 & & & \\ & a_1 + a_2 + a_5 & & \\ & & & \\ & & & \end{array}\right.$$

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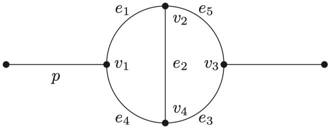

$$\left(\begin{array}{llll} a_1 + a_4 & & & \\ & a_1 + a_2 + a_5 & & \\ & & a_3 + a_5 & \\ & & & \end{array}\right.$$

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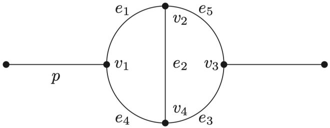

$$\begin{pmatrix} a_1 + a_4 & & & \\ & a_1 + a_2 + a_5 & & \\ & & a_3 + a_5 & \\ & & & a_2 + a_3 + a_4 \end{pmatrix}$$

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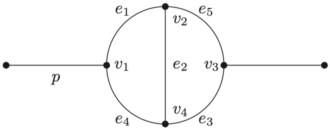

$$\begin{pmatrix} a_1 + a_4 & -a_1 & & \\ -a_1 & a_1 + a_2 + a_5 & & \\ & & a_3 + a_5 & \\ & & & a_2 + a_3 + a_4 \end{pmatrix}$$

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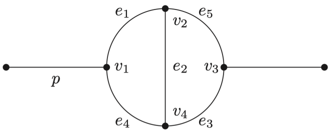

$$\begin{pmatrix} a_1 + a_4 & -a_1 & 0 & -a_4 \\ -a_1 & a_1 + a_2 + a_5 & -a_5 & -a_2 \\ 0 & -a_5 & a_3 + a_5 & -a_3 \\ -a_4 & -a_2 & -a_3 & a_2 + a_3 + a_4 \end{pmatrix}$$

Matrix-tree theorem

We write $L[i]$ for the submatrix of L obtained by deleting the ith row and column.

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In particular, for every internal vertex v_i one has $\mathcal{K}_{int} = \det L_{int}[i]$. One can then recover the first Symanzik polynomial $\mathcal{U}$ from $\mathcal{K}_{int}$ using

$$\mathcal{U}(a_1, \ldots, a_{n_{int}}) = a_1 \ldots a_{n_{int}} \mathcal{K}_{int} \left(\frac{1}{a_1}, \ldots, \frac{1}{a_{n_{int}}} \right).$$

Why matrix-tree?

The number of monomials in the Kirchhoff polynomial $\mathcal{K}$ of G is equal to the number of spanning trees of G. Thus, evaluating the Kirchhoff polynomial at the vector $(1, \dots, 1)$ counts this number. The matrix-tree theorem relates this to determinants of minors of the Laplacian.

Moving on to the second Symanzik polynomial

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Moving on to the second Symanzik polynomial

We denote by $\mathcal{T}_k^{I,J}$ the set of all spanning k-forests such that each tree in a forest contains exactly one vertex v_{i_α} and one vertex v_{j_β}.

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All-minors matrix-tree theorem

Theorem

$$\det L[I,J] = (-1)^{|I|+|J|} \sum_{F \in \mathcal{T}_k^{I,J}} \operatorname{sign}(\pi_F) \prod_{e_j \in F} a_j.$$

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Consider a graph G with n_{int} internal edges $(e_1, \ldots, e_{n_{int}})$, r_{int} internal vertices $(v_1, \ldots, v_{r_{int}})$, n_{ext} external edges $(e_{n_{int}+1}, \ldots, e_{n_{int}+n_{ext}})$ and n_{ext} external vertices $(v_{r_{int}+1}, \ldots, v_{r_{int}+n_{ext}})$.

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Now consider the polynomial

$$\mathcal{W}(a_1, \ldots, a_{n_{int}}, b_1, \ldots, b_{n_{ext}}) = \det L(G)[r_{int} + 1, \ldots, r_{int} + n_{ext}].$$

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We write $\mathcal{W}$ as a sum of polynomials homogeneous in the variables b_j: $\mathcal{W} = \mathcal{W}^{(0)} + \ldots + \mathcal{W}^{(m)}$, where $\mathcal{W}^{(k)}$ is homogeneous of degree k in the variables b_j.

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Furthermore, we write

$$\mathcal{W}^{(k)} = \sum_{(j_1,\ldots,j_k)} \mathcal{W}^{(k)}_{(j_1,\ldots,j_k)}(a_1, \ldots, a_{n_{int}}) b_{j_1} \ldots b_{j_k},$$

where the sum is over all indices with $1 \leqslant j_1 < \ldots < j_k \leqslant n_{ext}$.

The $\mathcal{W}^{(k)}_{(j_1,\dots,j_k)}$ are homogeneous polynomials of degree $r_{int} - k$ in the variables a_i. One finds that

$$\mathcal{W}^{(0)} = 0, \mathcal{W}^{(1)} = \mathcal{K}_{int}(a_1, \dots, a_{n_{int}}) \sum_{j=1}^{n_{ext}} b_j.$$

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Therefore,

$$\mathcal{U} = a_1 \ldots a_{n_{int}} \mathcal{W}^{(1)}_{(j)} \left(\frac{1}{a_1}, \ldots, \frac{1}{a_{n_{int}}} \right)$$

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For $\mathcal{F}_0$ one has

$$\mathcal{F}_0 = a_1 \ldots a_{n_{int}} \sum_{(j,k)} \left(\frac{p_j \cdot p_k}{\mu^2} \right) \cdot \mathcal{W}^{(2)}_{(j,k)} \left(\frac{1}{a_1}, \ldots, \frac{1}{a_{n_{int}}} \right).$$

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Then $\mathcal{F}$ is obtained by

$$\mathcal{F}(a) = \mathcal{F}_0(a) + \mathcal{U}(a) \sum_{i=1}^{n_{int}} a_i \frac{m_i^2}{\mu^2}.$$

Deletion and contraction of edges

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The Laplacian of the graph behaves nicely under these operations. This allows to define graph polynomials $\mathcal{U}$ and $\mathcal{F}$ recursively. For any regular egde e_k we have

$$\mathcal{U}(G) = \mathcal{U}(G/e_k) + a_k\mathcal{U}(G - e_k),$$

$$\mathcal{F}_0(G) = \mathcal{F}_0(G/e_k) + a_k\mathcal{F}_0(G - e_k).$$

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$$\mathcal{U} = a_{r_{int}} \dots a_{n_{int}}, \mathcal{F}_0 = a_{r_{int}} \dots a_{n_{int}} \sum_{j=1}^{r_{int}-1} a_j \left(\frac{-q_j^2}{\mu^2} \right),$$

where the parameters $a_{r_{int}}, \dots, a_{n_{int}}$ correspond to independent loop momenta.