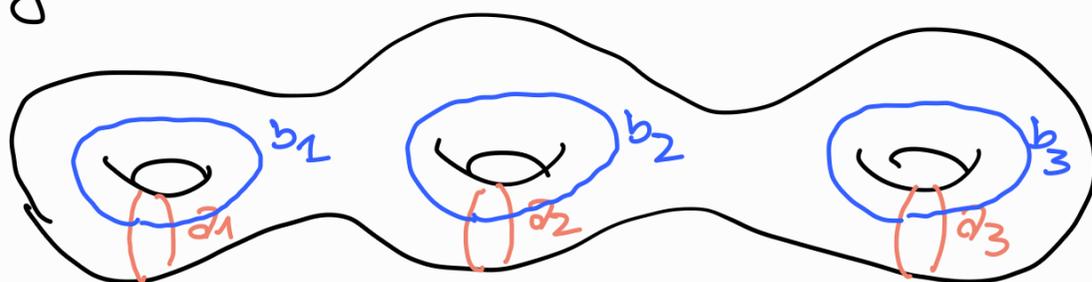


JACOBIANS and the TORELLI MAP

Let C be a smooth projective curve of genus g :



C as a Riemann surface

We know:

$H^0(C, \omega_C)$: space of holomorphic differentials, $\dim H^0(C, \omega_C) = g$

$$H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

In particular, let's choose a symplectic basis of the homology: this is just a basis

$$a_1, b_1, a_2, b_2, \dots, a_g, b_g \quad \text{s.t.}$$

$$a_i \cdot b_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \quad \begin{matrix} a_i \cdot a_j = 0 \\ b_i \cdot b_j = 0 \end{matrix}$$

Riemann proved that there exists a basis

$$\omega_1, \dots, \omega_g \quad \text{of } H^0(C, \omega_C) \quad \text{s.t.}$$

i.e. the matrix of A-periods

$$\int_{a_j} \omega_i = \delta_{ij}$$

$$A = \left(\int_{a_j} \omega_i \right) = \text{Id}_{g \times g}$$

Then he also proved that the matrix of B-periods

$$\tau = \begin{pmatrix} \int \omega_i \\ b_{ij} \end{pmatrix}$$

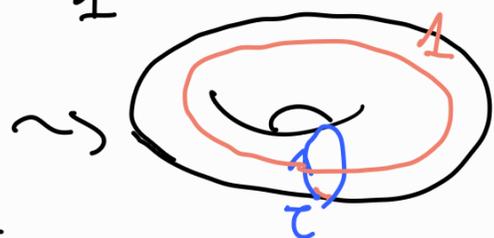
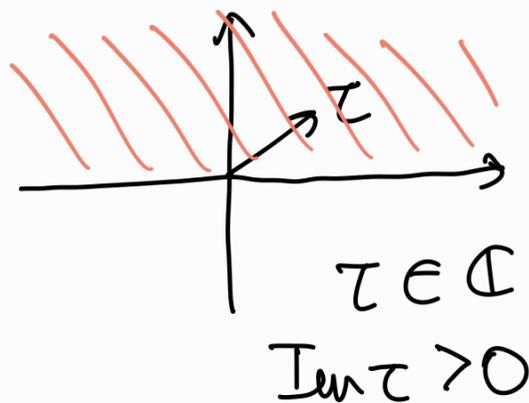
is a complex, symmetric $g \times g$ matrix with positive definite imaginary part

Riemann matrix

This procedure associates to C a Riemann matrix τ .

Example: $E =$ elliptic curve = curve of genus one

$$E = \mathbb{C} / \mathbb{Z} + \mathbb{Z}\tau$$



If we take the natural differential dz on E

then

$$\int_1 dz = 1, \quad \int_\tau dz = \tau$$

What have we done geometrically?

Consider the \mathbb{C} -vector space $H^0(C, \omega_C)^\vee$

Since we can integrate forms against cycles, we get a map

$$\begin{array}{ccc} H_1(C, \mathbb{Z}) & \longrightarrow & H^0(C, \omega_C)^\vee \\ \gamma & \longmapsto & \int_\gamma : \omega \longmapsto \int_\gamma \omega \end{array}$$

The image is a sublattice (additive subgroup of maximal rank $2g$) and the quotient

$H^0(C, \omega_C)^\vee / H_1(C, \mathbb{Z})$ is a complex torus, called the JACOBIAN of C

What we have done before is just writing the Jacobian in coordinates

$$H^0(C, \omega_C)^\vee / H_1(C, \mathbb{Z}) \cong \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g$$

So $C \mapsto \tau$ is the explicit representation of the map $C \mapsto J(C)$.

Rmk; This is computed explicitly if we
 now for example a plane model of the curve

$$C = \{ f(x, y) = 0 \}$$

for example in SAGE.

Explicit computation :

$H^0(C, \omega_C)^{\vee}$ has a basis $\omega_1^{\vee}, \omega_2^{\vee}, \dots, \omega_g^{\vee}$

Let's take a cycle $\gamma = n_1 \partial_1 + \dots + n_g \partial_g$
 $+ m_1 b_1 + \dots + m_g b_g$

we want $\int_{\gamma} \in H^0(C, \omega_C)^{\vee}$ in terms of the dual basis

$$\begin{aligned} \int_{\gamma} &= \sum_{i=1}^g \left(\int_{\gamma} \omega_i \right) \omega_i^{\vee} \\ &= \sum_{i=1}^g \left[\sum_{a=1}^g n_a \int_{\partial_a} \omega_i + \sum_{k=1}^g m_k \int_{b_k} \omega_i \right] \omega_i^{\vee} \end{aligned}$$

So if I write the basis $\partial_1, \dots, \partial_g, b_1, \dots, b_g$ of H_1 in coords

$$\begin{pmatrix} \int_{\partial_1} \omega_1 & \int_{\partial_2} \omega_1 & & \int_{b_1} \omega_1 \\ \int_{\partial_1} \omega_2 & \int_{\partial_2} \omega_2 & \dots & \vdots \\ \vdots & \vdots & & \vdots \\ \int_{\partial_1} \omega_g & \int_{\partial_2} \omega_g & & \int_{b_g} \omega_g \end{pmatrix} = (\text{Id}_{g \times g} \mid \tau)$$

Let's phrase this in terms of moduli spaces:

C moves in the moduli space \mathcal{M}_g

• What is the moduli space for $J(C)$?

It turns out that the Jacobian is a
PRINCIPALLY POLARIZED ABELIAN VARIETY

This means that it is a projective variety which
is at the same time a group

$$J(C) = \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g$$

is a group and
also a projective
variety

There is an explicit way to put $J(C)$ inside
projective space by using explicit functions
called THETA FUNCTIONS

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \tau n + 2\pi i n^t z)$$

$z \in \mathbb{C}^g$ τ is a Riemann
matrix, $\tau \in \mathcal{H}_g$

$$\vartheta: \mathbb{C}^g \times \mathcal{H}_g \longrightarrow \mathbb{C}$$

There are variants called theta functions with characteristic $\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z, \tau)$ s.t. the map

$$\varphi: \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g \longrightarrow \mathbb{P}^{3^g - 1}$$

$$[z] \longmapsto [\dots, \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z, \tau), \dots]$$

is well-defined and an embedding.

So the appropriate moduli space for jacobians is the moduli space of R.P. ABELIAN VARIETIES

$$\mathcal{A}_g = \mathcal{H}_g = \left\{ \tau \in \mathbb{C}^{g \times g} \mid \begin{array}{l} \tau \text{ symmetric} \\ \text{Im } \tau > 0 \end{array} \right\}$$

$$\Big/ \text{Sp}(2g, \mathbb{Z})$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (C\tau + D)^{-1} \cdot (A\tau + B)$$

The map that associates to each curve its Jacobian is a map of moduli spaces called the **Torelli map**

$$\begin{array}{ccc} t: \mathcal{M}_g & \longrightarrow & \mathcal{A}_g \\ \mathbb{C} & \longmapsto & \mathcal{J}(\mathbb{C}) \end{array}$$

Thm; [TORELLI] The Torelli map is
 injective: a curve is determined up
 to isomorphism by its Jacobian

So often \mathcal{M}_g can be identified with the
 image $t(\mathcal{M}_g) \subseteq \mathcal{A}_g$, called the
 SCHOTTKY LOCUS.

Then it's important to know

SCHOTTKY PROBLEM: What is the image of the
 Torelli map? Which ab. var. are Jacobians
 of curves.

$$\dim \mathcal{M}_g = 3g - 3$$

$$\dim \mathcal{A}_g = \frac{g(g+1)}{2}$$

g	$\dim \mathcal{M}_g$	$\dim \mathcal{A}_g$
$g=0$	1	1
$g=1$	$\dim \mathcal{M}_{1,1} = 1$	1
$g=2$	3	3
$g=3$	6	6

} no Schottky
 problem

g	$\dim \mathcal{M}_g$	$\dim \mathcal{A}_g$
$g=4$	9	10
$g \geq 5$	Schottky open	

→ Schottky-Igusa
equation which
defines the Schottky
locus