RIEMANN SURFACES and ALGEBRAIC CURVES

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REFERENCES:
- Notes on the webpage
- CAVALIERI and MILES, Riemann Surfaces and algebraic curves
- FULTON, Algebraic curves.
- KIRWAN, Complex Algebraic curves
- MIRANDA, Algebraic curves and Riemann surfaces

.... others on the webpage.

EXERCISES: • On the webpage.

§ 0 : MOTIVATION: ABELIAN INTEGRALS

Say we want to compute the integral

$$\int_a^b \frac{1}{(x-1)(x-2)(x-3)}\,dx .$$ How do we do it?

Well, we can compute a decomposition

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{1}{2}\frac{1}{(x-1)} - \frac{1}{(x-2)} + \frac{1}{2}\frac{1}{(x-3)}$$

and then it is easy to compute the primitive

$$\int \frac{1}{(x-1)}\,dx = \log(x-1), \quad \int \frac{1}{(x-2)}\,dx = \log(x-2), \dots$$

The same strategy works for any integral of a RATIONAL FUNCTION $\frac{p(x)}{q(x)}$, where $p(x), q(x)$ are two polynomials.

We can always write

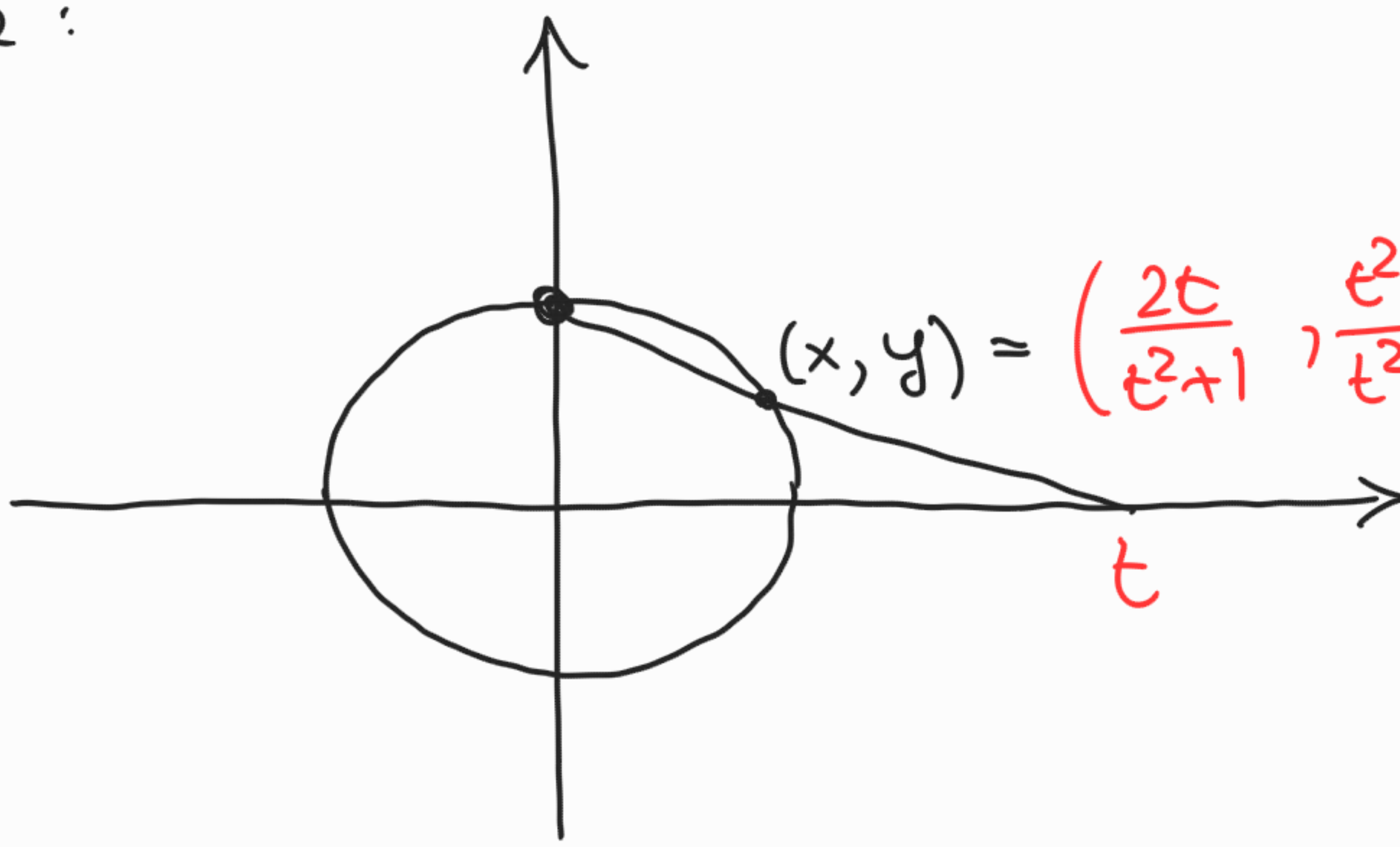
$$\frac{p(x)}{q(x)} = r(x) + a_1 (x-b_1)^{-k_1} + \dots + a_m (x-b_m)^{-k_m}$$

where $r(x)$ is a polynomial, and we know how to integrate these.

Let's consider now the integral

$$\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx$$

We set $y = \sqrt{1+x^2}$, so that $y^2 + x^2 = 1$.
This is an algebraic curve in the plane, the familiar circle:



This curve can be parametrized by a rational parameter t via projection from the point $(1,0)$:

$$x = \frac{2t}{t^2+1}, \quad y = \frac{t^2-1}{t^2+1}$$

Then the integral becomes

$$\int \frac{1}{y}\,dx = \int \frac{t^2+1}{t^2-1}\, d\left(\frac{2t}{t^2+1}\right) = \int \frac{\cancel{t^2+1}}{t^2-1}\, \frac{2(t^2+1)-4t^2}{(t^2+1)^2}\,dt$$

$$= \int \frac{-2t^2+2}{(t^2-1)(t^2+1)}\,dt = -2\int \frac{1}{t^2+1}\,dt$$

and this is again the integral of a rational function.

The same idea works when we try to compute integrals of the form

$$\int R(x, y)\, dx \quad \text{where } y = \sqrt{ax^2+bx+c}$$

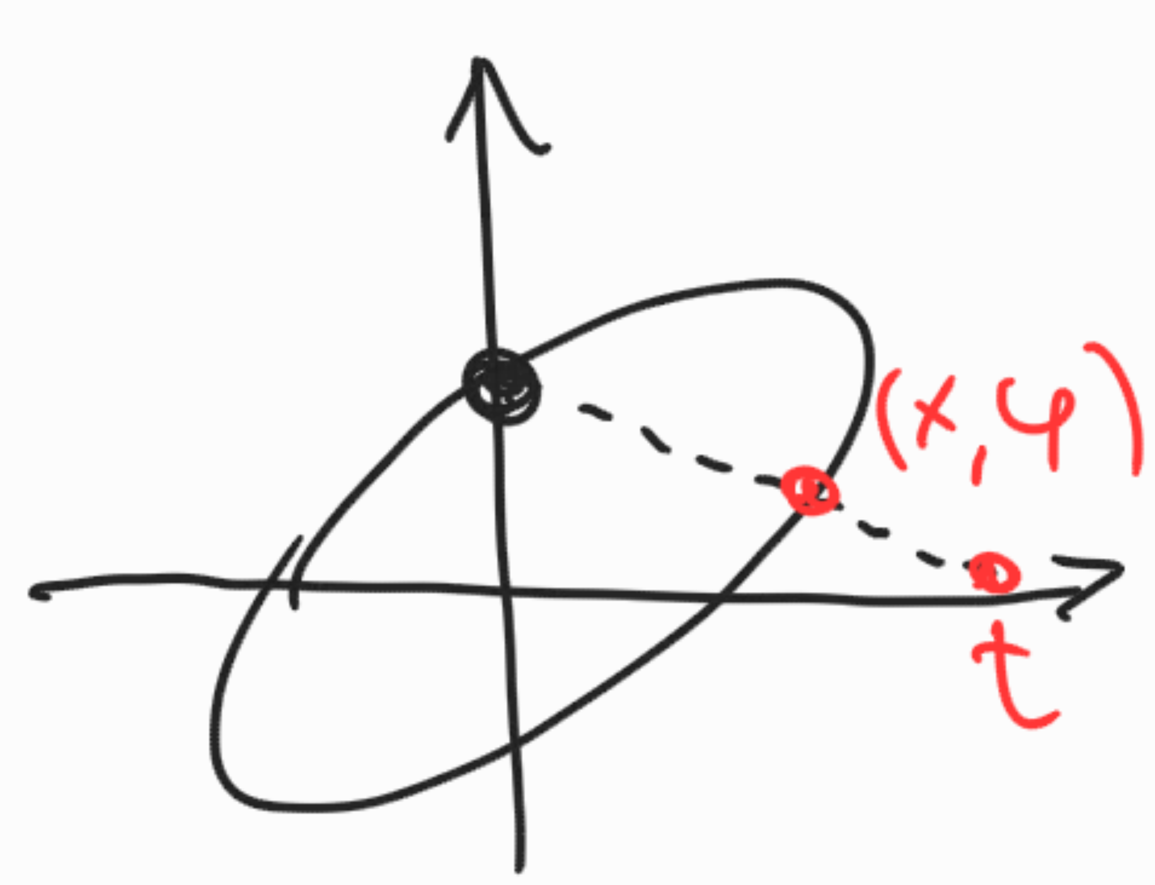
Why does this work? Well the idea is that all curves

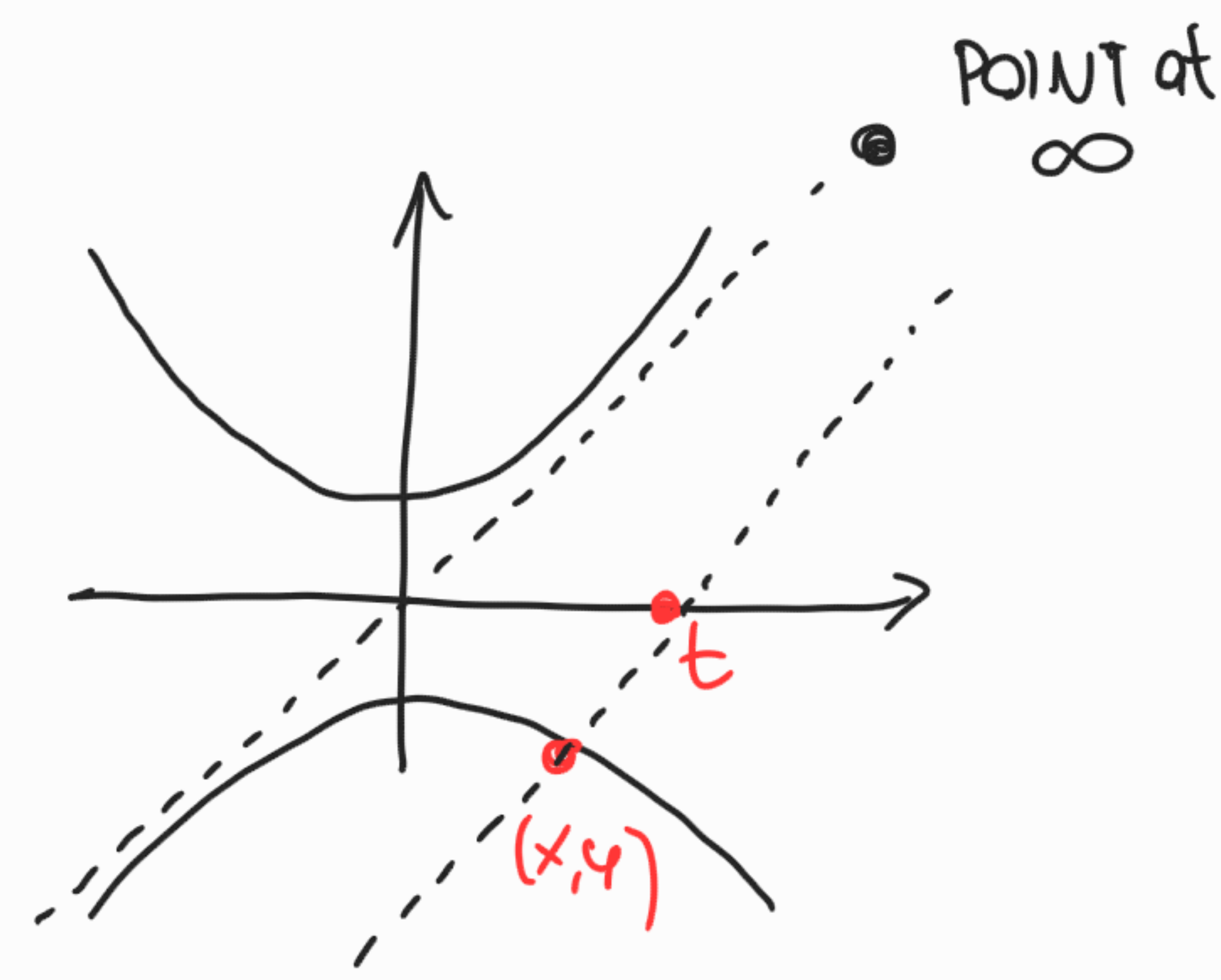
$$\{ y^2 = ax^2+bx+c \}$$

can be parametrized in the form

$$x = x(t), \quad y = y(t)$$ so that $x(t), y(t)$ are rational functions of t.

The parametrization is by projection from a point, just like before





This means that the algebraic curve

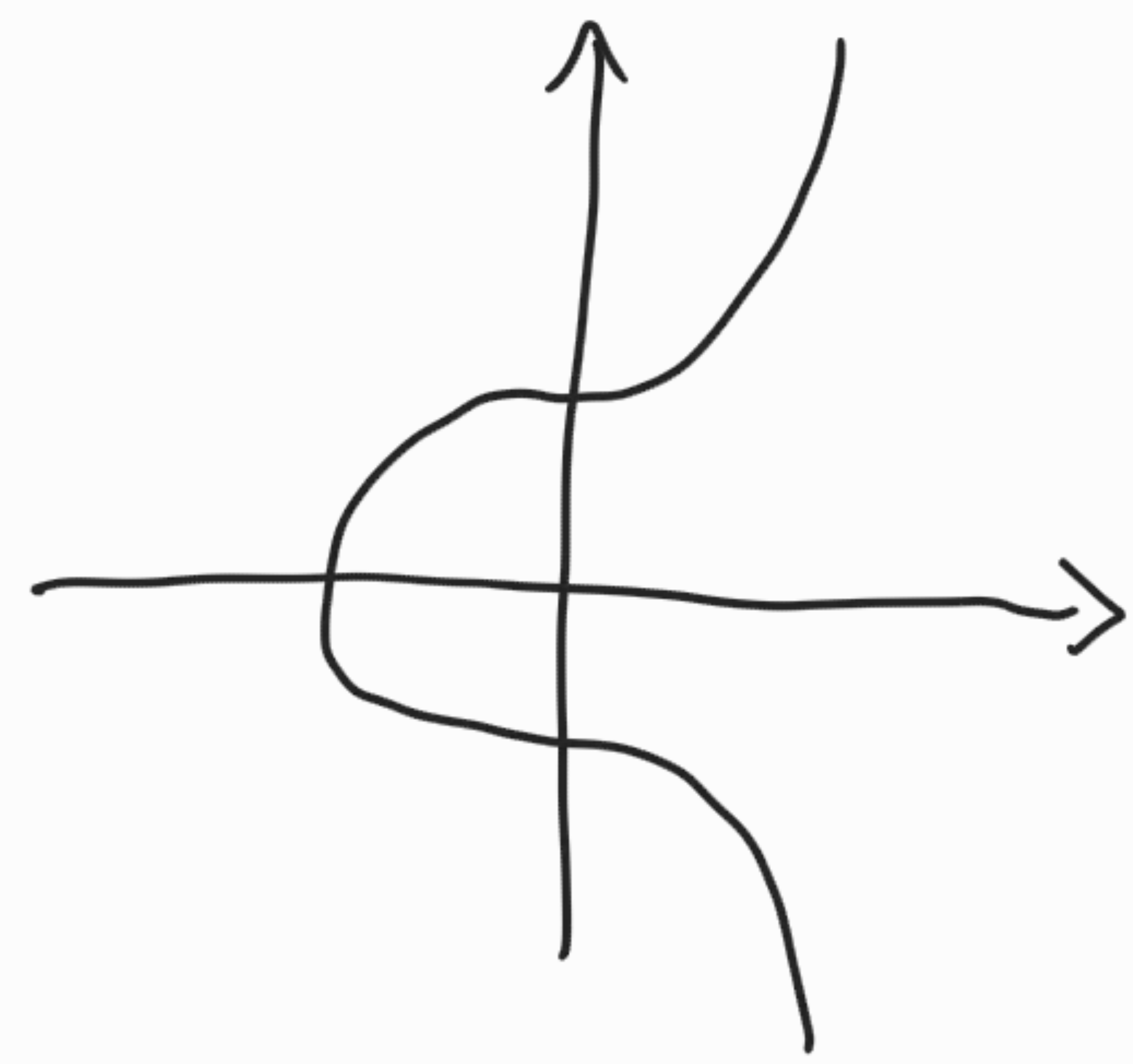
$C = \{ y^2 = ax^2+bx+c \}$ is RATIONAL, i.e. it can be parametrized by rational functions of t.

What about the integral

$$\int \frac{1}{\sqrt{x^3+x+1}}\, dx \quad ?$$

Can we do it in the same way? It turns out that the answer is NO! That's because the curve

$$C' = \{ y^2 = x^3+x+1 \}$$



is not rational. This is an example of elliptic curve.

Much of the theory of Riemann surfaces and algebraic curves arose from these integrals, and later in the course we are going to see how to compute them.

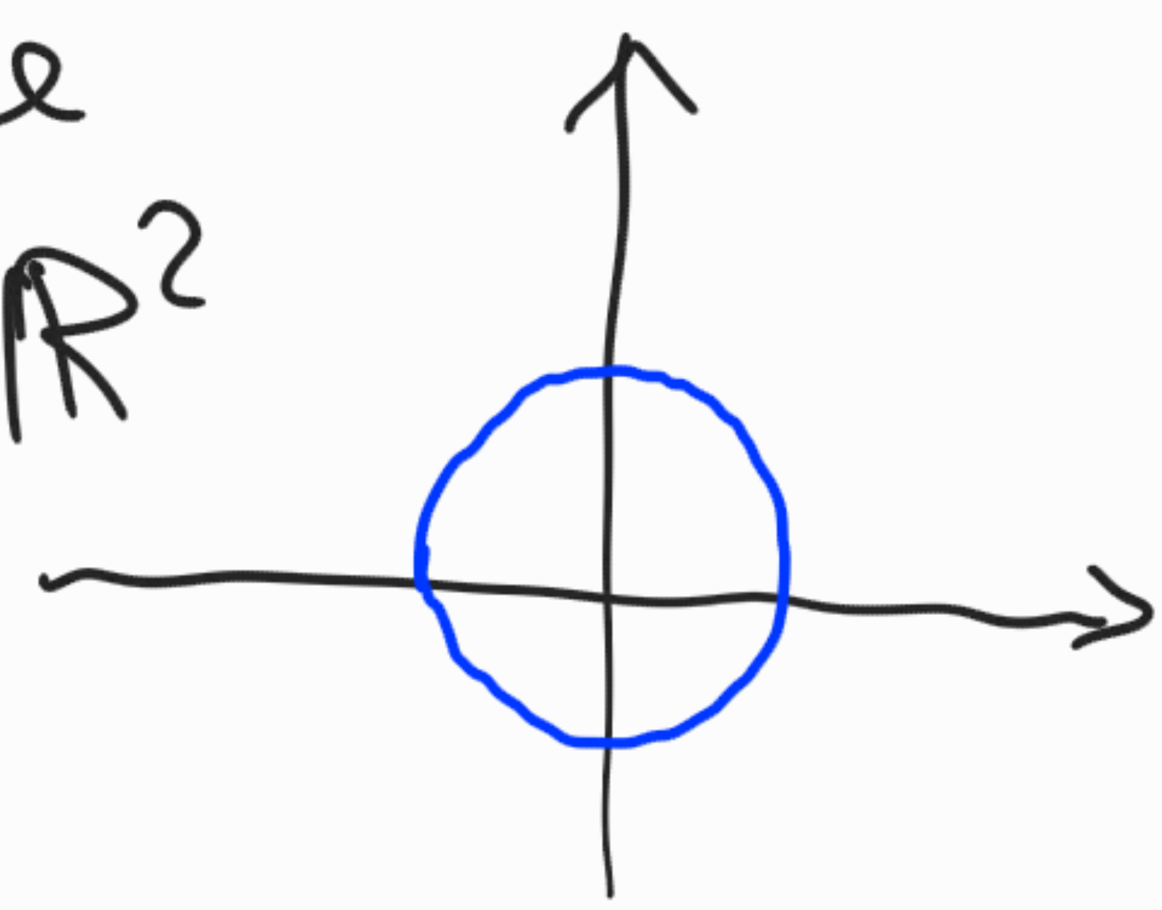
§1 : COMPLEX MANIFOLDS and HOLOMORPHIC FUNCTIONS.

The concept of MANIFOLD is one of the fundamental insights of $\geq$ 19th century mathematics.

Before that, geometric objects existed mostly only extrinsecally, usually as subsets of the affine space $\mathbb{R}^n$:

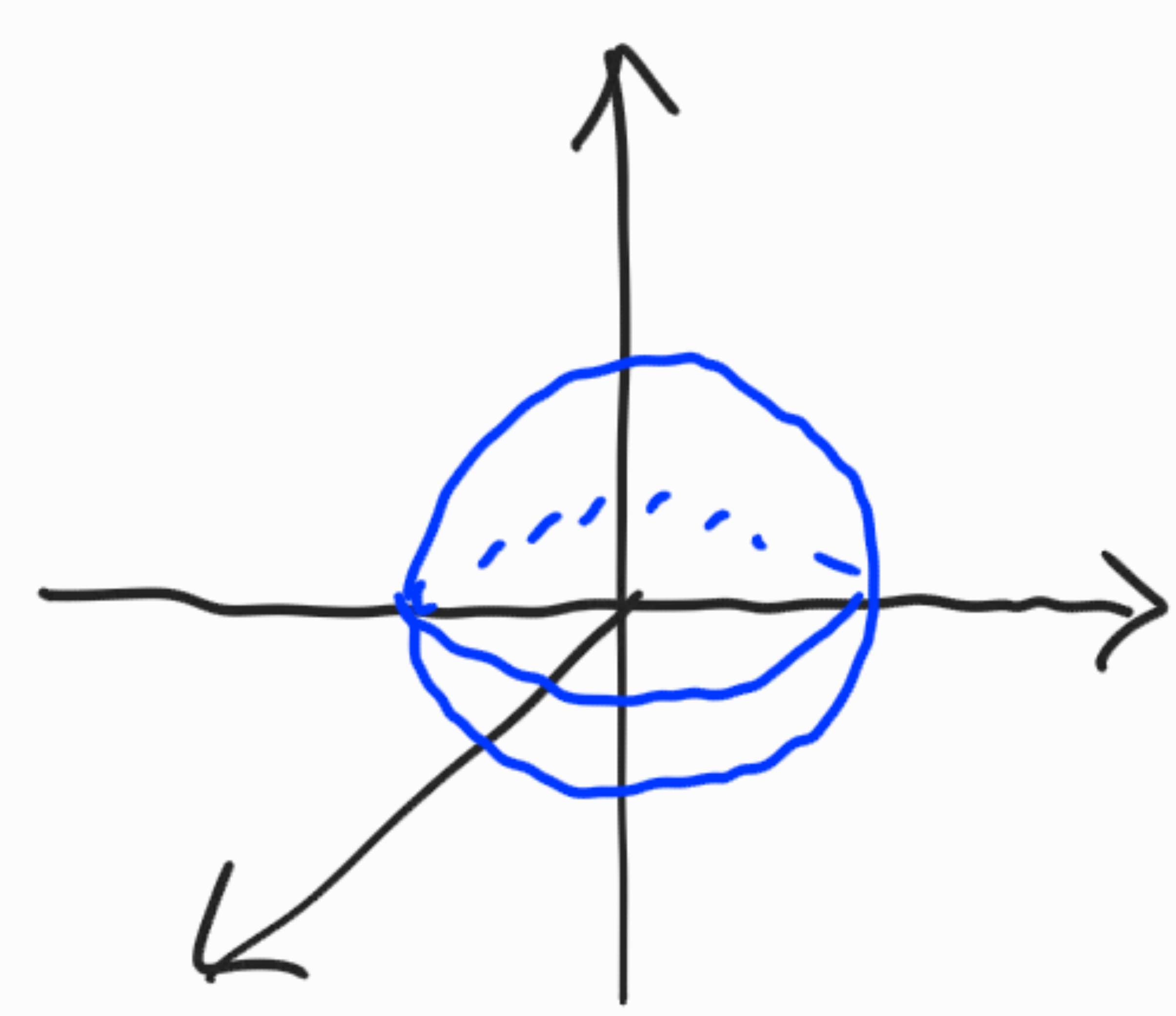
<u>Examples:</u> (1) The circle

$$\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\} \subseteq \mathbb{R}^2$$



(2) The sphere

$$\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$



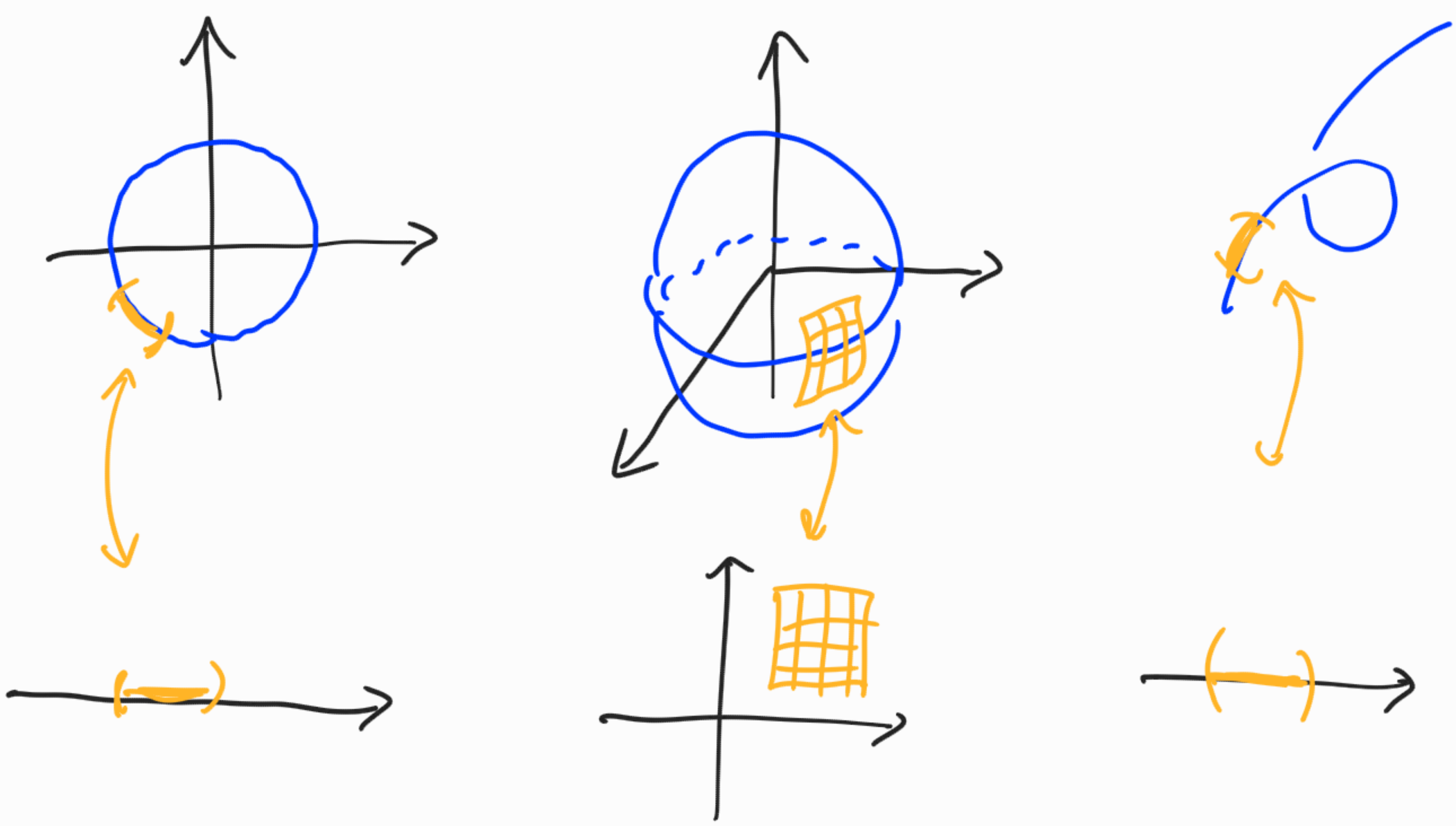
(3) A parametric curve

$$\{(t,t^2,t^3)\in\mathbb{R}^3 \mid t\in\mathbb{R}\}$$

However, RIEMANN realized that geometric objects can exist INTRINSICALLY, that is, without the need of an AMBIENT SPACE.

This leads to the concept of (SMOOTH) MANIFOLD:

SMOOTH MANIFOLD of dimension n = a topological space that <u>locally</u> looks like the affine space $\mathbb{R}^n$.



Alternatively, this means that on a manifold of dimension n we have local coordinates $(x_1, x_2, \dots, x_n)$ around each point.

In particular we can make CALCULUS (i.e. differentiate and integrate on such manifolds).

Let's now give the formal definition of a manifold.

Def: SMOOTH MANIFOLD

A smooth manifold is a second countable and Hausdorff topological space X together with an open cover

$$X = \bigcup_{i \in I} U_i \quad \text{s.t.}$$

(i) there are homeomorphisms, called CHARTS

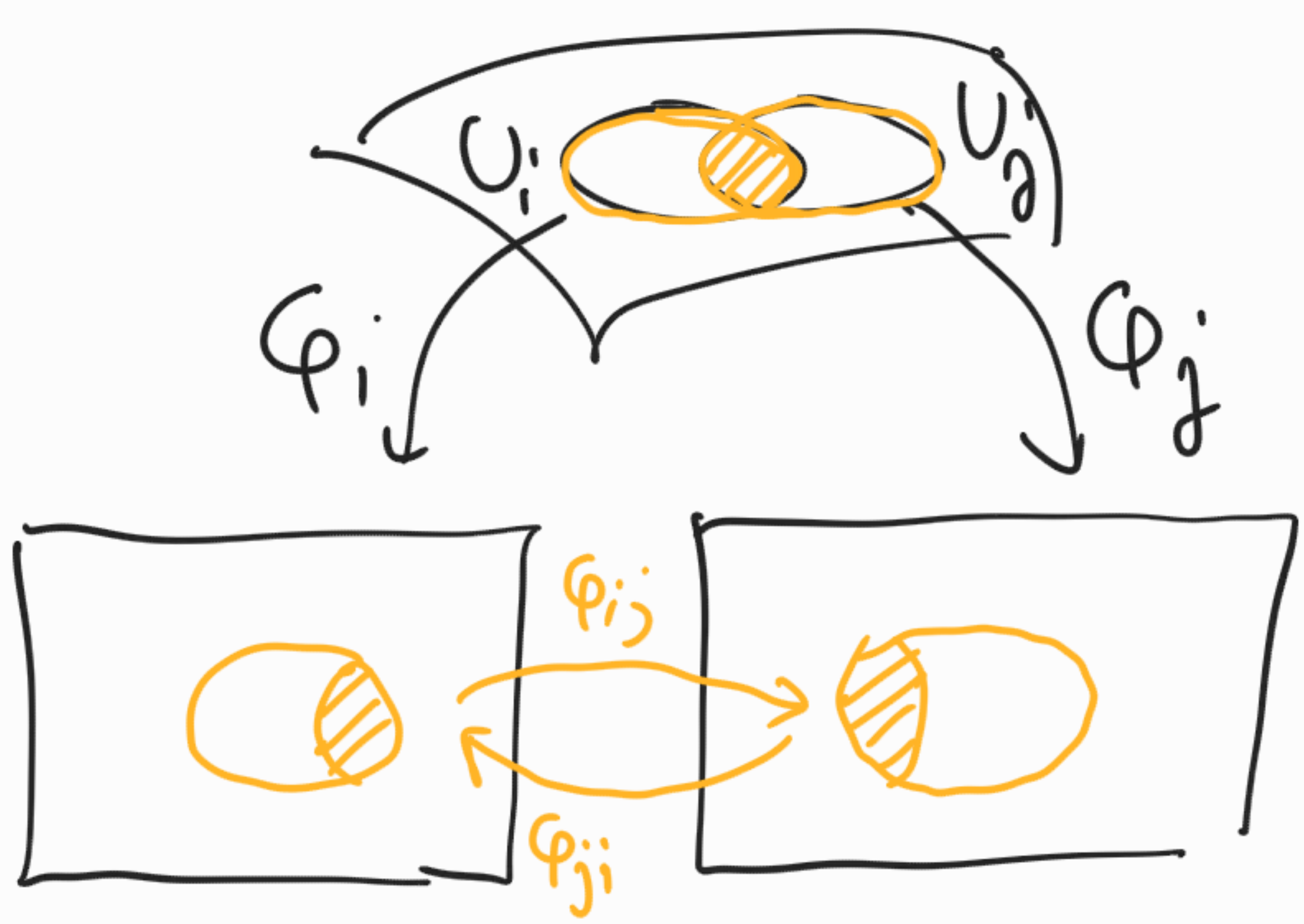
$$\varphi_i : U_i \xrightarrow{\sim} V_i$$

where $V_i \subseteq \mathbb{R}^n$ is an open subset

(ii) these CHARTS are compatible, meaning that the CHANGE of COORDINATES MAPS

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

are SMOOTH $(= C^\infty)$



Example: PROJECTIVE SPACE $\mathbb{P}^n$

This is an extremely important example for us. For starters, it is a manifold that does not appear naturally as a subset of something else.

$$\mathbb{P}^n_{\mathbb{R}} = \{ [x_0, \dots, x_n] \} \quad \text{space of } (n+1)\text{-tuples s.t.}$$

$$\begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \neq 0, \quad \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix} \overset{\text{def}}{\iff} \exists \lambda \in \mathbb{R}^* \text{ s.t. } \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$$

Why is this a manifold? Consider the charts

$$U_i = \left\{ \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_i \neq 0 \right\}.$$

These have homeomorphisms

$$\varphi_i : U_i \xrightarrow{\sim} \mathbb{R}^n \qquad \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} x_0/x_i \\ \vdots \\ \widehat{x_i/x_i} \\ \vdots \\ x_n/x_i \end{bmatrix}$$

from U_i to the affine space $\mathbb{R}^n$ with coordinates $(x_{0i}, x_{1i}, \dots, \hat{x}_{ii}, \dots, x_{ni})$

So, these charts put on each U_i the coordinates

$$x_{ai} = \frac{X_a}{X_i}$$

Now, we should check that the changes of coordinates are smooth. Why should we expect this? Well, on the intersection

$$U_i \cap U_j = \{x_i \neq 0, x_j \neq 0\}$$

we have the two sets of coordinates

$$(x_{0i}, x_{1i}, \dots, x_{ni}) \text{ with } x_{ji} \neq 0$$

$$\text{and } (x_{0j}, x_{1j}, \dots, x_{nj}) \text{ with } x_{ij} \neq 0$$

How do we change from one to the other? We see that

$$x_{aj} = \frac{X_a}{X_j} = \frac{X_a}{X_i} \cdot \frac{X_i}{X_j} = \frac{\left(\frac{X_a}{X_i}\right)}{\left(\frac{X_j}{X_i}\right)} = \frac{x_{ai}}{x_{ji}}$$

Thus, to go from one set of coordinates to the other, we simply apply multiplication by $1/x_{ji}$ which is clearly a smooth map.

In our course we will not be concerned with smooth manifolds but, rather, with COMPLEX MANIFOLDS. These are spaces that locally look like the complex affine space $\mathbb{C}^n$. Then, we will have to deal with functions that are differentiables in the complex sense: HOLOMORPHIC FUNCTIONS.

§ 1.2 : HOLOMORPHIC FUNCTIONS

Recall the definition

Def: HOLOMORPHIC FUNCTION

Let $U \subseteq \mathbb{C}$ be open. A function

$$f : U \longrightarrow \mathbb{C}$$

is holomorphic if at each point $z \in U$ the limit

$$f'(z) \overset{\text{def}}{:=} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, where h is a complex number.

This is similar to the definition of differentiable functions. However the condition of being holomorphic is <u>much stronger</u>: indeed

HOLOMORPHIC FUNCTIONS are ANALYTIC

Let $f : U \to \mathbb{C}$ be an holomorphic function. Then for each $z_0 \in U$ there is a disk around z_0 $z_0 \in \Delta \subseteq U$ s.t.

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n \quad \forall z \in \Delta$$

Conversely, any analytic function is holomorphic.

In the Exercises, we will see some of the consequences
One is:

<u>Fact</u>: Holomorphic functions are determined on an open set:

Let $U \subseteq \mathbb{C}$ be an open connected subset and $f, g : U \to \mathbb{C}$ two holomorphic functions. Then TFAE:

(i) $f = g$ on U.

(ii) $f = g$ on a nonempty open subset of U.

(iii) $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all $k \geq 0$ and one $z_0 \in U$.

<u>Proof</u>: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (i) Replacing f with $f - g$ we can assume that $g = 0$. Then consider the set

$$\Omega = \{ z \in U \mid f^{(k)}(z) = 0 \ \forall k \geq 0 \}$$

If we show that this is open, closed and nonempty, then since U is connected it must be that $\Omega = U$ and we are done.

- Ω is closed because $\Omega = \bigcap_{k \geq 0} \{ z \mid f^{(k)}(z) = 0 \}$ is an intersection of closed.
- Ω is nonempty because $z_0 \in \Omega$.
- Ω is open: suppose $w \in \Omega$. Then in a neighborhood of w we can write

$$f(z) = \sum f^{(k)}(w)(z-w)^k = 0. \qquad \square$$

§ 2 : RIEMANN SURFACE

A Riemann surface is a complex manifold of dimension one:

Def: RIEMANN SURFACE

A Riemann surface is a Hausdorff and second countable topological space S together with an open cover of charts $U_i \subseteq S$

$$\varphi_i : U_i \longrightarrow V_i \subseteq \mathbb{C} \quad \text{homeomorphisms}$$

such that the changes of coordinates are holomorphic:

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j).$$

Of course, nothing stops us from generalizing our previous definition of smooth manifolds to complex manifolds of arbitrary dimension, the only thing we need is the notion of a holomorphic function in multiple complex variables, but this is just a map

$$F : U \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^m$$
$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \longmapsto \begin{pmatrix} F_1(z_1, \dots, z_n) \\ \vdots \\ F_m(z_1, \dots, z_n) \end{pmatrix}$$

where each coordinate function F_i is holomorphic in each variable z_j.

Examples: (1) $\mathbb{P}^1(\mathbb{C})$ or the Riemann sphere.

This is the most basic Riemann surface. In general the calculations of before, show that $\mathbb{P}^n(\mathbb{C})$ is an n-dimensional complex manifold for any n. Hence, $\mathbb{P}^1(\mathbb{C})$ is a Riemann surface.
In particular, if $\mathbb{P}^1(\mathbb{C}) = \left\{ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \right\}$ we have two charts
$U_0 = \{x_0 \neq 0\}$ with local coordinate $z = \frac{x_1}{x_0}$
$U_1 = \{x_1 \neq 0\}$ with local coordinate $w = \frac{x_0}{x_1} = \frac{1}{z}$
Since $\mathbb{P}^1(\mathbb{C}) = U_0 \cup \{[0,1]\} \cong \mathbb{A}^1_z \cup \{\infty\}$
the point $[0,1]$ is called point at infinity (w.r.t. the coordinate z).

(2) AFFINE PLANE CURVES

Let $f(z_1, z_2)$ be a nontrivial polynomial in two variables. We consider the affine curve

$$C = \{ f(z_1, z_2) = 0 \} \subseteq \mathbb{C}^2$$

A point $p \in C$ is regular if

$$\left(\frac{\partial f}{\partial z_1}(p), \frac{\partial f}{\partial z_2}(p) \right) \neq 0$$

Then around each regular point the affine curve C

is locally a Riemann surface. More precisely if

$\frac{\partial f}{\partial z_1}(p) \neq 0$, then z_2 is a local coordinate around p

$\frac{\partial f}{\partial z_2}(p) \neq 0$, then z_1 is a local coordinate around p

This works as in the case of smooth manifolds, via the implicit function theorem.

(3) PROJECTIVE PLANE CURVES

Let $F(X_0, X_1, X_2) \in \mathbb{C}[X_0, X_1, X_2]_d$ be an homogeneous polynomial of degree d, such that

$$\left\{ F = \frac{\partial F}{\partial X_0} = \frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial X_2} \right\} \subseteq \mathbb{P}^2 \text{ is empty}$$

Then the zero locus $C = \{F = 0\} \subseteq \mathbb{P}^2$ is a Riemann surface. This can be c cked easily on the standard affine charts of $\mathbb{P}^2$.

<u>Rmk</u>: It turns out that the curve $C = \{F = 0\}$ is also connected, but we will prove this later.

(4) HYPERELLIPTIC CURVE

Let's consider a polynomial of even degree $2g+2 \geq 2$

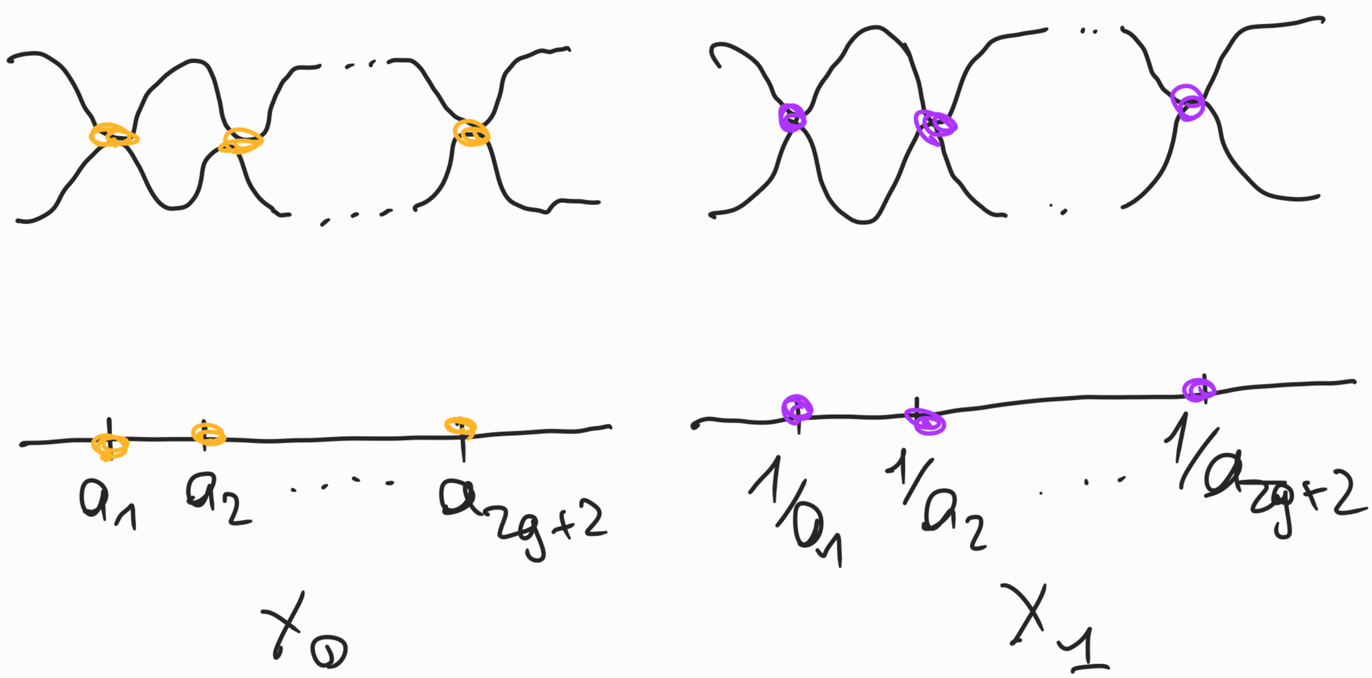
$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_{2g+2}) \quad a_i \text{ distinct}$$

The hyperelliptic curve associated to this polynomial is obtained by gluing two open sets as follows:
First consider the polynomial

$$g(u) = u^{2g+2} \cdot f\left(\frac{1}{u}\right) = (1 - a_1 \cdot u) \cdot \ldots \cdot (1 - a_{2g+2} \cdot u)$$

then we have two affine curves with open subsets

$$X_0 = \{ y^2 = f(x) \} \supseteq U_0 = \{ y^2 = f(x),\ x \neq 0 \}$$
$$X_1 = \{ v^2 = g(u) \} \supseteq U_1 = \{ v^2 = g(u),\ u \neq 0 \}$$



Then we can glue X_0, X_1 along U_0, U_1 via the maps

$$\varphi : U_0 \longrightarrow U_1 \qquad \psi : U_1 \longrightarrow U_0$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1/x \\ y/x^{g+1} \end{pmatrix} \qquad \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} 1/u \\ v/u^{g+1} \end{pmatrix}$$

Indeed, observe that if $y^2 = f(x)$ and $x \neq 0$, then

$$y^2 = f(x) \iff y^2 = x^{2g+2} g(1/x) \iff \left(\frac{y}{x^{g+1}}\right)^2 = g\left(1/x\right)$$

The resulting space $X = X_0 \cup X_1$ is a Riemann surface.

Moreover, the two natural maps

$$U_0 \to \mathbb{A}^1_x \quad (x,y) \mapsto x$$

$$U_1 \to \mathbb{A}^1_u \quad (u,v) \mapsto u$$

glue together to a global map

$$f : X \to \mathbb{P}^1.$$

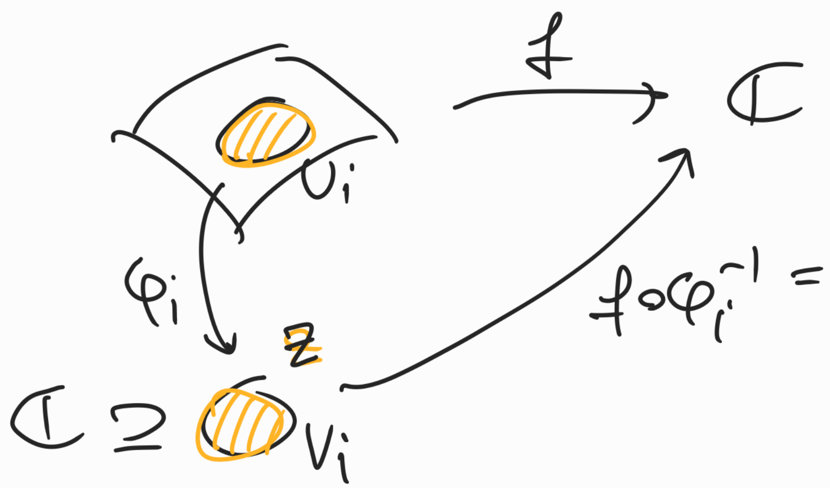
§ 2.1 : FUNCTIONS on RIEMANN SURFACES

Since on a Riemann surface we have local coordinates we can speak of holomorphic functions

Def : HOLOMORPHIC FUNCTION

Let S be a Riemann surface. A function $f : S \to \mathbb{C}$ is holomorphic if it is holomorphic in each local coordinate on S.

More formally, this means the following: we have an open cover of charts $S = \bigcup U_i$, $\varphi_i : U_i \xrightarrow{\sim} V_i \subseteq \mathbb{C}$



$f \circ \varphi_i^{-1}$ = holomorphic function in the local coordinate z on V_i

Then f is holomorphic if each composition $f|_{U_i} \circ \varphi_i^{-1}$ is holomorphic. This can be generalized to maps between any two Riemann surfaces

Def : HOLOMORPHIC MAP

A map $f : S_1 \to S_2$ between two Riemann surfaces is holomorphic if it is holomorphic in each local coordinate on S_1 and S_2.

Let $f: S_1 \to S_2$ be an holomorphic map between Riemann surfaces and let $p \in S_1$. Then Exercise 1.2 says that there are local coordinates around p and $f(p)$ such that f has the form

$$f(z) = z^e \qquad e > 0$$

Observe that e does not depend on the local coordinate. For example we can also define it as

$$e = \max\{ n \mid f^{(0)}(p) = f^{(1)}(p) = \dots = f^{(n-1)}(p) = 0 \}$$

Def: MULTIPLICITY of a MAP
With the above notation, we define the MULTIPLICITY of f at the point p as

$$\mathrm{mult}_p(f) = e.$$

Example: An holomorphic function $f: S \to \mathbb{C}$ has a zero at p if $f(p) = 0$ and $\mathrm{mult}_p(f) = e$.

Def: RAMIFICATION POINT
We say that an holomorphic map $f: S_1 \to S_2$ is ramified at p, if $\mathrm{mult}_p(f) > 1$.

§ 2.2 : MAPS of COMPACT RIEMANN SURFACES.

We will mostly care about COMPACT CONNECTED RIEMANN SURFACES, since these correspond to projective curves.

Holomorphic maps $f: S_1 \to S_2$ between two connected compact Riemann surfaces enjoy many nice properties:

(1) Any holomorphic map $f: S_1 \to S_2$ is either constant or surjective.

proof : if f is not constant then it is open (Exercise 1.3). Moreover f is closed because S_1 is compact. Hence $f(S_1)$ is both open and closed and as S_2 is connected it follows that $f(S_1) = S_2$

(2) Any nonconstant holomorphic map $f: S_1 \to S_2$ has finite fibers.

proof : from the local form of an holomorphic function, we can see that the fibers $f^{-1}(p)$ are discrete. Since S_1 is compact, $f^{-1}(p)$ must be compact and discrete, hence finite.

(3) Any nonconstant holomorphic map $f: S_1 \to S_2$ has finitely many ramification points.

proof : as for (2).

Now we can define the most important invariant of an holomorphic map between compact Riemann surfaces.

<u>Prop/Def</u> Let $f : S_1 \longrightarrow S_2$ be a nonconstant holomorphic map between compact connected Riemann surfaces.

Then for any $q \in S_2$ the number of points in the fiber, counted with multiplicity, is constant:

$$\boxed{d(q) = \sum_{p \in f^{-1}(q)} \mathrm{mult}_p(f)}$$

This is called the <u>DEGREE</u> of the map.

<u>proof</u>: We will show that d is locally constant. Let $q \in S_2$ and let $f^{-1}(q) = \{p_1, \dots, p_n\}$. Then we can find charts $U_i \subseteq S_1$ around each of the p_i and $V \subseteq S_2$ around q such that in the corresponding local coordinates, f looks like $f(z) = z^{e_i}$ around each p_i.

Now, since S_1, S_2 are both compact, by shrinking V if needed, we can assume that $f^{-1}(V) \subseteq U_1 \cup \dots \cup U_n$ (exercise in set topology).

Now we show that the function is constant on V. Let $q' \in V$ then $f^{-1}(q) \subseteq U_1 \cup \dots \cup U_n$. Then we can simply count the points of the fiber in each U_i: on each of those the map looks like $f(z) = z^{e_i}$

and it is now clear that $f^{-1}(a) = \begin{cases} \text{one pt of mult } e_i & , \text{if } a = 0 \\ e_i \text{ distinct pts} & , \text{if } a \neq 0 \end{cases}$

In any case, the sum of pts with multiplicity is e_i. Hence $d(q') = \sum e_i \; \forall q' \in V$. $\square$

§ 3. TOPOLOGY of RIEMANN SURFACE

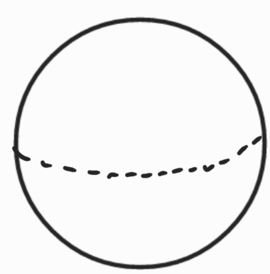
The topology of Riemann surfaces has been completly classified. Indeed, any compact Riemann surface is homeomorphic to a TORUS with g HOLES:

Def: TOPOLOGICAL GENUS

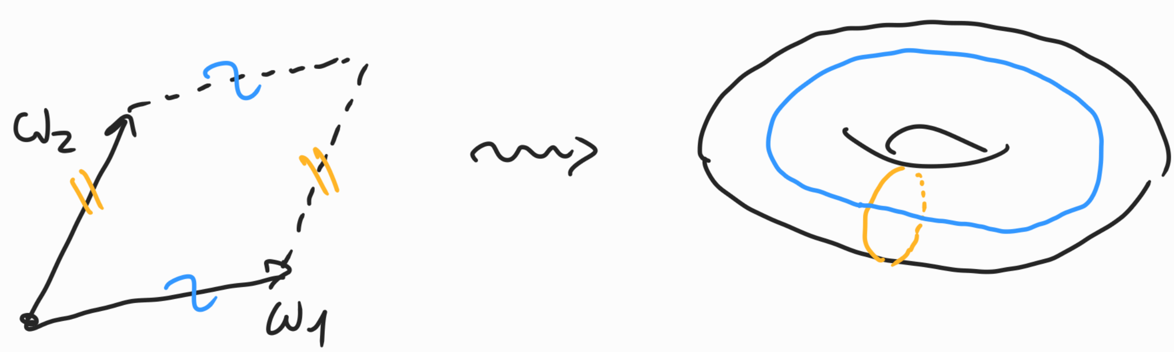
The (topological) genus of a compact Riemann surface X is the number g of holes

Examples: (1) $\mathbb{P}^1$: the projective line is homeomorphic to a sphere. This has zero holes so that the genus is 0.

$\mathbb{P}^1 \cong$



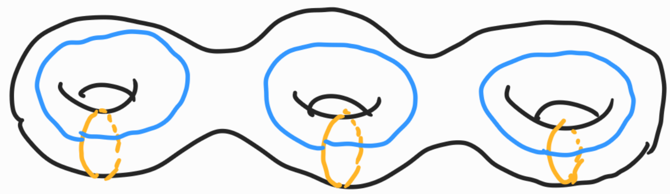
(2) A complex tores is a torus with one hole. Hence it has genus 1.



This number has other topological incarnations

- <u>HOMOLOGY</u>: The first homology group is free of rank $2g$:

$$H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$$



- <u>EULER CHARACTERISTIC</u>: The Euler characteristic is

$$\chi(X) = 2-2g.$$

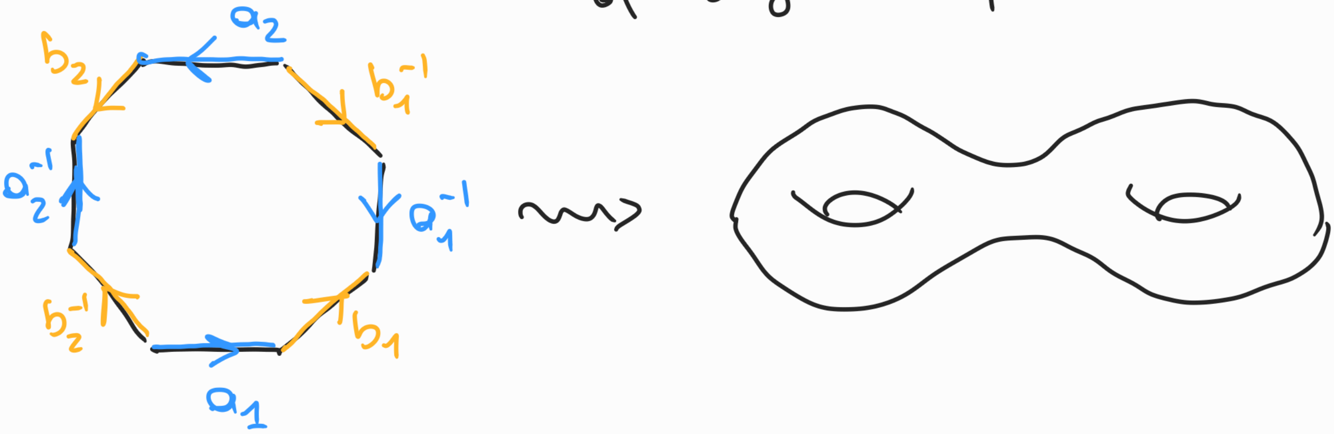
Indeed:

$$\chi(X) = \operatorname{rk} H^0(X,\mathbb{Z}) - \operatorname{rk} H^1(X,\mathbb{Z}) + \operatorname{rk} H^2(X,\mathbb{Z})$$
$$= 1-2g+1 = 2-2g.$$

- <u>TOPOLOGICAL MODEL</u>: a Riemann surface of genus g can be obtained by taking a $4g$-gon with edges

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$$

and identifying them. This generalizes the parallelogram of the genus 1 picture.



The genus, even if topological, is intimately related to the holomorphic structure of a Riemann surface. One of the most important examples is the Riemann-Hurwitz formula:

Thm: RIEMANN-HURWITZ FORMULA

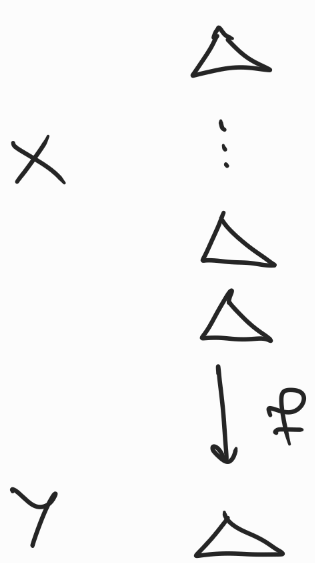
Let $f : X \to Y$ be a map of degree d between compact Riemann surfaces. Then

$$\boxed{2g(X)-2 = d\cdot(2g(Y)-2) + \sum_{p\in X}(\mathrm{mult}_p(f)-1).}$$

Rmk: Observe that $(\mathrm{mult}_p(f)-1)>0$ if and only if p is a ramification point for f. We know that these points are finitely many so that this sum makes sense.

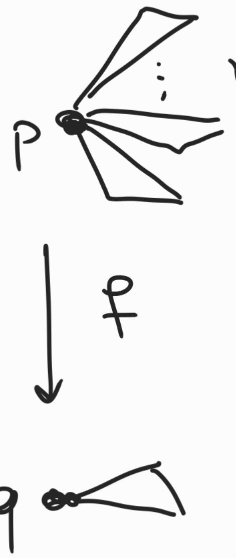
Proof: we will prove this in terms of Euler characteristics. Let's take a very fine triangulation on the Riemann surface Y, such that the branch points of $f: X \to Y$ are vertices.

The idea is to pull back this triangulation on X.



If a triangle on Y does not contain any branch point, then the preimage consists of d disjoint triangles.

Thus, the number of vertices, edges and faces is multiplied by d.



Suppose instead that a triangle has a vertex in a branch point $q \in Y$ and let $p \in f^{-1}(q)$ be a corresponding ramification point with $m = \mathrm{mult}_p(f) > 1$.

Then the map locally looks like

$$z \longmapsto z^m.$$

Then the preimage consists of m triangles meeting at the point p.

In this case, the number of vertices, edges and faces is multiplied by d, except for the vertices over a branch point, whose number is

$$d - \sum_{p \in f^{-1}(q)} \left(\mathrm{mult}_p(f) - 1 \right).$$

Hence, we see that

$$\begin{aligned} \chi(X) &= |V_X| - |E_X| + |F_X| \\ &= d|V_Y| - \sum_{q \text{ branch}} \sum_{p \in f^{-1}(q)} \left(\mathrm{mult}_p(f) - 1\right) - d|E_Y| + d|F_Y| \\ &= d\,\chi(Y) - \sum_{p \in X} \left(\mathrm{mult}_p(f) - 1\right). \end{aligned}$$

which is exactly the Riemann-Hurwitz formula. $\square$

As an application, let us compute the genus of a smooth plane curve:

<u>Example</u>: <u>GENUS of a SMOOTH PLANE CURVE</u>

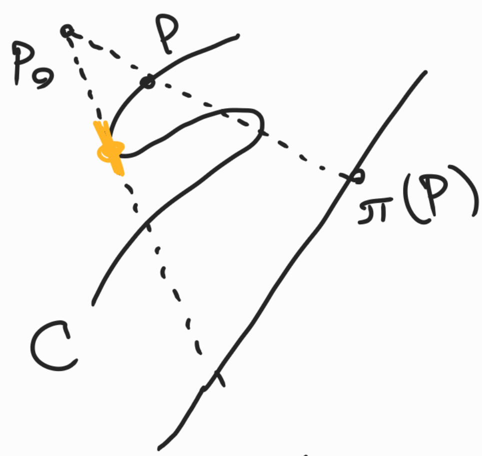
Let $C = \{F(X,Y,Z) = 0\}$ be a smooth plane curve of degree d, meaning that F is an homogeneous polynomial of degree d such that

$$\left\{ F = \frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0 \right\} = \emptyset.$$

We claim that the genus of C is

$$\boxed{g(C) = \frac{(d-1)(d-2)}{2}}$$

Consider a general point $P_0 \in \mathbb{P}^2$ and let's consider the projection from P_0 onto a line $L \subseteq \mathbb{P}^2$



$$\pi : C \longrightarrow L \cong \mathbb{P}^1$$

What is the degree of this map?

If $Q \in L$ is a point, then

$$\pi^{-1}(Q) = \mathrm{line}(P_0, Q) \cap C$$

So that $\pi^{-1}(Q)$ consists of d points, counted with multiplicity. Moreover we see that the ramification points are exactly those $P \in C$ such that $\mathrm{line}(P_0, P)$ is tangent to C at P.

Now, one can show (in the Exercises :)) that, since P_0 is general, every line passing through P_0 is tangent to C with multiplicity at most two. In other words $mult_p(\pi) \leq 2$, so that $\sum(mult_p(\pi)-1)$ is given by the number of points $P \in C$ such that T_pC passes through P_0

Let's write $P_0 = [a_0, b_0, c_0]$. The tangent line to C at $P \in C$ is

$$T_pC = \left\{ \frac{\partial F}{\partial x}(P)X + \frac{\partial F}{\partial y}(P)Y + \frac{\partial F}{\partial z}(P)Z = 0 \right\}$$

Then this line passes through P_0 iff

$$\frac{\partial F}{\partial x}(P)a_0 + \frac{\partial F}{\partial y}(P)b_0 + \frac{\partial F}{\partial z}(P)c_0 = 0.$$

Hence,

$$\{P \in C \mid T_pC \ni P_0\} = \{F=0\} \cap \left\{ \frac{\partial F}{\partial x}\cdot a_0 + \frac{\partial F}{\partial y}\cdot b_0 + \frac{\partial F}{\partial z} c_0 = 0 \right\}$$

Since F has degree d and the derivatives degree $d-1$, BEZOUT THEOREM says that there are $d(d-1)$ such points. Now we use Riemann-Hurwitz:

$$\begin{aligned} 2g(C)-2 &= d\,(2g(L)-2) + \sum_{P \in C}(mult_p(\pi)-1) \\ &= -2d + d(d-1), \end{aligned}$$

and then the formula for the genus follows. $\square$

§ 4: MEROMORPHIC FUNCTIONS

Recall that a meromorphic function on an open set $U \subseteq \mathbb{C}$ is a quotient of two holomorphic functions

$$f = \frac{g(z)}{h(z)} \qquad g, h \text{ holomorphic on } U, \; h \neq 0$$

__Rmk__: So, a meromorphic function is not a function, but a formal quotient of holomorphic functions. We will see later that they can be interpreted as maps $f : U \to \mathbb{P}^1$.

We can add and multiply meromorphic functions in the usual way; and with these operations, the set

$$\mathbb{C}(U) = \left\{ \begin{array}{c} \text{meromorphic} \\ \text{functions on } U \end{array} \right\} \quad \text{is a field.}$$

Now, let f be an holomorphic function and $z_0 \in U$. Then in a neighborhood of z_0 we can write

$$f = \frac{g(z)}{h(z)} = \frac{(z-z_0)^e g_0(z)}{(z-z_0)^f h_0(z)} \qquad g_0, h_0 \text{ holomorphic}, \; g_0(z_0), h_0(z_0) \neq 0$$

$$= (z-z_0)^{e-f} \left(\frac{g_0(z)}{h_0(z)} \right)$$

$$= (z-z_0)^m f_0(z) \qquad m \in \mathbb{Z}, \; f_0 \text{ holomorphic and } f_0(z_0) \neq 0.$$

<u>Def</u>: ORDER of a MEROMORPHIC FUNCTION

With the above notation, we define the order of the meromorphic function f at z_0 as

$$\mathrm{ord}_{z_0}(f) = m \qquad \left[\mathrm{ord}_{z_0}(0) \overset{\text{def}}{:=} \infty\right]$$

- $m > 0$: z_0 is called a ZERO of f of order m.
- $m < 0$: z_0 is called a POLE of f of order $-m$.

<u>Rmk</u>: The map

$$\mathrm{ord}_{z_0} : \mathbb{C}(U) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

is a DISCRETE VALUATION. This means that

1) $\mathrm{ord}_{z_0}(f \cdot g) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$.
2) $\mathrm{ord}_{z_0}(f+g) \geq \min\{\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)\}$

as one can easily prove.

<u>Rmk</u>: Any holomorphic function $f(z)$ gives a meromorphic function $\frac{f(z)}{1}$ without poles. Conversely, any meromorphic function without poles comes from an holomorphic function.

We know that holomorphic functions have a local expansion into a power series. What about meromorphic functions?

Let f be meromorphic and $z_0 \in U$. Then we can write

$$f = (z-z_0)^m f_0(z) \qquad \begin{array}{l} f_0(z) \text{ holomorphic} \\ f_0(z_0) \neq 0 \end{array}$$

$$= (z-z_0)^m \sum_{n=0}^{\infty} a_{n,0} (z-z_0)^n$$

$$= \sum_{n=m}^{\infty} a_n (z-z_0)^n \qquad [a_n = a_{0,n-m}]$$

$$= a_m (z-z_0)^m + a_{m+1} (z-z_0)^{m+1} + \dots$$

This is a power series with finitely many terms with negative exponent (when $m<0$, i.e. when z_0 is a pole). Such a series is called a LAURENT SERIES.

All of this can be generalized to any Riemann surface

<u>Def</u>: MEROMORPHIC FUNCTION on a RIEMANN SURFACE
A meromorphic function on a Riemann surface S is a collection $f_i = \frac{g_i}{h_i}$ of meromorphic functions on each chart $U_i \subseteq S$. On $U_i \cap U_j$ we require that $f_i = f_j$ meaning $g_i h_j = g_j h_i$.

All the other notions extend to this general case:

$$\mathbb{C}(S) = \left\{ \begin{array}{c} \text{meromorphic functions} \\ \text{on } S \end{array} \right\} \quad \text{field}$$

$p \in S \quad \mathrm{ord}_p : \mathbb{C}(S) \to \mathbb{Z} \cup \{\infty\}$ discrete valuation.

§4 MEROMORPHIC FUNCTIONS

Recall: meromorphic function on an open set $U \subseteq \mathbb{C}$ is the quotient of two holomorphic functions

$$f = \frac{g}{h} \quad , \; h \neq 0$$

order of f at z_0

$$= (z-z_0)^m \underbrace{f_0(z)}_{\text{homomorphic at } z_0,\ f_0(z_0) \neq 0} \quad \text{locally around } z_0 \in U$$

$\rightsquigarrow$ $\mathbb{C}(U) = \{\text{meromorphic functions on } U\}$

$$\mathrm{ord}_{z_0} : \mathbb{C}(U) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

is a <u>discrete valuation</u>

<u>DEF:</u> (MEROMORPHIC FUNCTIONS ON RS)
A meromorphic function on a Riemann surface X is a collection $f_i = \frac{g_i}{h_i}$ of meromorphic functions on each chart $U_i \subseteq X$ such that $f_i = f_j$ on $U_i \cap U_j$
(that means $h_i g_j = h_j g_i$)

$\rightsquigarrow$ $\mathbb{C}(X) = \{\text{meromorphic function on } X\}$

& for all $p \in X$: $\mathrm{ord}_p : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}$

<u>Exm:</u> $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(z)$

MEROMORPHIC FUNCTIONS & HOLOMORPHIC MAPS TO $\mathbb{P}^1$

Let X be a Riemann surface and let f be a meromorphic function on X

$$f = g_i / h_i \text{ on } U_i \subseteq X$$

Define $F : X \to \mathbb{P}^1$ by gluing

$$F : U_i \to \mathbb{P}^1, \quad F(z) = [1, g_i/h_i(z)] = $$

$$= [h_i/g_i(z), 1]$$

$$= [h_i(z), g_i(z)]$$

Well-defined: locally at $z_0 \in U_i$

$$g_i(z) = (z-z_0)^e g_0(z) \quad \text{with } g_0(z_0) \neq 0$$
$$h_i(z) = (z-z_0)^f h_0(z) \quad\quad h_0(z_0) \neq 0$$

Set $m = e - f$

$$F(z) = \left[1, (z-z_0)^m \frac{g_0(z)}{h_0(z)}\right] = \left[(z-z_0)^{-m}, \frac{g_0(z)}{h_0(z)}\right] =$$

$$= [h_0(z), (z-z_0)^m g_0(z)] = [(z-z_0)^{-m} h_0(z), g_0(z)]$$

A pole of f gets mapped to $[0,1]$
a zero of f gets mapped to $[1,0]$

FROM MAPS TO $\mathbb{P}^1$ TO MEROMORPHIC FUNCTIONS

Let X be a Riemann surface and
$F: X \to \mathbb{P}^1$ be a holomorphic map
Let $p \in X$ and let $\Delta \subseteq X$ be a neighborhood of p, $\Delta \cong$ open neighborhood of $0 \in \mathbb{C}$
$p \cong 0$
Let z be a local coordinate

① $F(p) \neq 0, \infty$: Then $F: \Delta \to \mathbb{C}$ holomorphic
$\rightsquigarrow$ meromorphic function $f = \frac{F}{1}$ on Δ

② $F(p) = 0$:
Then $f = \frac{F}{1} = z^m \frac{F_0(z)}{1}$ with

$F_0(0) \neq 0$:
$\Rightarrow \mathrm{ord}_p(f) = \mathrm{mult}_0(F)$

③ $F(p) = \infty$: $F: \Delta \to \mathbb{P}^1$, $U_1 = \{x_1 \neq 0\} \subseteq \mathbb{P}^1$

$$F(z) = z^m F_0(z), \quad F_0(0) \neq 0$$

On $\Delta \setminus \{0\}$: $\quad \Delta \setminus \{0\} \to U_1 \setminus \{\infty\} \cong U_0 \setminus \{0\}$

$$z \mapsto \underbrace{\frac{1}{z^m F_0(z)}}_{f(z)}$$

meromorphic function f

$$\mathrm{ord}_p(f) = -\mathrm{mult}_0(F)$$

These local representations of f glue to a meromorphic function on X.

$\boxed{\text{FACT: For any Riemann surface } X, \text{ there is a correspondence } \left\{\begin{array}{c}\text{meromorphic functions}\\ \text{on } X\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{holomorphic maps}\\ X \to \mathbb{P}^1 \text{ (that are not identically } \infty)\end{array}\right\}}$

$$\text{ZEROES OF } f \leftrightarrow F^{-1}([1,0])$$

$$\text{POLES OF } f \leftrightarrow F^{-1}([0,1])$$

$$\mathrm{ord}_p(f) = \begin{cases} \mathrm{mult}_p(F) & \text{if } F(p) = [1,0] \\ -\mathrm{mult}_p(F) & \text{if } F(p) = [0,1] \end{cases}$$

Prop: Let X be a compact Riemann surface,

be

Let $f \in \mathbb{C}(X)^{\vee}$ nonconstant

$$\sum_{p \in X} \operatorname{ord}_p(f) = 0$$

Proof: $\sum_{p \in X} \operatorname{ord}_p(f) = \sum_{\substack{p \in X \\ \operatorname{ord}_p(f) > 0}} \operatorname{ord}_p(f) - \sum_{\substack{p \in X \\ \operatorname{ord}_p(f) < 0}} |\operatorname{ord}_p(f)|$

$$= \sum_{p \in F^{-1}([1,0])} \operatorname{mult}_p(F) - \sum_{p \in F^{-1}([0,1])} \operatorname{mult}_p(F) =$$

$$= \deg(F) - \deg(F) = 0 \qquad \square$$

Lemma: Let X be a compact Riemann surface and let f be a nonconstant meromorphic function on f. Then f has at least 1 pole.

Proof: $F: X \to \mathbb{P}^1$ holomorphic, non-constant

$\Rightarrow F$ is surjective. $\square$

Lemma: Let X be a compact Riemann surface with a meromorphic function with exactly one pole (counted with multiplicity). Then $X \cong \mathbb{P}^1$

EXAMPLES: RATIONAL FUNCTIONS ON $\mathbb{P}^1$

→ Functions with arbitrary zeroes and poles

$f = \frac{(z-a)}{1}$ simple root at a and a simple pole at ∞

$f = \frac{1}{(z-a)}$ simple pole at a and a simple root at ∞

§5 DIVISORS

DEF (DIVISORS):

Let X be a compact Riemann surface

$\mathrm{Div}(X)$: free abelian group on X

$$= \left\{ \sum_{p\in X} n_p \cdot p \;\middle|\; n_p \in \mathbb{Z},\ n_p = 0 \text{ for almost all } p \in X \right\}$$

An element of $\mathrm{Div}(X)$ is called DIVISOR on X.

The DEGREE of a divisor $D = \sum_{p\in X} n_p \cdot p \in \mathrm{Div}(X)$ is $\deg(D) = \sum_{p \in X} n_p \in \mathbb{Z}$.

The map $\deg: \mathrm{Div}(X) \to \mathbb{Z}$ is a group homomorphism.

Let f be a meromorphic function on X

Define $\mathrm{div}(f) = \sum_{p \in X} \mathrm{ord}_p(f) \cdot p$

Every divisor of this form is called a

PRINCIPAL DIVISOR.

The set $\mathrm{PDIV}(X)$ of principal divisors is a subgroup of $\mathrm{DIV}(X)$

DIVISOR of ZEROES: $\mathrm{div}_0(f) = \sum_{\substack{p \in X \\ \mathrm{ord}_p(f) > 0}} \mathrm{ord}_p(f) \cdot p$

DIVISOR of POLES: $\mathrm{div}_\infty(f) = - \sum_{\substack{p \in X \\ \mathrm{ord}_p(f) < 0}} \mathrm{ord}_p(f) \cdot p$

$\leadsto \mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$

<u>Lemma:</u> Let f, g be meromorphic functions on X

(a) $\mathrm{div}(f \cdot g) = \mathrm{div}(f) + \mathrm{div}(g)$

(b) $\mathrm{div}(1/f) = -\mathrm{div}(f)$

(c) $\operatorname{div}\left(\frac{f}{g}\right) = \operatorname{div}(f) - \operatorname{div}(g)$

Proof: ord_p is a discrete valuation.

LINEAR EQUIVALENCE

Def:
Two divisors $D_1, D_2 \in \operatorname{Div}(X)$ are **LINEARLY EQUIVALENT**, written $D_1 \sim D_2$, if their difference is a principal divisor i.e.

$$D_1 - D_2 = \operatorname{div}(f) \text{ for some } f \in \mathbb{C}(X).$$

Exm: linear equivalence on $\mathbb{P}^1$:

Every divisor of degree 0 on $\mathbb{P}^1$ is principal:

$$D = \sum_{\lambda_i \in \mathbb{C}} n_{\lambda_i} \lambda_i + e_\infty \cdot \infty$$

$$f = \prod_{n_{\lambda_i} \neq 0} (z - \lambda_i)^{n_{\lambda_i}} \Rightarrow \operatorname{div}(f) = D.$$

$$\Rightarrow \operatorname{Div}(\mathbb{P}^1) / \operatorname{PDiv}(\mathbb{P}^1) \cong \mathbb{Z}$$

Lemma: Let X be a compact Riemann surface

(a) Linear equivalence is an equivalence relation on $\mathrm{Div}(X)$

(b) $D \sim 0 \iff D \in \mathrm{PDiv}(X)$

(c) $D_1 \sim D_2 \Rightarrow \deg(D_1) = \deg(D_2)$

RAMIFICATION AND BRANCH DIVISORS

Let $F: X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces

The RAMIFICATION DIVISOR of F, denoted R_F, is

$$R_F = \sum_{p \in X} (\mathrm{mult}_p(F) - 1) \cdot p \in \mathrm{DIV}(X)$$

The BRANCH DIVISOR of F, denoted B_F, is

$$B_F = \sum_{q \in Y} \left(\sum_{p \in F^{-1}(q)} (\mathrm{mult}_p(F) - 1) \right) \cdot q \in \mathrm{DIV}(Y)$$

Hurwitz formula:

$$2g(Y)-2 = \deg(F)\,(2g(X)-2) + \deg(R_F)$$

INTERSECTION DIVISORS

Let X be a smooth projective curve (in $\mathbb{P}^2$)
Let $G(x,y,z) \neq 0$ be a homogeneous polynomial of degree d.

We define the INTERSECTION DIVISOR $\operatorname{div}(G)$ on X

Fix a point $p \in X$ where G vanishes and choose a polynomial H with $\deg(G) = \deg(H) = d$ which does not vanish at p
Then G/H is a meromorphic function on X

Define n_p to be the order of G/H at p

and $$\operatorname{div}(G) = \sum_{\substack{p \in X \\ G(p)=0}} n_p \cdot p$$

Lemma: well-defined...

MEROMORPHIC FUNCTIONS ON COMPLEX TORI

$\mathbb{P}^1$: meromorphic functions are rational: $\frac{p}{q}$ homogeneous of the same deg.

$$\mathbb{P}^1 = \mathbb{C}^2/\mathbb{C}^* \quad : \quad \frac{p(\lambda x)}{q(\lambda x)} = \frac{\lambda^d p(x)}{\lambda^d q(x)} = \frac{p(x)}{q(x)}$$

Complex torus: $\mathbb{C}/L$, where $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\mathbb{R}$-linearly independent $\omega_1, \omega_2 \in \mathbb{C}$

① Let f be an L-periodic holomorphic function, i.e. $f(x+z) = f(x)$ for all $z \in L$.
Then f induces a holomorphic function $\bar{f} : \mathbb{C}/L \to \mathbb{C}$
$\overset{\mathbb{C}/L \text{ compact}}{\Longrightarrow}$ $\bar{f}$ is constant

② But every meromorphic function on $\mathbb{C}/L$ is a quotient of <u>theta functions</u>

Without loss of generality, we can assume $L = \mathbb{Z} + \mathbb{Z}\tau$ with $\operatorname{Im}(\tau) > 0$

<u>Def:</u> (JACOBI) THETA FUNCTIONS

The theta function of $L = \mathbb{Z} + \mathbb{Z}\tau$ is

$$\theta_\tau(z) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i \left(n^2\tau + 2nz\right)\right)$$

<u>Claim 1:</u> This series converges absolutely and uniformly on compact subsets of $\mathbb{C} \times \underbrace{\mathbb{H}}_{=\{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}}$

Proof: Suppose $|\mathrm{Im}(z)| < c \in \mathbb{R}$ and $\mathrm{Im}(\tau) > \varepsilon \in \mathbb{R}$

$$|\exp(\pi i\,(n^2\tau + 2nz))| = |\exp(-\pi\,(n^2\,\mathrm{Im}(\tau) + 2n\,\mathrm{Im}(z)))|$$
$$< \exp(-\pi\varepsilon n^2)\cdot\exp(2\pi c\cdot n) = \exp(-\pi\varepsilon)^{n^2}\cdot\exp(2\pi c)^n$$
$$= \exp(-\pi\varepsilon)^{n(n-n_0)}\cdot\left[\exp(-\pi c)^{n_0}\cdot\exp(2\pi c)\right]^n$$

Choose $n_0 \in \mathbb{N}$ such that $\exp(-\pi c)^{n_0}\cdot\exp(2\pi c) < 1$

so that $|\exp(\pi i\,(n^2\tau + 2nz))| < \exp(-\pi\varepsilon)^{n(n-n_0)}$

For large enough n, this is small enough.

series convergence $\Rightarrow$ $\sum_{n=-\infty}^{\infty} \exp(\pi i\,(n^2\tau + 2nz))$ converges absolutely and uniformly on compact sets. $\square$

So $\Theta_\tau(z) : \mathbb{C} \to \mathbb{C}$ is holomorphic.

Claim 2: $\Theta_\tau(z)$ is quasi-periodic with respect to $L = \mathbb{Z} + \mathbb{Z}\tau$

(a) $\Theta_\tau(z+1) = \Theta_\tau(z)$

(b) $\Theta_\tau(z+\tau) = \exp(-\pi i\tau - 2\pi i z)\,\Theta_\tau(z)$

Proof: (b) $\Theta_\tau(z+1) = \sum_{n=-\infty}^{\infty} \exp(\pi i\,(n^2\tau + 2n(z+\tau)))$

$$n^2\tau + 2n(z+\tau) = n^2\tau + 2n\tau + 2nz = (n+1)^2\tau - \tau + 2(n+1)z - 2z$$

(a) $\Theta_\tau(z+\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i\,(n^2\tau + 2n(z+1)))$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\exp(2n\pi i)}_{=1}\,\exp(\pi i\,(n^2\tau + 2nz)) = \Theta_\tau(z).$$
$\square$

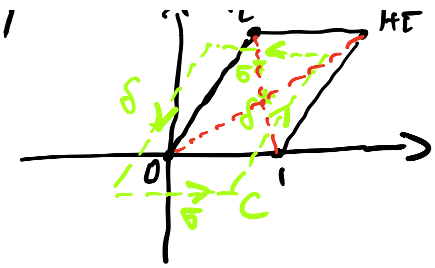
ZEROES OF THE THETA FUNCTION

Step 1: Count the zeroes in a parallelogram.

If f is holomorphic function, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \text{number of zeroes in the region bounded by } C$$

$$\log(\Theta(z+\tau))' = \log\left[\exp(-\pi i\tau - 2\pi i z)\,\Theta(z)\right]'$$
$$= \quad -2\pi i \quad + \left[\log(\Theta(z))\right]'$$



$$\oint_C \log(\Theta)'(z)\,dz$$

$$\int_\delta \frac{\Theta'}{\Theta}dz + \int_{\delta^*} \frac{\Theta'}{\Theta}dz = \int_\delta \frac{\Theta'(z)}{\Theta(z)}dz - \int_\delta \frac{\Theta'(z+1)}{\Theta(z+1)}dz = 0$$

$$\int_\sigma \frac{\Theta'(z)}{\Theta(z)}dz + \int_{\sigma^*} \frac{\Theta'(z)}{\Theta(z)}dz = \int_\sigma \frac{\Theta'(z)}{\Theta(z)}dz - \int_\sigma \frac{\Theta'(z+\tau)}{\Theta(z+\tau)}dz$$
$$= \int_\sigma \frac{\Theta'(z)}{\Theta(z)}dz - \int_\sigma -2\pi i\,dz - \int_\sigma \frac{\Theta'(z)}{\Theta(z)}dz$$
$$= 2\pi i$$

$\Rightarrow$ $\Theta_\tau(z)$ has one simple root in the fundamental parallelogram.

Step 2: Find the root.

$$\vartheta_{1/2,1/2}(-z,\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i\,(n+\tfrac{1}{2})^2\tau + 2\pi i\,(n+\tfrac{1}{2})(-z+\tfrac{1}{2})\right) =$$

$$\overset{m=-n-1}{=} \sum_{m=-\infty}^{\infty} \exp\left(\pi i\,(-m-\tfrac{1}{2})^2\tau + 2\pi i\,(-m-\tfrac{1}{2})(-z+\tfrac{1}{2})\right) =$$

$$= \sum_{m=-\infty}^{\infty} \exp\left(\pi i\,(m+\tfrac{1}{2})^2\tau + 2\pi i\,(m+\tfrac{1}{2})(z+\tfrac{1}{2}) + 2\pi i\,(m+\tfrac{1}{2})\right)$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{\exp(2\pi i(m+\tfrac{1}{2}))}_{=-1} \cdot \exp\left(\pi i\,(m+\tfrac{1}{2})^2\tau + 2\pi i\,(m+\tfrac{1}{2})(z+\tfrac{1}{2})\right)$$

$$= \sum_{m=-\infty}^{\infty} -\exp\left(\pi i\,(m+\tfrac{1}{2})^2\tau + 2\pi i\,(m+\tfrac{1}{2})(z+\tfrac{1}{2})\right)$$

$$= -\vartheta_{\frac{1}{2},\frac{1}{2}}(z,\tau)$$

$z=0$: $\Rightarrow$ $\underbrace{\vartheta_{1/2,1/2}(0,\tau)}_{} = 0$ $\quad\Rightarrow \Theta_\tau(\tfrac{1}{2}+\tfrac{1}{2}\tau) = 0$

$$= \theta_\tau\left(\tfrac{1}{2}+\tfrac{1}{2}\tau\right)\cdot \exp\left(\pi i \tfrac{1}{4}\tau + 2\pi i \tfrac{1}{4}\right)$$

SUMMARY: $\theta_\tau(z)$ has one simple zero inside the fundamental parallelogram and that zero is $\frac{1}{2}+\frac{1}{2}\tau$.

So: $\theta_\tau(z) = 0 \iff z = \frac{1}{2}+\frac{1}{2}\tau + a + b\tau$ for some $a, b \in \mathbb{Z}$.

(and all those roots are simple)

Def: We write $\theta_\tau^{(x)}(z)$ for the theta function

$$\theta_\tau^{(x)}(z) = \theta\left(z - \tfrac{1}{2} - \tfrac{1}{2}\tau - x\right)$$

for any $x \in \mathbb{C}$.

The zeroes of $\theta_\tau^{(x)}(z)$ are $x + L$.

Exercise: $\theta_\tau^{(x)}(z+\tau) = -\exp(-2\pi i (z-x))\, \theta_\tau^{(x)}(z)$

Consider $R(z) = \dfrac{\prod_i \theta_\tau^{(x_i)}(z)}{\prod_j \theta_\tau^{(y_j)}(z)}$ $\to$ meromorphic function on $\mathbb{C}$

Is it L-periodic?

$R(z+1) = R(z)$: ✓

$$R(z+\tau) = \prod_{i=1}^{m} \theta_\tau^{(x_i)}(z+\tau) \Big/ \prod_{j=1}^{n} \theta_\tau^{(y_j)}(z+\tau) =$$

$$= \underbrace{\cancel{(-1)^{m-n}} \exp\left(\cancel{-2\pi i (m-n) z} + 2\pi i \left[\sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j\right]\right)}_{=1 \text{ for all } z \in \mathbb{C}} R(z)$$

$$\Rightarrow \boxed{m = n,\ \sum_{i=1}^{m} x_i - \sum_{j=1}^{m} y_j \in \mathbb{Z}}$$

We have shown:

Prop: Fix an integer $d \in \mathbb{N}$ and complex numbers $x_1, \dots, x_d \in \mathbb{C}$ and $y_1, \dots, y_d \in \mathbb{C}$ such that $\sum x_i - \sum y_j \in \mathbb{Z}$.
Then the ratio
$$R(z) = \prod_i \theta_\tau^{(x_i)}(z) \Big/ \prod_j \theta_\tau^{(y_j)}(z)$$

of translated theta functions is L-periodic & therefore induces a meromorphic function on $\mathbb{C}/L$.
The divisor of this function is $\sum 1\overline{x_i} - \sum 1\overline{y_j} \in \mathrm{Div}(\mathbb{C}/L)$

Prop: Any meromorphic function on a complex torus $\mathbb{C}/L$ is a ratio of translated theta functions.

Proof: Let f be a meromorphic function on $\mathbb{C}/L$. Then f has finitely many zeroes and poles, say $p_1, \dots, p_n$; $q_1, \dots, q_n$.

First, we show that $\sum p_i = \sum q_j$ in $\mathbb{C}/L$ as a group:
Suppose not and choose $p_0, q_0 \in X = \mathbb{C}/L$ such that
$$\sum_{i=0}^{n} p_i = \sum_{j=0}^{n} q_j.$$

Set $R(z) = \prod_i \theta_\tau^{(x_i)}(z) \Big/ \prod_j \theta_\tau^{(y_j)}(z)$

for points $x_i, y_j \in \mathbb{C}$ with $\overline{x_i} = p_i$, $\overline{y_j} = q_j$ and
$$\sum x_i - \sum y_j \in \mathbb{Z}.$$

Then $g = R/f$ is a meromorphic function on X with exactly one zero (p_0) and one pole (q_0) ↯ because $g(X) = 1$.

With the same argument we get that $f = R$ because their divisors are equal. $\square$

§7 : LINEAR SYSTEMS

X = compact Riemann surface

Rmk: Merom. functions on X are determined by their divisors, up to a nonzero constant. So we can study them via their divisors.

proof: Let $f_1, f_2 \in \mathbb{C}(X)$ merom. functions s.t. $\mathrm{div}(f_1) = \mathrm{div}(f_2)$. Then $\mathrm{div}\left(\frac{f_1}{f_2}\right) = 0$

So $\frac{f_1}{f_2}$ has no poles, so it is holomorphic but since X is compact $\frac{f_1}{f_2} = \lambda \in \mathbb{C}^*$. $\square$

Some terminology:

A divisor $D = \sum n_p \cdot p$ on X is called EFFECTIVE if $n_p \geq 0$, and we write $D \geq 0$.

For two divisors D_1, D_2 we write $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$, i.e., $D_1 - D_2$ is effective.

Def: $H^0(X, D)$

Let X be a Riemann surface and D a divisor:

$$H^0(X,D) = \{ f \in \mathbb{C}(X)^* \mid \mathrm{div}(f) \geq -D \} \cup \{0\}$$

So these sets tell us about meromorphic functions with bounded poles.

<u>Example</u>: (1) $X = \mathbb{P}^1$, $D = d\cdot\infty$ $d\in\mathbb{Z}$

$$H^0(\mathbb{P}^1, d\cdot\infty) = \{ f\in\mathbb{C}(\mathbb{P}^1) \mid \operatorname{div}(f) \geq -d\cdot\infty \}$$

So elements here are merom. functions with poles only at ∞ and of order at most d (if $d>0$) and with zeroes of order at least d (if $d<0$).

- $d<0$: $H^0(\mathbb{P}^1, d\cdot\infty) = 0$.

  Because any function in there has no poles, so it is constant. But it has also zeroes, so it must be zero.

- $d\geq 0$: $H^0(\mathbb{P}^1, d\cdot\infty) = \mathbb{C}[z]_{\leq d} = \mathbb{C}[s,t]_d$ $\quad z = t/s$

  If $f\in H^0(\mathbb{P}^1, d\cdot\infty)$ then it has poles only at ∞. Since any meromorphic function on $\mathbb{P}^1$ is rational it must be that f is a polynomial. Since the order of the pole at ∞ is at most d, this is a polynomial of degree $\leq d$.

We consider some properties of these sets:

(1) $H^0(X,D)$ is a vector space (subspace of $\mathbb{C}(X)$).

<u>proof</u>: because ord_p is a discrete valuation. $\square$

(2) If $D_1 \sim D_2$ ($\sim$ = linear equivalence) then there is a "canonical" isomorphism $H^0(X,D_1) \cong H^0(X,D_2)$.

proof: Let $g \in \mathbb{C}(X)^*$ s.t.

$$\operatorname{div}(g) = D_1 - D_2$$

Then

$$\begin{aligned} H^0(X,D_1) &\xrightarrow{\cdot g} H^0(X,D_2) \\ f &\longmapsto f \cdot g \end{aligned}$$

is an isomorphism. If $f \in H^0(X,D_1)$

$$\begin{aligned} \operatorname{div}(f \cdot g) &= \operatorname{div}(f) + \operatorname{div}(g) \\ &= \operatorname{div}(f) + D_1 - D_2 \\ &\geq -D_1 + D_1 - D_2 = -D_2 \end{aligned}$$

So the map is well defined and the inverse is $(\cdot \frac{1}{g})$. $\square$

(3) There is a canonical identification

$$\mathbb{P}(H^0(X,D)) \cong |D| \overset{\text{def}}{:=} \left\{ \begin{array}{c} \text{all effective divisors} \\ E \sim D \end{array} \right\}$$

proof: Let $f \in H^0(X,D)$, $f \neq 0$. We can define an effective divisor

$$\boxed{\operatorname{div}^D(f) := \operatorname{div}(f) + D} \geq 0$$

This is also lin. equiv to D

$$\operatorname{div}^D(f) - D = \operatorname{div}(f)$$

Moreover if $\lambda \in \mathbb{C}^*$, then $\operatorname{div}^D(f) = \operatorname{div}^D(\lambda \cdot f)$

So we have a map

$$\begin{array}{ccc} \mathbb{P}(H^0(X,D)) & \longrightarrow & |D| \\ [f] & \longmapsto & \operatorname{div}^D(f) = \operatorname{div}(f) + D \end{array}$$

- INJECTIVITY: if $\operatorname{div}^D(f_1) = \operatorname{div}^D(f_2)$, then $\operatorname{div}(f_1) = \operatorname{div}(f_2)$ and then we know that $f_1 = \lambda \cdot f_2$, $\lambda \in \mathbb{C}^*$.

- SURJECTIVITY: E effective, $E \sim D$. Then $E - D = \operatorname{div}(f)$ and $f \in H^0(X,D)$ because E is effective. Then $E = \operatorname{div}^D(f)$.

Example: (1) On $\mathbb{P}^1$, if $d \geq 0$

$$|d \cdot \infty| = \left\{ \begin{array}{l} \text{all effective divisors of} \\ \text{degree } d \text{ on } \mathbb{P}^1 \end{array} \right\}$$

$$= \mathbb{P}(\mathbb{C}[z]_{\leq d}) = \mathbb{P}(\mathbb{C}[s,t]_d)$$

(2) Let $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, and $p \in E$.

$|p| = \{p\}$ [Exercise 5.3.(c)]

$$H^0(E,p) = \{ f \mid \operatorname{div}(f) \geq -p \}$$
$$= \langle 1 \rangle$$

(4) If D is effective then $1 \in H^0(X,D)$ and indeed $D \in |D|$.

(5) If $\deg D < 0$, then $H^0(X,D) = 0$.

proof: $\mathbb{P}(H^0(X,D)) = \left\{ \begin{array}{c} \text{effective divisors} \\ E \sim D \end{array} \right\}$

But if $E \sim D$ then $\deg(E) = \deg(D) < 0$ impossible if E is effective. $\square$

(6) If $\deg D = 0$;

$$H^0(X,D) \cong \begin{cases} \mathbb{C} & \text{if } D \sim 0 \\ 0 & \text{if } D \not\sim 0 \end{cases}$$

proof: $\mathbb{P}(H^0(X,D)) = |D|$

So if $E \sim D$ and E is effective then $\deg(E) = 0$, so it must be $E = 0$. So

$$|D| = \begin{cases} \{0\} & \text{, if } D \sim 0 \\ \phi & \text{, if } D \not\sim 0 \end{cases}$$

Otherwise if $D \sim 0$ then

$$H^0(X,D) \cong H^0(X,0) = \{ f \mid \text{div}(f) \geq 0 \}$$
$$= \{ \text{hol. functions on } X \} = \mathbb{C}.$$

(7) If $\deg D = 1$ then

$$\dim H^0(X,D) = \begin{cases} 2 & \text{if } X \cong \mathbb{P}^1 \\ \leq 1 & \text{if } X \not\cong \mathbb{P}^1 \end{cases}$$

proof: If $X \cong \mathbb{P}^1$, we saw this before:

$H^0(X,D) \cong H^0(X,\infty) = \mathbb{C}[z]_{\leq 1}$

On the other hand, suppose $\dim H^0(X,D) \geq 2$.

then there is an effective divisor E lin. equiv. to D and $E = p$. So

$$\dim H^0(X,p) = \dim H^0(X,D) \geq 2.$$

We always have $1 \in H^0(X,p)$ so there must be a nonconstant $f \in H^0(X,p)$. which has exactly one pole. So $f: X \longrightarrow \mathbb{P}^1$ is an isomorphism. $\square$

(8) If $p \in X$, then $H^0(X,D-p) \subseteq H^0(X,D)$.

In general if E is effective $H^0(X,D-E) \subseteq H^0(X,D)$.

proof: $H^0(X,D) = \{ f \mid \operatorname{div}(f) \geq -D \}$

$H^0(X,D-p) = \{ f \mid \operatorname{div}(f) \geq -D+p \}$. $\square$

Let's try to understand $H^0(X, D-p) \subseteq H^0(X,D)$:

Suppose first that p is not in the support of D

$\left(D = \sum n_p \cdot p \qquad \mathrm{Supp}(D) = \{ p \mid n_p \neq 0 \} \right)$

Then each $f \in H^0(X,D)$ is holomorphic around p and then

$$H^0(X, D-p) = \{ f \in H^0(X,D) \mid \mathrm{div}(f) \geq -D+p \}$$
$$= \{ f \in H^0(X,D) \mid f(p) = 0 \}$$

Suppose in general that p appears in $\mathrm{Supp}(D)$

$$D = n_p \cdot p + \text{sum of pts distinct from } p$$

then

$$H^0(X,D) = \{ f \mid \mathrm{div}(f) \geq -n_p \cdot p + \dots \}$$
$$H^0(X,D-p) = \{ f \mid \mathrm{div}(f) \geq -(n_p - 1) p + \dots \}$$

Fix a local coordinate z around p, then if $f \in H^0(X,D)$ we can write a Laurent series expansion

$$f = a_{-n_p} z^{-n_p} + a_{-(n_p-1)} z^{-(n_p-1)} + \dots\dots$$

$$H^0(X, D-p) = \{ f \in H^0(X,D) \mid a_{-n_p} = 0 \}$$

(9) $H^0(X, D-p)$ has codimension ≤ 1 inside $H^0(X, D)$.

(10) In particular $h^0(X,D) \overset{\text{def}}{:=} \dim H^0(X,D)$ we see that this is finite, and moreover

$$h^0(X,D) = \deg D + 1 \quad , \text{ if } X \cong \mathbb{P}^1 \text{ and } \deg D \geq 0$$
$$h^0(X,D) \leq \deg D \quad , \text{ if } X \not\cong \mathbb{P}^1$$

<u>proof</u>: let $d = \deg D$. If $X \cong \mathbb{P}^1$, then

$$H^0(X,D) \cong H^0(X, d\cdot\infty) \cong \mathbb{C}[z]_{\leq d}$$

and it has dim $d+1$ if $d \geq 0$ and 0 otherwise. Suppose $X \not\cong \mathbb{P}^1$. Then we have a chain of inclusions

$$H^0(X, D-(d-1)p) \subseteq \quad \cdots\cdots \quad \subseteq H^0(X, D-p) \subseteq H^0(X,D)$$

We know that

$$h^0(X, D-(d-1)p) \leq 1$$

because $\deg(D-(d-1)p) = 1$ and $X \not\cong \mathbb{P}^1$ and at each inclusion in the chain the dimension grows at most by 1, so

$$h^0(X,D) \leq h^0(X, D-(d-1)p) + \#\{\text{inclusions}\}$$
$$\leq 1 + d - 1 = d. \qquad \square$$

Def : LINEAR SYSTEM (of DIVISORS)

The proj space $|D|$ is called the complete linear system associated to D. A linear system is a linear subspace $\Delta \subseteq |D|$. The degree of D is the degree of the linear system.

The dimension $r = \dim \Delta$ is the dimension of the linear system

§8: LINEAR SYSTEMS and MAPS to PROJECTIVE SPACE

Let X be a (compact) Riemann surface.
We want to see that linear systems give a powerful language to describe holomorphic maps

$$\varphi : X \longrightarrow \mathbb{P}^r$$

We will assume that these maps are NONDEGENERATE meaning that the image $f(X)$ is not contained in any hyperplane.

Any map $\varphi : X \to \mathbb{P}^n$ is the composition of a nondegenerate map $X \to \mathbb{P}^r$ and a linear embedding $\mathbb{P}^r \hookrightarrow \mathbb{P}^n$.

First a little digression:

- PULLBACKS of DIVISORS: Let $\varphi : X \to \mathbb{P}^r$ be a Riemann surface and $D \subseteq \mathbb{P}^r$ an hypersurface (or divisor on $\mathbb{P}^r$) such that $\varphi(X) \not\subseteq D$. For any point $p \in X$ choose an affine chart $V \cong \mathbb{A}^r$ around $\varphi(p)$ and a local coordinate z in a neighborhood Δ around p. Then we write φ as

$$\varphi|_\Delta : \Delta \to V \qquad f_i \text{ holomorphic on } V$$
$$z \longmapsto (\varphi_1(z), \dots \varphi_r(z))$$

and in the affine chart $D = \{ F(x_1, \dots, x_r) = 0 \}$
So we define

$$(\varphi^* D)_p := \operatorname{ord}_p F(f_1(z), \dots, f_r(z))$$

$$\varphi^* D := \sum_{p \in X} (\varphi^* D)_p \cdot p$$

This is an effective divisor: $(\varphi^* D)_p \geq 0$ and $(\varphi^* D)_p = 0$ iff $\varphi(p) \in \varphi(X) \cap D$. This is a finite set because it is discrete and X is compact.

Now we proceed with our correspondence:

- <u>From MAPS to LINEAR SYSTEMS</u>

Let $\varphi : X \longrightarrow \mathbb{P}^r$ be a holomorphic map.
On $U_i = \varphi_i^{-1}(\{ X_i \neq 0 \})$ we can write

$$\varphi_{|U_i}(p) = [f_{i0}(p), f_{i1}(p), \dots, f_{ir}(p)]$$

for some holomorphic functions f_{ik} on U_i ($f_{ii} = 1$). We can rewrite this as

$$\varphi_{|U_i} = \left[1, \frac{f_{i1}}{f_{i0}}, \frac{f_{i2}}{f_{i0}}, \dots, \frac{f_{ir}}{f_{i0}} \right]$$

In particular, on $U_i \cap U_j$ we get that

$$[1, \frac{f_{i1}}{f_{i0}}, \dots, \frac{f_{ir}}{f_{i0}}] = [1, \frac{f_{j1}}{f_{j0}}, \dots, \frac{f_{jr}}{f_{j0}}]$$

so $\frac{f_{ik}}{f_{i0}} = \frac{f_{jk}}{f_{j0}}$ on $U_i \cap U_j$. But this means that the meromorphic functions $\left(\frac{f_{ik}}{f_{i0}}\right)$ on U_i glue together to a meromorphic function f_k on X. So we can describe φ as a map

$$\varphi : X \longrightarrow \mathbb{P}^r \qquad \varphi = [1, f_1, f_2, \dots, f_r]$$

where the f_i are meromorphic on X.

This map is nondegenerate iff all the $1, f_1, \dots, f_r$ are linearly independent. We keep this assumption for what follows.

We also denote the coordinate hyperplanes by

$$H_i = \{x_i = 0\} \subseteq \mathbb{P}^r$$

Prop: Let $\varphi : X \to \mathbb{P}^r$, $\varphi = [f_0, f_1, \dots, f_r]$ $(f_0 = 1)$ be a nondegenerate map as before and let
$$H = \{ a_0 X_0 + \dots + a_r X_r = 0 \} \subseteq \mathbb{P}^r$$
Then
$$\operatorname{div}\left(\sum a_i f_i \right) = \varphi^* H - \varphi^* H_0$$

proof: Let $p \in X$. Suppose first that none of the f_i has a pole at p. Then $\varphi(p) \in \{ X_0 \neq 0 \}$ and in this affine chart H and H_0 have equation
$$H \cap \{X_0 \neq 0\} = \{ a_0 + a_1 x_1 + \dots + a_r x_r = 0 \}$$
$$H_0 \cap \{X_0 \neq 0\} = \{ 1 = 0 \} = \emptyset, \text{ so}$$

$$(\varphi^* H)_p - (\varphi^* H_0)_p = \operatorname{ord}_p (a_0 + a_1 f_1 + \dots + a_r f_r).$$

Suppose instead one of the f_i has a pole at p and let $\operatorname{ord}_p(f_j) = \min\{\operatorname{ord}_p(f_i)\}$. If we choose a local coordinate z around p, in this neighborhood we have
$$\varphi(z) = [1, f_1(z), \dots, f_r(z)]$$
$$= \left[\frac{1}{f_j}, \frac{f_1}{f_j}, \dots, \frac{f_r}{f_j} \right]$$
and the f_i/f_j are holomorphic at p. $\left(\frac{f_j}{f_j} = 1 \right)$.

So we see $\varphi(p) \in \{X_j \neq 0\}$ and in this affine chart

$$H \cap \{X_j \neq 0\} = \{a_0 x_0 + \dots + a_j + \dots + a_r x_r = 0\}$$
$$H_0 \cap \{X_j \neq 0\} = \{x_0 = 0\}$$

So

$$(\varphi^* H)_p - (\varphi^* H_0)_p =$$
$$= \mathrm{ord}_p \left(a_0 \frac{1}{f_j} + \dots + a_j + \dots + a_r \frac{f_r}{f_j} \right) - \mathrm{ord}_p \left(\frac{1}{f_j} \right)$$
$$= \mathrm{ord}_p (a_0 + a_1 f_1 + \dots + a_r f_r).$$

$\square$

This tells us that

$\boxed{\text{$f_0, f_1, \dots, f_r$ are linearly independent elements in $H^0(X, \varphi^* H_0)$ which span a subspace $V \subseteq H^0(X, \varphi^* H_0)$ of dimension $r+1$}}$

Equivalently

$\boxed{\text{$\{\varphi^* H \mid H \subseteq \mathbb{P}^r \text{ hyperplane}\}$ is a linear system of dimension $r+1$ in $|\varphi^* H_0|$. It can be considered as the projectivization of the space V.}}$

Rmk: There is nothing special about the coordinate X_0 in all this. We could as well have written

$$\varphi = [f_0, \dots, f_r] \quad \text{with } \sum a_i f_i = 1$$

and then everything would go through replacing H_0 with $H = \{\sum a_i X_i = 0\}$. The language of linear systems gets rid of this ambiguity: indeed the linear system $\{\varphi^* H \mid H \subseteq \mathbb{P}^r \text{ hyperplane}\}$ is clearly independent of the choice of an hyperplane.

Rmk: If $\varphi = [f_0, \dots, f_r]$ as before and we compose it with a linear change of coordinates $\mathbb{P}^r \to \mathbb{P}^r$ the effect is that of replacing $(f_0, \dots, f_r)$ with another basis of V.

Moreover the linear systems obtained in this way are special, in the sense that they are BASE-POINT-FREE.

Def: BASE-POINT-FREE

Let D be a divisor on X. A linear system

$$\Lambda \subseteq |D|$$

Has a base point at $p \in X$ iff $p \in D \ \forall D \in \Lambda$.
A linear system without base pts is base-point-free.

Rmk: What does this mean in terms of meromorphic functions? Suppose that Λ corresponds to a subspace $V \subseteq H^0(X,D)$ then the corresponding linear system is $\Lambda = \{ \operatorname{div}(f) + D \mid f \in \mathbb{P}(V) \}$ and there is a base point at p if and only if $p \in \operatorname{div}(f) + D$ for each $f \in V$, which can be rephrased as saying

$$\operatorname{ord}_p(f) + \operatorname{ord}_p(D) > 0 \quad \forall f \in V.$$

Prop: Let $\varphi : X \to \mathbb{P}^r$, $\varphi = [1, f_1, \ldots, f_r]$ be a nondegenerate map as before. Then the corresponding linear system is base point free.

proof: Take $p \in X$ and choose an hyperplane $H \subseteq \mathbb{P}^r$ that does not pass through $\varphi(p)$. Then $\varphi^* H$ does not contain p. $\square$

- From LINEAR SYSTEMS to MAPS

Conversely, suppose we have a divisor D and a base-point-free subspace $V \subseteq H^0(X, D)$ of dimension $r+1$, which corresponds to a base-point-free linear system $\Lambda \subseteq |D|$.

If we choose a basis $f_0, \dots, f_r$ of V, we get a map

$$\varphi : X \longrightarrow \mathbb{P}^r \qquad \varphi = [f_0, \dots, f_r]$$

out of this map we can get again a linear system by taking $\{ \varphi^* H \mid H \subseteq \mathbb{P}^r \}$ hyperplane and this will be the same as the original linear system:

Prop: In the above notation

$$\Lambda = \{ \varphi^* H \mid H \subseteq \mathbb{P}^r \}.$$

proof: Exercise.

In conclusion, this gives us a correspondence

$$\left\{\begin{array}{c}\varphi : X \longrightarrow \mathbb{P}^r \\ \text{nondegenerate} \\ \text{map up} \\ \text{to linear change} \\ \text{of coordinates}\end{array}\right\} \leftrightarrows \left\{\begin{array}{c}\text{base-point-free} \\ \text{linear systems} \\ \text{of dimension } r\end{array}\right\}$$

To get one actual map, not just up to change of coordinates, we choose a vector space $V \subseteq H^0(X,D)$ that induces the linear system and a basis of V.

- - - - - - - - - - - - -

<u>Prop</u>: A complete linear system $H^0(X,D)$ is bpf iff

$$h^0(X, D-p) = h^0(X,D) - 1 \quad \forall p \in X.$$

<u>proof</u>: We claim that $p \in X$ is a base point iff

$$h^0(X,D-p) = h^0(X,D).$$

∘ <u>DIVISORS</u>: $\{E \in |D| \mid p \in E\} = |D-p| + p$

$\boxed{\subseteq}$ $E = E' + p$, E' effective.

$$E \sim D \Rightarrow E' \sim D-p \Rightarrow E' \in |D-p|.$$

$\boxed{2}$ $E' \sim D-p$ and E' effective. Then $E = E'+p$ is effective and $E \sim D$.

Then p is a basepoint iff

$$|D-p|+p = |D|$$

$\Leftrightarrow \dim(|D-p|+p) = \dim|D|$

$\Leftrightarrow h^0(D-p)-1 = h^0(D)-1$

$\Leftrightarrow h^0(D-p) = h^0(D).$

- FUNCTIONS: So, p is a basepoint iff

$\Leftrightarrow \mathrm{ord}_p(f)+\mathrm{ord}_p(D) > 0 \quad \forall f \in H^0(X,D)$

$\Leftrightarrow \mathrm{ord}_p(f)+\mathrm{ord}_p(D) \geq 1$

$\Leftrightarrow \mathrm{ord}_p(f)+\mathrm{ord}_p(D-p) \geq 0$

$\Leftrightarrow f \in H^0(X, D-p) \qquad \forall f \in H^0(X,D)$

If $q \neq p$ $\mathrm{ord}_q(D) = \mathrm{ord}_q(D-p)$.

So iff $\quad H^0(X,D-p) = H^0(X,D) \qquad \square$

A divisor D s.t. $H^0(X,D)$ is bpf is called bpf.

§ 9: DIFFERENTIAL FORMS and RIEMANN-ROCH

We shift gears for a bit.

First, let $U \subseteq \mathbb{C}$ be an open subset. An HOLOMORPHIC DIFFERENTIAL FORM ω on U has the form

$$\omega = f(z)\,dz \quad \text{where } f \text{ is a holomorphic function.}$$

When f is meromorphic instead, we talk of a MEROMORPHIC DIFFERENTIAL FORM.

This notion makes sense also on any Riemann surface X. An HOLOMORPHIC FORM ω on X can be described by a collection of holomorphic forms

$$\omega_i = f_i(z_i)\,dz_i \quad \text{on each chart } U_i$$

which coincide on the intersections $U_i \cap U_j$. The same for meromorphic differential forms.

The space of all holomorphic forms on a Riemann surface is a complex vector space that we denote by $H^0(X, \omega_X)$, or $H^0(X, \Omega^1_X)$.

Example: (1) HOLOMORPHIC FORMS on $\mathbb{P}^1$

Take the two usual charts U_0, U_1 of $\mathbb{P}^1$ with affine coordinates x and z respectively, s.t. $z = 1/x$ on $U_0 \cap U_1$. Then a holomorphic form is

$$\omega_0 = f_0(x)\,dx \qquad f_0 \text{ holomorphic}$$
$$\omega_1 = f_1(z)\,dz \qquad f_1 \text{ holomorphic}$$

such that $\omega_0|_{U_0\cap U_1} = \omega_1|_{U_0\cap U_1}$. On this intersection we see

$$\omega_0|_{U_0\cap U_1} = f_0(x)\,dx = f_0\left(\tfrac{1}{z}\right) d\left(\tfrac{1}{z}\right)$$
$$= -\frac{f_0(1/z)}{z^2}\,dz$$

So the condition is that $f_1(z) = -\frac{1}{z^2}\cdot f_0\left(\frac{1}{z}\right)$

However this is impossible because the expression on the right has always a pole at $z=0$.

Hence, on $\mathbb{P}^1$ the only holomorphic differential form is the zero form.

$$H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}) = 0$$

(2) HOLOMORPHIC FORMS on a COMPLEX TORUS

Take a complex torus $X_\tau = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ as usual.

Then on $\mathbb{C}$ we have the differential form dz which is invariant under the action of the lattice

$$d(z+m+\tau n) = dz$$

Then one can see that this induces a nonzero holomorphic form dz on X. Furthermore this is essentially the only one; morally, if ω is a differential form on $\mathbb{C}$ then it must have the form

$$\omega = f(z)\,dz$$

with f holomorphic on $\mathbb{C}$ and s.t. $f(z+m+\tau n) = f(z)$, but any such f is constant. Hence

$$H^0(X_\tau, \omega_{X_\tau}) = \mathbb{C}\cdot dz$$

(3) HOLOMORPHIC FORMS ON PLANE CURVES

Suppose $X = \{f(x,y)=0\}$ is a smooth plane curve and set $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$.

On the set $U_1 = \{f_x \neq 0\}$ we can define the form

$$\omega_1 = \frac{1}{f_x}\,dy$$

Can we extend this to a form also on the chart $U_2 = \{f_y \neq 0\}$? Morally we do it like this: since $f(x,y) = 0$ on X, we have that

$$f_x \, dx + f_y \, dy = 0$$

so

$$1/f_x \, dy = - 1/f_y \, dx$$

and this extends ω_1 to a form ω on the whole of X. As an exercise, make this rigorous.

- FORMS and DIVISORS

Let ω be a meromorphic differential form. Then on a collection of charts U_i we can write $\omega_{|U_i} = f_i(z_i) \, dz_i$ for $f_i(z_i)$ meromorphic on U_i. We then define

$$\boxed{\mathrm{ord}_p(\omega) := \mathrm{ord}_p f_i} \quad \text{if } p \in U_i$$

This is well defined: if $p \in U_i \cap U_j$ then

$$f_i(z_i) \, dz_i = f_j(z_j) \, dz_j \quad \text{on } U_i \cap U_j$$

Let $z_i = \varphi(z_j)$ be the change of coordinate. Then

$$f_i(z_i)dz_i = (f_i \circ \varphi)(z_j)\, d\varphi(z_j)$$
$$= (f_i \circ \varphi)(z_j) \cdot \dot{\varphi}(z_j)\, dz_j$$

hence $f_j(z_j) = (f_i \circ \varphi)(z_j)\, \dot{\varphi}(z_j)$

and since $\dot{\varphi}(z_j) \neq 0$, the two orders coincide. The same argument works when we choose another collection of representatives for ω.

<u>Def</u>: CANONICAL DIVISOR

For any meromorphic differential form ω we can define the divisor

$$\boxed{\operatorname{div}(\omega) = \sum_{p \in X} \operatorname{ord}_p(\omega) \cdot p}$$

and any divisor of this form is called a canonical divisor on X, sometimes denoted by K

<u>Facts</u>: (1) Any two canonical divisors are linearly equivalent.

<u>proof</u>: Let ω_1, ω_2 be two meromorphic forms. Then on a collection of charts U_i we have:

$$\omega_{1|U_i} = f_{1,i}(z_i)\, dz_i$$
$$\omega_{2|U_i} = f_{2,i}(z_i)\, dz_i$$

We claim that the collection $\left(\frac{f_{1,i}(z_i)}{f_{2,i}(z_i)}\right)$ is a meromorphic function on X.

Indeed, on $U_i \cap U_j$ let $z_i = \varphi(z_j)$ be the change of coordinates. Then

$$f_{1,j}(z_j) = f_{1,i}(\varphi(z_j))\,\dot{\varphi}(z_j)$$
$$f_{2,j}(z_j) = f_{2,i}(\varphi(z_j))\,\dot{\varphi}(z_j)$$

so
$$\frac{f_{1,j}(z_j)}{f_{2,j}(z_j)} = \frac{f_{1,i}(z_i)\cdot\cancel{\dot{\varphi}(z_j)}}{f_{2,i}(z_i)\,\cancel{\dot{\varphi}(z_j)}} = \frac{f_{1,i}(z_i)}{f_{2,i}(z_i)}.$$

We denote this meromorphic function by

$$\left(\frac{\omega_1}{\omega_2}\right)$$

Then it is easy to see that

$$\mathrm{div}(\omega_1) - \mathrm{div}(\omega_2) = \mathrm{div}(f). \qquad \square$$

(2) If K is any canonical divisor, then

$$H^0(X, \omega_X) \cong H^0(X, K)$$

<u>proof</u>: suppose $K = \mathrm{div}(\omega_0)$ for a meromorphic differential ω_0. Then for any $\omega \in H^0(X, \omega_X)$ we see that

$$\mathrm{div}\left(\frac{\omega}{\omega_0}\right) = \mathrm{div}(\omega) - \mathrm{div}(\omega_0)$$
$$= \mathrm{div}(\omega) - K \geqslant -K$$

because $\mathrm{div}(\omega)$ is effective. Hence we have

$$H^0(X, \omega_X) \longrightarrow H^0(X, K)$$
$$\omega \longmapsto \left(\frac{\omega}{\omega_0}\right)$$

We also have the inverse map: if $f \in H^0(X, K)$ we can define a meromorphic differential $f \cdot \omega_0$ in a collection of charts U_i as

$$f \cdot \omega_0|_{U_i} = f|_{U_i}(z_i)\, \omega_0|_{U_i}$$

As an exercise, show that the map

$$H^0(X, K) \longrightarrow H^0(X, \omega_X)$$
$$f \longmapsto f \cdot \omega_0$$

is well-defined and an inverse of the previous one.

(3) More generally, for any divisor D on X we can define

$$H^0(X, \omega_X(D)) = \left\{ \omega \ \begin{matrix}\text{merom.}\\ \text{diff.}\end{matrix} \ \middle|\ \operatorname{div}(\omega) \geq -D \right\}$$

The same proof of before shows that

$$H^0(X, \omega_X(D)) \cong H^0(X, K_X + D)$$

where K_X is any canonical divisor on X.

- PULLBACKS of FORMS

We can pull back differentials along maps of Riemann surfaces.

Locally, consider a map $F : U \longrightarrow V$ with $U, V \subseteq \mathbb{C}$ open sets.
Let $\omega = f(w)dw$ be a meromorphic diff. on V

$$F^*\omega := f(F(z))\, dF(z) = f(F(z))\, \dot{F}(z)\, dz$$

and this generalizes via charts to maps $F : X \to Y$ of Riemann surfaces.
We can interpret the Riemann-Hurwitz theorem in this way. Let $F : X \longrightarrow Y$ be a map of compact Riemann surfaces. Recall:

- If $q \in Y$, we define the divisor
$$F^*q = \sum_{p \in F^{-1}(q)} \text{mult}_p(F) \cdot p$$
If $D = m_1 q_1 + \dots + m_r q_r$ is any divisor on Y we define $F^*D = m_1 F^*q_1 + \dots + m_r F^*q_r$. Observe that $\deg(F^*D) = \deg F \cdot \deg D$.
- The ramification divisor of F is $R = \sum_{p \in X} (\text{mult}_p(F) - 1) \cdot p$

Thm: [RIEMANN-HURWITZ II]

Let $F: X \to Y$ be a map of compact Riemann surfaces of degree d. Let ω be a meromorphic differential on Y. Then

$$\operatorname{div}(F^*\omega) = F^*(\operatorname{div}\omega) + R$$

Proof: Let $p \in X$ and $q = f(p) \in Y$.
Locally around p and q the map has the shape $F: U \to V$, $F(z) = z^m$, where $m = \operatorname{mult}_p F$. If $\omega = f(w)dw$ on V then
$F^*\omega = f(F(z))dz^m = m \cdot f(z^m) \cdot z^{m-1} dz$.
Let $e = \operatorname{ord}_q(\omega) = \operatorname{ord}_q f(w)$. Then
$f(w) = a \cdot w^e + \dots$ and $f(z^m) = a \cdot z^{me} + \dots$
hence $\operatorname{ord}_q f(z^m) = m \cdot e$. In conclusion

$$\begin{aligned}\operatorname{ord}_p F^*\omega &= \operatorname{ord}_p(f(z^m)) + \operatorname{ord}_p(z^{m-1}) \\ &= m \cdot \operatorname{ord}_q(f(w)) + (m-1) \\ &= \operatorname{ord}_p(F^*(\operatorname{div}\omega)) + \operatorname{ord}_p R\end{aligned}$$

which proves what we wanted. $\square$

Rmk: We can write this as $K_X = F^*K_Y + R$.

- **RIEMANN-ROCH**

We can finally state one of the most important results in the theory of Riemann surfaces and algebraic curves.

Thm: RIEMANN-ROCH
Let X be a compact Riemann surface of genus g and D a divisor on X.

$$\boxed{h^0(X,D) - h^0(X,\omega_X(-D)) = \deg D + 1 - g}$$

This has a lot of consequences. Let's see some.

(1) If $\deg D \geq g-1+n$, then $h^0(X,D) \geq n$.

proof: $h^0(X,D) = \deg D + 1 - g + h^0(X,\omega_X(-D))$
$\geq n + h^0(X,\omega_X(-D)) \geq n.$ $\square$

(2) X has a nonconstant meromorphic function

proof: choose a divisor of degree $g+1$. $\square$

(3) X has a nonzero meromorphic differential, and

$$\deg K_X = 2g-2$$

<u>proof</u>: let f be a nonconstant meromorphic function and $f: X \longrightarrow \mathbb{P}^1$ the corresponding map.

If ω is any nonzero meromorphic diff on $\mathbb{P}^1$ then $F^*\omega$ is a nonzero meromorphic diff on X.

By Riemann-Hurwitz II we know that

$$\begin{aligned}\deg K_X &= \deg F^*K_{\mathbb{P}^1} + \deg R \\ &= (\deg F)\cdot(\deg K_{\mathbb{P}^1}) + \deg R \\ &= -2\cdot \deg F + \deg R\end{aligned}$$

By Riemann-Hurwitz I we know that

$$2g(X)-2 = -2\deg F + \deg R. \qquad \square$$

(4) Let D be a divisor on X.

- If $\deg D \geqslant 2g-1$, then

$$h^0(X,D) = \deg D + 1 - g.$$

- If $\deg D \geqslant 2g$, then D is base-point-free.
- If $\deg D \geqslant 2g+1$, then D is very ample.

proof: • If $\deg D \geq 2g-1$, then $\deg K_X - D < 0$, so $h^0(X, \omega_X(-D)) = h^0(X, K_X - D) = 0$. The rest follows from Riemann-Roch.

• Recall that D is base-point-free iff $h^0(X, D-p) = h^0(X, D) - 1$ for all $p \in X$. But this follows from the previous point.

• Recall that D is very ample iff $h^0(X, D-p-q) = h^0(X, D) - 2$ for all $p, q \in X$. This follows again from the first point. $\square$

(5) Any compact Riemann surface is isomorphic to a smooth projective curve.

proof: Let D be a divisor of degree $2g+1$. Then D is very ample and it induces an embedding $\varphi : X \hookrightarrow \mathbb{P}^r$. $\square$

(6) $h^0(X, \omega_X) = g$.

proof: $h^0(X, K_X) = g - 1 + h^0(X, \mathcal{O}_X) = g$. $\square$

PROJECTIVE GEOMETRY OF CURVES

CANONICAL CURVES.

Let X be an alg. curve (compact (connected) Riemann surface) and $K = (\omega)$ a canonical divisor

Prop: Either $|K|$ is very ample or X is hyperelliptic

Remark: If $|K|$ is very ample, then the genus of X is at least 3.

Def: A canonical curve is an alg. curve $X \overset{j}{\hookrightarrow} \mathbb{P}^{g-1}$ that is not hyperelliptic and such that a hyperplane section $\mathrm{div}_X(H) = j^*(H)$ is a canonical divisor of X.

Def: The degree $\deg(X)$ of a nondegenerate alg. curve $X \subset \mathbb{P}^n$ is the degree of the linear system $|\mathrm{div}_X(H)|$, where $H \subseteq \mathbb{P}^n$ is a hyperplane.

Prop: A canonical curve $X \subseteq \mathbb{P}^{g-1}$ has degree $2g-2$.

Classification of curves of low genus.

① $g=0$: $X \cong \mathbb{P}^1$

② $g=1$: $X \cong \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ for a $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau)>0$

$X \cong$ smooth plane cubic

③ $g=2$: X is hyperelliptic : affine model $\{y^2 = f(x)\}$ for a polynomial f of degree 6 with distinct roots

④ $g=3$:

Case 1: X is hyperelliptic

Case 2: $\phi_K : X \hookrightarrow \mathbb{P}^2$ is an embedding.

$\phi_K(X) \subseteq \mathbb{P}^2$ has degree 4

What is the ideal of $\phi_K(X)$ in $\mathbb{C}[x_1,x_2,x_3]$? $\mathbb{P}^2 = \{(x_1,x_2,x_3)\}$

For every degree d fix a homogeneous polynomial F_0 in $\mathbb{C}[x_1,\dots,x_g]$ (e.g. $F_d = F_1^d$):

We get a map $R_d: \mathbb{C}[x_1,\dots,x_g]_d \longrightarrow H^0(X, d\cdot K)$

$$F \longmapsto \frac{F}{F_d}$$

$$\operatorname{div}\left(\frac{F}{F_d}\right) = \operatorname{div}(F) - \underbrace{\operatorname{div}(F_d)}_{=d\cdot K} = \operatorname{div}(F) - dK$$

Dimension counting: $\dim \mathbb{C}[x_1,\dots,x_g]_d = \binom{g+d-1}{d}$

$\dim H^0(X, dK) \underset{d\geq 2}{=} d(2g-2)+1-g = (2d-1)(g-1)$

$\underline{g=3:\ d=4.}$ $h^0(X, 4K) = 2\cdot 7 = 14$

$\dim(\mathbb{C}[x_1,x_2,x_3]_4) = \binom{6}{2} = 15$

Prop: A curve of genus 3 is either hyperelliptic or a smooth quartic plane curve.

(I) $\underline{g=4:}$ Case 1: X is hyperelliptic.

Case 2: $\phi_K: X \hookrightarrow \mathbb{P}^3$, curve of degree 6.

$\dim \ker(R_2: \mathbb{C}[x_1,\dots,x_4]_2 \to H^0(X,2K)) \geq 10 - 3\cdot 3 = 1.$ $\Rightarrow$ There is a quadric Q in $I(X)$

$\dim \ker(R_3: \mathbb{C}[x_1,\dots,x_4]_3 \to H^0(X,3K)) \geq \binom{6}{3} - 5\cdot 3 = 20-15 = 5$

$\Rightarrow$ There is also a cubic C in $I(X)$ that is not a multiple of Q

$\rightsquigarrow$ $I(X) = \langle Q, C\rangle$

Prop: A curve of genus 4 is hyperelliptic or the complete intersection of a quadric and a cubic in $\mathbb{P}^3$

GEOMETRIC FORM OF RIEMANN-ROCH

Let $X \hookrightarrow \mathbb{P}^{g-1}$ be a canonical curve.

Let $D = P_1 + \dots + P_d$ be an effective divisor of degree d on X

If the points are distinct, then span(D) is the smallest linear space in $\mathbb{P}^{g-1}$ that contains $P_1, \dots, P_d$.

Def: A hyperplane $H \subseteq \mathbb{P}^{g-1}$ 'contains a divisor D' if

$$\mathrm{div}_X(H) \geq D.$$

H contains $P \in X \iff P \in H$

The linear system of hyperplanes in $\mathbb{P}^{g-1}$ that contain D is isomorphic to $|K-D|$

Def: The span of a divisor on $X \hookrightarrow \mathbb{P}^{g-1}$ is the intersection of all hyperplanes $H \subseteq \mathbb{P}^{g-1}$ that contain D.

By definition of the span of a divisor D, we have

$$\dim(\mathrm{span}(D)) + h^0(X, K-D) = g-1$$

$$\dim(\underbrace{\widehat{\mathrm{span}(D)}}_{\subseteq \mathbb{C}^g}) + h^0(X, K-D) = g$$

$$\mathbb{P}(\widehat{\mathrm{span}(D)}) = \mathrm{span}(D)$$

Thm: Let $X \hookrightarrow \mathbb{P}^{g-1}$ be a canonical curve.

For any divisor D on X we have

$$\dim |D| = \deg(D) - 1 - \dim \mathrm{span}(D)$$

Proof:

$$h^0(X,D) \cancel{-1} = \deg(D) \cancel{-1} - (g-1-h^0(X,K-D))$$

$$h^0(X,D) = \deg(D) \quad -g+1 + h^0(X, K-D) \qquad \square$$

⑥ $g=5$: $\phi_K : X \hookrightarrow \mathbb{P}^4$

$$R_2 : \mathbb{C}[x_0, \dots, x_4]_2 \longrightarrow H^0(X, 2K)$$

dim: $\quad 15 \qquad\qquad 3\cdot 4$

$\Rightarrow \dim(\ker(R_2)) \geq 3$

Can 3 points on X be collinear? Take $P_1, P_2, P_3 \in X$, $D = P_1 + P_2 + P_3$.

$\Rightarrow \dim |D| = 3 - 1 - \underbrace{\dim \operatorname{span}(D)}$

if this is 1, then $\dim |D| = 1$

Suppose such three points exist, then $h^0(D) = 2$.

Then there is a 1-dimensional family of lines that intersect X in 3 points and their union is a surface: rational normal scroll.

Such canonical curves of genus 5 are called trigonal.

CAYLEY-BACHARACH.

Thm: Let $X_1, X_2 \subseteq \mathbb{P}^2$ be two curves of degree d and e respectively. Suppose $d \geq e$ and X_1 is smooth. Suppose that $X_1 \cap X_2$ consists of $d\cdot e$ many distinct points. $P_1, \ldots, P_{d\cdot e}$

If $C \subseteq \mathbb{P}^2$ is any curve of degree $d+e-3$ containing all but one point of $X_1 \cap X_2$, then it contains all points of $X_1 \cap X_2$.

Fact 1: Let H be the intersection divisor of a line $L \subseteq \mathbb{P}^2$ with X_1. Then $K_{X_1} \sim (d-3)\cdot H$

Cor (Riemann-Roch): Every effective divisor on X that is linearly equivalent to $(d-3)\cdot H + P$ for $P \in X$ actually contains P.

Proof of Cayley-Bacharach:

$$\mathrm{div}_{X_1}(C) = C.X_1 = P_1 + \dots + P_{d\cdot e-1} + Q_1 + \dots + Q_{d\cdot(d-3)+1}$$

$$\left[(d+e-3)\cdot d = d\cdot e - 1 + d\cdot(d-3)+1\right.$$

$$K_{X_1} \sim (d-3)\cdot H \quad \text{and} \quad P_1 + \dots + P_{d\cdot e} \sim e\cdot H$$

$$C.X_1 \sim (d+e-3)\cdot H$$

$$\Rightarrow (d+e-3)\cdot H \sim e\cdot H - P_{d\cdot e} + Q_1 + \dots + Q_{d(d-3)+1}$$

$$\Rightarrow (d-3)\cdot H \sim Q_1 + \dots + Q_{d(d-3)+1} - P_{de}$$

$$(d-3)\cdot H + P_{d\cdot e} \sim Q_1 + \dots + Q_{d(d-3)+1}$$

$$\Rightarrow P_{d\cdot e} \in \{Q_1, \longrightarrow Q_{d(d-3)+1}\} \qquad \square$$

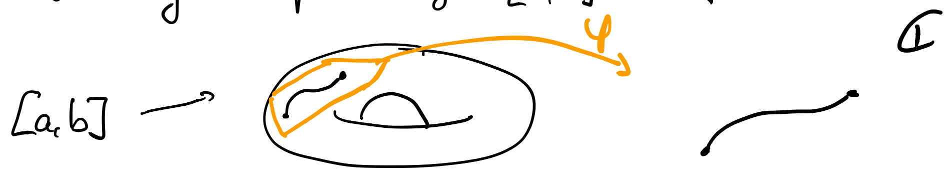
ABEL'S THEOREM

① Integration on Riemann surfaces.

Let ω be a holomorphic differential on a compact Riemann surface X

Locally: $\omega = f(z)\,dz$

Integrate ω along a path $\gamma : [a,b] \to X$

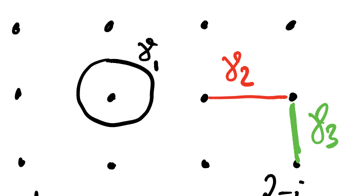


$$\int_\gamma \omega = \int_a^b f(\varphi(\gamma(t)))\cdot(\varphi\circ\gamma)'(t)\,dt \in \mathbb{C}$$

$z = \varphi(\gamma(t))$

Exm: $X = \mathbb{C}/L$, $L = \mathbb{Z} \oplus \mathbb{Z}\tau$, $\mathrm{Im}(\tau) > 0$

$\tau = i$



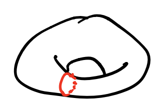
$\gamma_1 : [0, 2\pi] \to \mathbb{C}$

$\gamma_1(t) = \frac{1}{2}(\cos(t), \sin(t)) = \frac{1}{2}e^{it}$

$\omega = dz$

$$\int_{\gamma_1} \omega = \int_0^{2\pi} i\,\frac{1}{2}e^{it}\,dt = \left[\frac{1}{2}e^{it}\right]_0^{2\pi} = 0$$

$\gamma_2 : [0,1] \to \mathbb{C}$, $\gamma_2(t) = 1 + t$



$$\int_{\gamma_2} \omega = \int_0^1 1\,dt = 1$$

$$\int_{\gamma_3} \omega = i$$

FACT: The integral $\int_\gamma \omega$ of ω along a closed path $\gamma : [0,1] \to X$ $(\gamma(0) = \gamma(1))$ depends only on the homology class of γ.

map $\mathbb{Z}^{2g} \cong H_1(X,\mathbb{Z}) \longrightarrow H^0(X,\omega_0)^* \cong \mathbb{C}^g$

$$[\gamma] \longmapsto \int_\gamma \omega$$

is a group homomorphism

$$\int_{\gamma_1+\gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$$

Def: A linear functional $\lambda : H^0(X,\omega) \to \mathbb{C}$ is a PERIOD if $\lambda = \int_{[\gamma]} \omega$ for some $[\gamma] \in H_1(X,\mathbb{Z})$.

FACT: The periods form a lattice in $H^0(X,\omega)^*$.

Def: The JACOBIAN of X is the quotient of $H^0(X,\omega)^*$ modulo the subgroup Λ of periods:

$$\mathrm{Jac}(X) = H^0(X,\omega)^* / \Lambda \underset{\text{as a group!}}{\cong} \mathbb{C}^g / \mathbb{Z}^{2g}$$

Exm: a) $X \cong \mathbb{P}^1$: $\mathrm{Jac}(X) = \{0\}$

b) $X = \mathbb{C}/L$: $\mathrm{Jac}(X) \cong X$

ABEL - JACOBI MAP

Fix a base point $P_0 \in X$. For any $P \in X$, choose a path γ_P on X from P_0 to P.

We get a map $A : X \to H^0(X,\omega)^*$

$$A(P) \mapsto \left(\omega \longmapsto \int_{\gamma_P} \omega\right)$$

This map depends on the choice of P_0 and the choice of the path γ_P from P_0 to P.

It depends on the choice of γ_P only up to periods!

$\gamma_P - \gamma'_P$ is a closed path, so

$\int_{\gamma_P - \gamma'_P} \omega$ is a period.

So we get a map

$$\boxed{\begin{aligned} A: \quad & X \to \mathrm{Jac}(X) \\ & A(P)(\omega) = \int_{\gamma_P} \omega \end{aligned}}$$

ABEL-JACOBI MAP

(still depends on the choice of P_0)

Choosing a basis $\omega_1, \dots, \omega_g$ of $H^0(X, \omega)$, this map is $A(P) = \left(\int_{\gamma_P} \omega_1, \dots, \int_{\gamma_P} \omega_g \right) \in \mathbb{C}^g / \Lambda$

The ABEL-JACOBI MAP extends linearly to divisors on X:

$$A : \mathrm{Div}(X) \longrightarrow \mathrm{Jac}(X)$$
$$\sum n_P P \longmapsto \underbrace{\sum n_P A(P)}_{\text{sum as elements of the group } \mathrm{Jac}(X)}$$

$A_0 : \mathrm{Div}_0(X) \to \mathrm{Jac}(X)$: map induced on divisors of degree 0.

<u>Prop:</u> The map $A_0 : \mathrm{Div}_0(X) \to \mathrm{Jac}(X)$ does not depend on the choice of the base point $P_0 \in X$.

<u>Proof:</u> Let P_0' be another point in X and choose a path γ from P_0 to P_0'.

$$A(P) = \left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right)$$

$$A'(P) = \left(\int_{P_0'}^{P} \omega_1, \dots, \int_{P_0'}^{P} \omega_g \right) =$$

$$\overset{\text{in } \mathrm{Jac}(X)}{=} \underbrace{\left(\int_{P_0'}^{P_0} \omega_1, \dots, \int_{P_0'}^{P_0} \omega_g \right)}_{j :=} + \underbrace{\left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right)}_{A(P)}$$

$$A'(\Sigma n_P \cdot P) = \underbrace{(\Sigma n_P)}_{=0} j + A(\Sigma n_P \cdot P) \qquad \square$$

<u>THM:</u> (ABEL)

Let D be a divisor of degree 0 on X.

Then D is a principal divisor on X if and only if

$$A_0(P) = 0 \quad \text{in } \mathrm{Jac}(X).$$

<u>Exm:</u> a) $X \cong \mathbb{P}^1$, $\mathrm{Jac}(X) = \{0\}$: Every divisor of degree 0 is principal ✓.

b) $X \cong \mathbb{C}/L$, $\mathrm{Jac}(X) \cong X$: $A_0(\Sigma n_P \cdot P)$ is $\Sigma n_P \cdot P$ (group law on X)

<u>Cor:</u> X has genus ≥ 1, Then $A : X \to \mathrm{Jac}(X)$ is an embedding (but depends on the choice of base point P_0).

<u>Proof:</u> A is a holomorphic map of complex manifolds. It is also injective: If $A(P) = A(P')$, then

$$\mathcal{A}(P-P') = 0 \overset{\text{ABEL}}{\Longrightarrow} P-P' \text{ is principal}$$

$$\Longrightarrow X \cong \mathbb{P}^1. \quad \text{if} \quad P \neq P'$$

$$\Longrightarrow P = P'. \qquad \square$$

<u>SCHOTTKY PROBLEM.</u> Which $\mathbb{C}^g/\Lambda$ are Jacobians of curves?

$$\exp\left((z_1, \dots, z_g)(A+iB)\begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix} \right)$$