§8: LINEAR SYSTEMS and MAPS to PROJECTIVE SPACE

Let X be a (compact) Riemann surface.
We want to see that linear systems give a powerful language to describe holomorphic maps

$$\varphi : X \longrightarrow \mathbb{P}^r$$

We will assume that these maps are NONDEGENERATE meaning that the image $f(X)$ is not contained in any hyperplane.

Any map $\varphi : X \to \mathbb{P}^n$ is the composition of a nondegenerate map $X \to \mathbb{P}^r$ and a linear embedding $\mathbb{P}^r \hookrightarrow \mathbb{P}^n$.

First a little digression:

- PULLBACKS of DIVISORS: Let $\varphi : X \to \mathbb{P}^r$ be a Riemann surface and $D \subseteq \mathbb{P}^r$ an hypersurface (or divisor on $\mathbb{P}^r$) such that $\varphi(X) \not\subseteq D$. For any point $p \in X$ choose an affine chart $V \cong \mathbb{A}^r$ around $\varphi(p)$ and a local coordinate z in a neighborhood Δ around p. Then we write φ as

$$\varphi|_\Delta : \Delta \to V$$
$$z \longmapsto (\varphi_1(z), .. \varphi_r(z))$$

f_i holomorphic on V

and in the affine chart $D = \{ F(x_1, .., x_r) = 0 \}$
So we define

$$(\varphi^* D)_p := \text{ord}_p \, F(f_1(z), .., f_r(z))$$

$$\varphi^* D := \sum_{p \in X} (\varphi^* D)_p \cdot p$$

This is an effective divisor: $(\varphi^* D)_p \geq 0$ and
$(\varphi^* D)_p = 0$ iff $\varphi(p) \in \varphi(X) \cap D$. This
is a finite set because it is discrete and X is compact.

Now we proceed with our correspondence:

- <u>From MAPS to LINEAR SYSTEMS</u>

Let $\varphi : X \longrightarrow \mathbb{P}^r$ be a holomorphic map.
On $U_i = \varphi_i^{-1}(\{X_i \neq 0\})$ we can write

$$\varphi_{|U_i}(p) = [f_{i0}(p), f_{i1}(p), .., f_{ir}(p)]$$

for some holomorphic functions f_{ik} on U_i
$(f_{ii} = 1)$. We can rewrite this as

$$\varphi_{|U_i} = \left[1, \frac{f_{i1}}{f_{i0}}, \frac{f_{i2}}{f_{i0}}, .., \frac{f_{ir}}{f_{i0}} \right]$$

In particular, on $U_i \cap U_j$ we get that

$$\left[1, \frac{f_{i1}}{f_{i0}}, \dots, \frac{f_{ir}}{f_{i0}}\right] = \left[1, \frac{f_{j1}}{f_{j0}}, \dots, \frac{f_{jr}}{f_{j0}}\right]$$

so $\frac{f_{ik}}{f_{i0}} = \frac{f_{jk}}{f_{j0}}$ on $U_i \cap U_j$. But this means that the meromorphic functions $\left(\frac{f_{ik}}{f_{i0}}\right)$ on U_i glue together to a meromorphic function f_k on X. So we can describe φ as a map

$$\varphi : X \longrightarrow \mathbb{P}^r \qquad \varphi = [1, f_1, f_2, \dots, f_r]$$

where the f_i are meromorphic on X.

This map is nondegenerate iff all the $1, f_1, \dots, f_r$ are linearly independent. We keep this assumption for what follows.

We also denote the coordinate hyperplanes by

$$H_i = \{x_i = 0\} \subseteq \mathbb{P}^r$$

Prop: Let $\varphi : X \to \mathbb{P}^r$, $\varphi = [f_0, f_1, \dots, f_r]$ $(f_0 = 1)$ be a nondegenerate map as before and let

$$H = \{ a_0 X_0 + \dots + a_r X_r = 0 \} \subseteq \mathbb{P}^r$$

Then

$$\mathrm{div}\left(\sum a_i f_i \right) = \varphi^* H - \varphi^* H_0$$

proof: Let $p \in X$. Suppose first that none of the f_i has a pole at p. Then $\varphi(p) \in \{X_0 \neq 0\}$ and in this affine chart H and H_0 have equation

$$H \cap \{X_0 \neq 0\} = \{ a_0 + a_1 x_1 + \dots + a_r x_r = 0 \}$$
$$H_0 \cap \{X_0 \neq 0\} = \{ 1 = 0 \} = \emptyset, \text{ so}$$

$$(\varphi^* H)_p - (\varphi^* H_0)_p = \mathrm{ord}_p (a_0 + a_1 f_1 + \dots + a_r f_r).$$

Suppose instead one of the f_i has a pole at p and let $\mathrm{ord}_p(f_j) = \min\{\mathrm{ord}_p(f_i)\}$. If we choose a local coordinate z around p, in this neighborhood we have

$$\varphi(z) = [1, f_1(z), \dots, f_r(z)]$$
$$= \left[\frac{1}{f_j}, \frac{f_1}{f_j}, \dots, \frac{f_r}{f_j} \right]$$

and the f_i/f_j are holomorphic at p. $\left(\frac{f_j}{f_j} = 1 \right)$.

So we see $\varphi(p) \in \{X_j \neq 0\}$ and in this affine chart

$$H \cap \{X_j \neq 0\} = \{ a_0 x_0 + \ldots + a_j + \ldots + a_r x_r = 0 \}$$
$$H_0 \cap \{X_j \neq 0\} = \{ x_0 = 0 \}$$

So

$$(\varphi^* H)_p - (\varphi^* H_0)_p =$$
$$= \operatorname{ord}_p \left(a_0 \frac{1}{f_j} + \ldots + a_j + \ldots + a_r \frac{f_r}{f_j} \right) - \operatorname{ord}_p \left(\frac{1}{f_j} \right)$$
$$= \operatorname{ord}_p (a_0 + a_1 f_1 + \ldots + a_r f_r). \qquad \square$$

This tells us that

$f_0, f_1, \ldots, f_r$ are linearly independent elements in $H^0(X, \varphi^* H_0)$ which span a subspace $V \subseteq H^0(X, \varphi^* H_0)$ of dimension $r+1$

Equivalently

$\{ \varphi^* H \mid H \subseteq \mathbb{P}^r \text{ hyperplane} \}$ is a linear system of dimension $r+1$ in $|\varphi^* H_0|$. It can be considered as the projectivization of the space V.

Rmk: There is nothing special about the coordinate x_0 in all this. We could as well have written

$$\varphi = [f_0, \dots, f_r] \quad \text{with} \quad \sum a_i f_i = 1$$

and then everything would go through replacing H_0 with $H = \{\sum a_i x_i = 0\}$. The language of linear systems gets rid of this ambiguity: indeed the linear system $\{\varphi^* H \mid H \in \mathbb{P}^r \text{ hyperplane}\}$ is clearly independent of the choice of an hyperplane.

Rmk: If $\varphi = [f_0, \dots, f_r]$ as before and we compose it with a linear change of coordinates $\mathbb{P}^r \to \mathbb{P}^r$ the effect is that of replacing $(f_0, \dots, f_r)$ with another basis of V.

Moreover the linear systems obtained in this way are special, in the sense that they are BASE-POINT-FREE.

Def: BASE-POINT-FREE

Let D be a divisor on X. A linear system

$$\Lambda \subseteq |D|$$

Has a base point at $p \in X$ iff $p \in D \ \forall D \in \Lambda$.
A linear system without base pts is base-point-free.

Rmk: What does this mean in terms of meromorphic functions? Suppose that Λ corresponds to a subspace $V \subseteq H^0(X,D)$ then the corresponding linear system is $\Lambda = \{\mathrm{div}(f) + D \mid f \in \mathbb{P}(V)\}$ and there is a base point at p if and only if $p \in \mathrm{div}(f) + D$ for each $f \in V$, which can be rephrased as saying

$$\mathrm{ord}_p(f) + \mathrm{ord}_p(D) > 0 \quad \forall f \in V.$$

Prop: Let $\varphi : X \to \mathbb{P}^r$, $\varphi = [1, f_1, \dots, f_r]$ be a nondegenerate map as before. Then the corresponding linear system is base point free

proof: Take $p \in X$ and choose an hyperplane $H \subseteq \mathbb{P}^r$ that does not pass through $\varphi(p)$. Then $\varphi^* H$ does not contain p. $\square$

- **From LINEAR SYSTEMS to MAPS**

Conversely, suppose we have a divisor D and a base-point-free subspace $V \subseteq H^0(X, D)$ of dimension $r+1$, which corresponds to a base-point-free linear system $\Lambda \subseteq |D|$.

If we choose a basis $f_0, .. f_r$ of V, we get a map

$$\varphi : X \longrightarrow \mathbb{P}^r \qquad \varphi = [f_0, .. f_r]$$

out of this map we can get again a linear system by taking $\{ \varphi^* H \mid H \subseteq \mathbb{P}^r \}$ hyperplane and this will be the same as the original linear system:

Prop: In the above notation

$$\Lambda = \{ \varphi^* H \mid H \subseteq \mathbb{P}^r \}.$$

proof: Exercise.

In conclusion, this gives us a correspondence

$$\left\{ \begin{array}{c} \varphi : X \longrightarrow \mathbb{P}^r \\ \text{nondegenerate} \\ \text{map up} \\ \text{to linear change} \\ \text{of coordinates} \end{array} \right\} \rightleftarrows \left\{ \begin{array}{c} \text{base-point-free} \\ \text{linear systems} \\ \text{of dimension } r \end{array} \right\}$$

To get one actual map, not just up to change of coordinates, we choose a vector space $V \subseteq H^0(X,D)$ that induces the linear system and a basis of V.

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<u>Prop</u>: A complete linear system $|H^0(X,D)|$ is bpf iff $h^0(X, D-p) = h^0(X,D) - 1 \quad \forall p \in X$.

<u>proof</u>: We claim that $p \in X$ is a base point iff
$$h^0(X,D-p) = h^0(X,D).$$

- <u>DIVISORS</u>: $\{ E \in |D| \mid p \in E \} = |D-p| + p$

$\boxed{\subseteq}$ $E = E' + p$, E' effective.

$$E \sim D \Rightarrow E' \sim D - p \Rightarrow E' \in |D-p|.$$

$\boxed{2}$ $E' \sim D-p$ and E' effective. Then $E = E'+p$ is effective and $E \sim D$.

Then p is a basepoint iff

$$|D-p|+p = |D|$$

$$\iff \dim(|D-p|+p) = \dim|D|$$

$$\iff h^0(D-p)-1 = h^0(D)-1$$

$$\iff h^0(D-p) = h^0(D).$$

- <u>FUNCTIONS</u>: So, p is a basepoint iff

$$\iff \mathrm{ord}_p(f) + \mathrm{ord}_p(D) > 0 \quad \forall f \in H^0(X,D)$$

$$\iff \mathrm{ord}_p(f) + \mathrm{ord}_p(D) \geq 1$$

$$\iff \mathrm{ord}_p(f) + \mathrm{ord}_p(D-p) \geq 0$$

$$\iff f \in H^0(X, D-p) \qquad \forall f \in H^0(X,D)$$

If $q \neq p$ $\mathrm{ord}_q(D) = \mathrm{ord}_q(D-p)$.

So iff $H^0(X,D-p) = H^0(X,D)$ $\square$

A divisor D s.t. $H^0(X,D)$ is bpf is called bpf.