COMPLEX TORI

Def : LATTICE

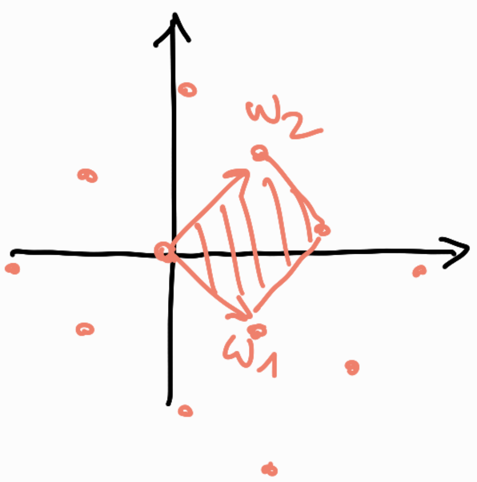
A lattice $\Lambda \subseteq \mathbb{C}$ is the subgroup generated by two $\mathbb{R}$-linearly independent complex numbers

$$\Lambda = \{ n_1 \omega_1 + n_2 \omega_2 \mid n_i \in \mathbb{Z} \} \qquad \begin{array}{l} \omega_1, \omega_2 \in \mathbb{C} \\ \mathbb{R}\text{-linearly indep} \end{array}$$

Def ; COMPLEX TORUS

A complex torus is a quotient $\mathbb{C}/\Lambda$ where $\Lambda \subseteq \mathbb{C}$ is a lattice acting by translations.

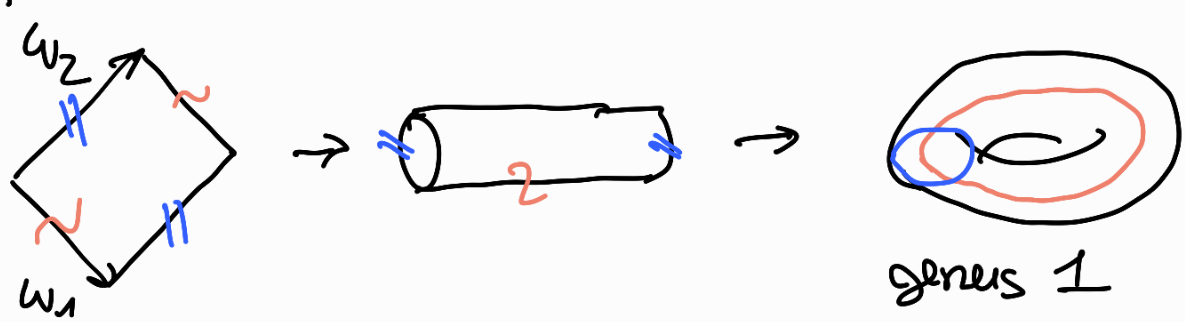
We put on the torus $E = \mathbb{C}/\Lambda$ the quotient topology so that E is a compact, Hausdorff space.



A fundamental parallelogram of the lattice $\Lambda = \{ n_1\omega_1 + n_2\omega_2 \}$ is

$$P = \{ \lambda_1 \omega_1 + \lambda_2 \omega_2 \mid \lambda_i \in [0,1] \}$$

Then, as topological spaces, $\mathbb{C}/\Lambda$ is homeomorphic to the parallelogram where the opposite sides are identified



So we see that a complex torus is also a topological torus: an orientable compact surface of genus 1.

Prop: A complex torus has a natural structure of a compact connected Riemann surface of genus 1.

proof: we know already that $E = \mathbb{C}/\Lambda$ is a compact connected topological surface of genus 1. We need to put the complex structure on it. Consider the projection map

$$\pi : \mathbb{C} \to E$$

and let $\mathring{P} = \{ \lambda_1 \omega_1 + \lambda_2 \omega_2 \mid \lambda_i \in (0,1)\}$ be the interior of a fundamental parallelogram. We claim that

$$\pi : \mathring{P} + v \to \pi(\mathring{P} + v)$$ is an homeomorphism onto the image for all $v \in \mathbb{C}$

The map π is open so we just need to prove that it is bijective. As it is clearly surjective, it is enough to show that it is injective: let $\lambda_i, \mu_i \in (0,1)$ s.t.

$$\pi(\lambda_1 \omega_1 + \lambda_2 \omega_2) = \pi(\mu_1 \omega_1 + \mu_2 \omega_2)$$

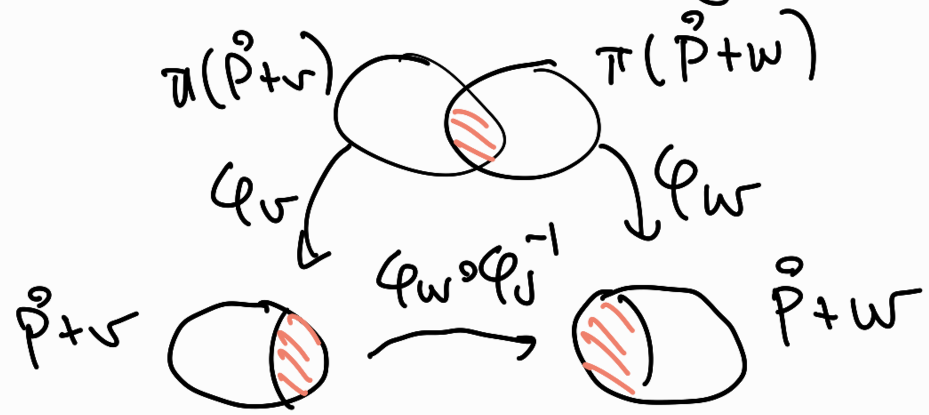
Then $(\lambda_1 - \mu_1)\omega_1 + (\lambda_2 - \mu_2)\omega_2 \in \Lambda$ so that

$$(\lambda_1 - \mu_1)\omega_1 + (\lambda_2 - \mu_2)\omega_2 = n_1 \omega_1 + n_2 \omega_2 \quad n_i \in \mathbb{Z}$$

so that $\lambda_i - \mu_i = n_i$. But then $n_i = 0$.

The open sets $\pi(\mathring{P} + v)$ clearly cover E. As charts we take $\varphi_v : \pi(\mathring{P} + v) \to \mathring{P} + v$, $\varphi_v = (\pi|_{\mathring{P}+v})^{-1}$.

We need to check compatibility:



However it is easy to check that this change of coordinates map is simply a translation, which is holomorphic.

$$\varphi_w \circ \varphi_v^{-1}: \quad \underset{\substack{\| \\ u}}{z+v} \longmapsto z+w$$
$$u \longmapsto u+(w-v)$$

$\square$

<u>Rmk</u>: With this complex structure on E, the map

$$\pi: \mathbb{C} \to E = \mathbb{C}/\Lambda \;, \; z \mapsto [z]$$

is holomorphic, as well as the maps

$$E \times E \to E \qquad\qquad E \to E$$
$$[z_1],[z_2] \mapsto [z_1+z_2] \qquad\qquad [z] \longmapsto [-z]$$

giving the group structure of E

Now we want to consider maps between complex tori. Consider first a linear complex map

$$\tilde{f}: \mathbb{C} \longrightarrow \mathbb{C} \;, z \longmapsto a \cdot z \quad a \in \mathbb{C}$$

s.t. $\tilde{f}(\Lambda_1) \subseteq \Lambda_2$ for two lattices $\Lambda_1, \Lambda_2 \subseteq \mathbb{C}$

Then $\tilde{f}$ induces a holomorphic map

$$f : \mathbb{C}/\Lambda_1 \longrightarrow \mathbb{C}/\Lambda_2$$
$$[z] \longmapsto [az]$$

which is also a group homomorphism.

[Exercise: check these assertions]

actually almost all maps are of this form.

Prop: Let $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ be a holomorphic map s.t. $f(0) = 0$. Then f is induced by a $\mathbb{C}$-linear map $\tilde{f} : \mathbb{C} \to \mathbb{C}$ s.t. $\tilde{f}(\Lambda_1) \subseteq \Lambda_2$ as before.

Proof: The map $\mathbb{C} \to \mathbb{C}/\Lambda_2$ is the universal cover, so the composition $\mathbb{C} \to \mathbb{C}/\Lambda_1 \xrightarrow{f} \mathbb{C}/\Lambda_2$ lifts to it, since $\mathbb{C}$ is simply connected.

We have a diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi"] \\ \mathbb{C}/\Lambda_1 \arrow[r, "f"] & \mathbb{C}/\Lambda_2 \end{tikzcd}$$

$\tilde{f}$ holomorphic

$\tilde{f}(0) = 0$

We claim that $\tilde{f}$ is a linear map: $\tilde{f} = a \cdot z$, or, equivalently, that the derivative $\tilde{f}'$ is constant. Fix $\lambda_1 \in \Lambda_1$ and consider the function $\tilde{f}(z + \lambda_1) - \tilde{f}(z)$

By construction, we have that
$[\tilde{f}(z+\lambda_1)] = [\tilde{f}(z)]$ in $\mathbb{C}/\Lambda_2$, hence
$\tilde{f}(z+\lambda_1) - \tilde{f}(z) \in \Lambda_2$. Since Λ_2 is discrete
it must be that there exists λ_2 s.t.

$$\tilde{f}(z+\lambda_1) - \tilde{f}(z) = \lambda_2 \quad \forall z \in \mathbb{C}$$

differentiating

$$\tilde{f}'(z+\lambda_1) - \tilde{f}'(z) = 0$$

this proves that $\tilde{f}'$ is Λ_1-invariant, hence
induces a holomorphic map $\tilde{f}': \mathbb{C}/\Lambda_1 \to \mathbb{C}$
which then must be constant, as $\mathbb{C}/\Lambda_1$ is
compact. This is what we wanted to show. $\square$

Cor: Any holomorphic map between complex tori is the composition of a homomorphism and a translation

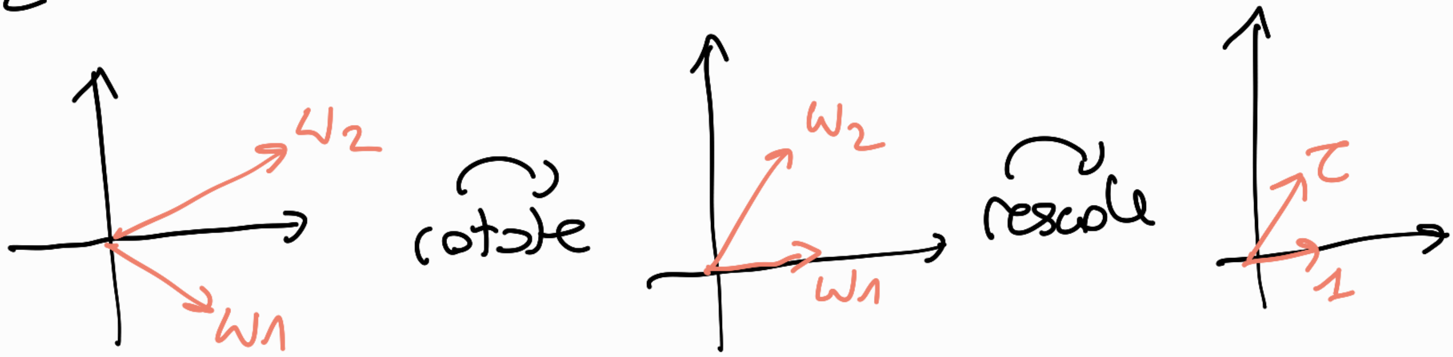
proof: $f: E_1 \to E_2$. Let $a = f(0)$
and consider the composition $t_{-a} \circ f : E_1 \to E_2$
where $t_{-a} : E_2 \to E_2$, $z \mapsto z - a$.
Then $(t_{-a} \circ f)(0) = 0$ so it is a homomorphism
by the previous proposition. Then

$$f = t_a \circ (\text{homomorphism})$$

$\square$

Lemma: Every complex torus is isomorphic to one of the form $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ where
$$\tau \in \mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$$

proof: take a lattice $\Lambda = \{n_1\omega_1 + n_2\omega_2\}$ and rotate it and rescale it until it is of the form $\mathbb{Z}+\tau\mathbb{Z}$. Rotation and rescale are $\mathbb{C}$-linear $\square$



Def: UPPER HALF SPACE
The space $\mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ is called upper half space.

This representation allows us to define a fundamental function

Def: THETA FUNCTION
Let $\tau \in \mathcal{H}$. The corresponding theta function is
$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i n\tau n + 2\pi i n z\right)$$

Prop: The theta function is an entire holomorphic function of z. Moreover it is even.

proof: we show that the series converges uniformly on compact sets. Suppose $Im(\tau) > \varepsilon > 0$, and $Im(z) \geq -c/2$ for a fixed $c > 0$. Then

$$|\exp(\pi i n \tau n + 2\pi i n z)| =$$

$$|\exp(\pi i n \cdot Re(\tau) \cdot n + 2\pi i n \cdot Re(z))| \cdot$$

$$\cdot |\exp(-\pi n Im(\tau) n - 2\pi n Im(z))|$$

$$= \exp(-\pi n Im(\tau) n - 2\pi i n Im(t))$$

$$< \exp(-\pi n^2 \varepsilon + \pi n c) = \exp(-\pi n(n\varepsilon - c))$$

Now choose $n_0 \in \mathbb{N}$ s.t. $n_0 \varepsilon - c > 1$

then $\forall n \geq n_0$

$$\exp(-\pi n(n\varepsilon - c)) =$$

$$\exp(-\pi n((n-n_0)\varepsilon + n_0\varepsilon - c) \leq$$

$$\leq \exp(-\pi n(n-n_0)\varepsilon) \quad \text{which converges}$$

To check that it is even

$$\Theta(-z,\tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i n \tau n + 2\pi i n(-z))$$

$$= \sum_{n \in \mathbb{Z}} \exp(\pi i(-n)\tau(-n) + 2\pi i(-n) z) = \Theta(z,\tau). \qquad \square$$

<u>Prop</u>: The Theta function is QUASIPERIODIC meaning that

$$\theta(z+m+\tau n,\tau)=\exp(-\pi i n\tau n-2\pi i n z)\,\theta(z,\tau)$$

for all $n,m\in\mathbb{Z}$, $z\in\mathbb{C}$.

<u>proof</u>: write for $k\in\mathbb{Z}$

$$\exp(\pi i k\tau k+2\pi i k(z+m+\tau n))=$$

$$=\exp(\pi i k\tau k+2\pi i kz+2\pi i k\tau n)\cdot\underbrace{\exp(2\pi i km)}_{1}$$

$$=\exp(\pi i(k+n)\tau(k+n)+2\pi i(k+n)z)$$

$\cdot\exp(-\pi i n\tau n-2\pi i n z)$. So, summing over all $k\in\mathbb{Z}$ we get

$$\theta(z+m+\tau n,\tau)=\exp(-\pi i n\tau n-2\pi i n z)\underbrace{\sum_{k\in\mathbb{Z}}\exp(\pi i(k+n)\tau(k+n)+2\pi i(k+n)z)}_{\theta(z,\tau)}$$

□

In particular, since the factor $\exp(-\pi i n\tau n-2\pi i n z)$ is never zero, this tells us that

$$\theta(z+m+\tau n,\tau)=0 \iff \theta(z,\tau)=0$$

so the set of zeroes is Λ-invariant.

We want to determine this explicitly.
To do this, it will be useful to consider the logarithmic derivative of Θ:

$$\frac{d\log\Theta}{dz} = \frac{\frac{d\Theta}{dz}}{\Theta}$$ meromorphic function

This obeys the formal rules of the logarithm in particular:

$$\Theta(z+m+\tau n,\tau) = \exp(-\pi i n\tau n - 2\pi i n z)\Theta(z,\tau)$$

$$\log\Theta(z+m+\tau n,\tau) = -\pi i n\tau n - 2\pi i n z + \log\Theta(z,\tau)$$

$$\frac{d\log\Theta}{dz}(z+m+\tau n,\tau) = -2\pi i n + \frac{d\log\Theta}{dz}(z,\tau)$$

$$\frac{d^2\log\Theta}{dz^2}(z+m+\tau n,\tau) = \frac{d^2\log\Theta}{dz^2}(z,\tau)$$

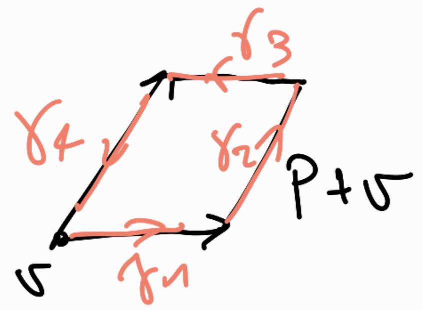
Given this, we can prove

Prop: The zeroes of the theta function are

$$\{z \mid \Theta(z,\tau)=0\} = \frac{1}{2}+\tau\frac{1}{2}+(\mathbb{Z}+\tau\mathbb{Z})$$

and each zero has multiplicity one.

proof: first we count them. Since the set of zeroes is discrete we can find a translate of the fundamental parallelogram that does not have any zeroes on the boundary. The zeroes inside this translate are counted, with multiplicity by



$$\sum_{z\in \mathring{P}+v} \mathrm{Res}_z\left(\frac{d\log\theta}{dz}\right) = \sum_{z\in \mathring{P}+v} \mathrm{ord}_z\,\theta$$

The sum of the residues can be computed via the residue theorem as

$$\frac{1}{2\pi i}\left(\int_{\gamma_1}\frac{d\log\theta}{dz}\,dz + \int_{\gamma_2}\frac{d\log\theta}{dz}\,dz + \int_{\gamma_3}\frac{d\log\theta}{dz}\,dz + \int_{\gamma_4}\frac{d\log\theta}{dz}\,dz\right)$$

However, since

$$\frac{d\log\theta}{dz}(z+m+\tau n,\tau) = -2\pi i n + \frac{d\log\theta}{dz}(z,\tau)$$

we see that

$$\int_{\gamma_3}\frac{d\log\theta}{dz}\,dz = 2\pi i - \int_{\gamma_1}\frac{d\log\theta}{dz}\,dz$$

$$\int_{\gamma_2}\frac{d\log\theta}{dz}\,dz = -\int \frac{d\log\theta}{dz}\,dz$$

Hence we see that the final result is 1. The theta function has one zero in $\mathring{P}+v$.

It is now enough to find one point where the function is zero.

<u>Claim</u>: $\theta\left(\frac{1}{2}+\tau\frac{1}{2},\tau\right)=0.$

How can we prove this? It would be easy if the function

$$z \longmapsto \theta\left(z+\frac{1}{2}+\tau\frac{1}{2},\tau\right)$$

were to be odd, since any odd function vanishes at zero. However

$$\theta\left(-z+\frac{1}{2}+\tau\frac{1}{2},\tau\right)=\theta\left(z-\frac{1}{2}-\tau\frac{1}{2},\tau\right) \quad [\theta \text{ is even}]$$

$$=\theta\left(z+\frac{1}{2}+\tau\frac{1}{2}-1-\tau,\tau\right)$$

$$=\exp\left(+\pi i\tau+2\pi i\left(z+\frac{1}{2}+\tau\frac{1}{2}\right)\right)\theta\left(z+\frac{1}{2}+\tau\frac{1}{2},\tau\right)$$

So, it is not exactly odd. However, we can fix by defining

$$\theta_{11}(z,\tau)=\exp\left(-\frac{1}{4}\pi i\tau-\pi i\left(z+\frac{1}{2}\right)\right)\theta\left(z+\frac{1}{2}+\frac{1}{2}\tau\right)$$

which is odd (exercise) and we conclude. $\square$

MEROMORPHIC FUNCTIONS on COMPLEX TORI

For every $x \in \mathbb{C}$ define the function

$$\Theta^{(x)}(z,\tau) := \Theta\left(z+\tfrac{1}{2}+\tfrac{1}{2}\tau - x, \tau\right)$$

Observe that

$$\Theta^{(x)}(z+m+\tau n,\tau) = \exp\left(-\pi i n \tau n - 2\pi i n\left(z+\tfrac{1}{2}+\tfrac{1}{2}\tau - x\right)\right)\Theta^{(x)}(z,\tau)$$
$$= \exp(2\pi i n x)\exp\left(-\pi i n\tau n - 2\pi i n\left(z+\tfrac{1}{2}+\tfrac{1}{2}\tau\right)\right)\Theta^{(x)}(z,\tau)$$

In particular, take $x_1,\dots,x_d,y_1,\dots,y_d \in \mathbb{C}$ s.t.

$$\boxed{x_1+\dots+x_d = y_1+\dots+y_d}$$

and define

$$\eta^{(\underline{x},\underline{y})}(z,\tau) = \frac{\Theta^{(x_1)}(z,\tau)\cdots\Theta^{(x_d)}(z,\tau)}{\Theta^{(y_1)}(z,\tau)\cdots\Theta^{(y_d)}(z,\tau)}$$

<u>Prop</u>: $\eta^{(\underline{x},\underline{y})}$ defines a meromorphic function on $E_\tau = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ with divisor

$$\boxed{\operatorname{div}\left(\eta^{(\underline{x},\underline{y})}\right) = [x_1]+\dots+[x_d]-[y_1]-\dots-[y_d]}$$

proof: we show that η is Λ_τ-invariant

$$\eta(z+m+\tau n,\tau) = \exp\left(2\pi i n\left(x_1+\dots+x_d-y_1-\dots-y_d\right)\right) \cdot \eta(z,\tau) = \eta(z,\tau)$$

hence η defines a meromorphic function on E_τ.
To show that the divisor is that one, it is enough to observe that the zeroes of $\Theta^{(x)}$ are

$$\{z \mid \Theta^{(x)}(z,\tau)=0\} = x+\Lambda$$

each one with multiplicity one. $\square$

This allows us to prove

Thm: ABEL'S THEOREM for COMPLEX TORI - I

Two divisors on a complex torus E are equivalent if and only if they have the same degree and the same sum of the corresponding points in E. More precisely

$$\boxed{\sum n_p \cdot P \sim \sum m_p \cdot P \iff \begin{array}{l} \cdot \sum n_p = \sum m_p \\ \quad \text{and} \\ \cdot \sum n_p \cdot P = \sum m_p \cdot P \\ \quad \text{where the sum is in } E \end{array}}$$

<u>Proof</u>: by the first formulation we mean this
take two divisors $D = \sum n_p \cdot p$ and $E = \sum m_p \cdot p$
then

$$D \sim E \quad \text{iff} \quad \begin{array}{c} \sum n_p = \sum m_p \\ \text{and} \\ \sum n_p p = \sum m_p \cdot p \end{array} \quad \begin{array}{c}\text{as a sum} \\ \text{in } E\end{array}$$

We can suppose that the torus is of the form
$\mathbb{C} / \mathbb{Z} + \tau\mathbb{Z}$ for $\tau \in \mathcal{H}$.

We can also assume that D, E are effective
of the same degree, so that

$$D = [x_1] + \dots + [x_d] \ , \quad E = [y_1] + \dots + [y_d]$$

Assume first that the sum of the points in
$\mathbb{C} / \mathbb{Z} + \tau\mathbb{Z}$ is the same i.e.

$$[x_1 + \dots + x_d] = [y_1 + \dots + y_d]$$

Then, up to changing representatives, we can
assume $x_1 + \dots + x_d = y_1 + \dots + y_d$ and
then the previous proposition gives us

$$[x_1] + \dots + [x_d] - [y_1] - \dots - [y_d] = \operatorname{div}\left(\eta^{(\underline{x}, \underline{y})}\right)$$

hence they are linearly equivalent.

Conversely, suppose that D and E are linearly equivalent and set

$$x_{d+1} = y_1 + \dots + y_d \,, \quad y_{d+1} = x_1 + \dots + x_d$$

then, thanks to what we already proved, we know that

$$[x_1] + \dots + [x_d] + [y_{d+1}] \sim [y_1] + \dots + [y_d] + [y_{d+1}]$$

but $[x_1] + \dots + [x_d] \sim [y_1] + \dots + [y_d]$ by assumption, so it must be that $[x_{d+1}] \sim [y_{d+1}]$. This means that there is a meromorphic function f s.t.

$$[x_{d+1}] - [y_{d+1}] = \mathrm{div}(f)$$

equivalently, there is a map $f : \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \to \mathbb{P}^1$ s.t. $f^{-1}(0) = [x_{d+1}]$, $f^{-1}(\infty) = [y_{d+1}]$ with multiplicity. But, if f is nonconstant then it has degree 1 and it is an isomorphism impossible because a complex torus has genus 1. Then f is constant and $[x_{d+1}] = [y_{d+1}]$ which is what we wanted to prove. $\square$

Thm: ABEL'S THEOREM - II

Let E be a complex torus.

For all fixed $p_0 \in E$ the map

$$\boxed{\begin{aligned} \alpha_{p_0} : E &\longrightarrow \mathrm{Pic}_0(E) \\ p &\longmapsto [p - p_0] \end{aligned} \\ \text{is an isomorphism of groups.}}$$

proof: α_{p_0} injective: if $p \sim p_0$, the previous result shows that $p = p_0$.

α_{p_0} surjective: let $D = \sum n_q \cdot q$ be any divisor of degree zero. We need to find a point $p \in E$ s.t.

$$p - p_0 \sim \sum n_q \cdot q \qquad \text{i.e.}$$

$$p \sim \sum n_q \cdot q + p_0$$

it is enough to take $p = \sum n_q q + p_0$ where the sum on the right is now on the complex torus E.