DIVISORS

Def: DIVISOR

Let S be a compact and connected Riemann surface. A DIVISOR is a formal finite sum of points in S:

$$\boxed{D = \sum_{p \in S} n_p \cdot p} \qquad n_p \in \mathbb{Z}, \quad n_p = 0 \text{ for all but finitely many points } p \in S$$

The set $\mathrm{Div}(S)$ of divisors is naturally an abelian group with the sum.

A divisor D as above is called EFFECTIVE if $n_p \geq 0 \ \forall p \in S$. We write $D \geq 0$ to say that D is effective. Furthermore, we write $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$.

Def: DEGREE of a DIVISOR

Let $D = \sum_{p \in S} n_p \cdot p$ be a divisor. Its degree is

$$\boxed{\deg(D) = \sum_{p \in S} n_p \in \mathbb{Z}}$$

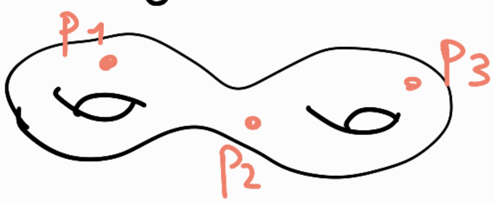
This defines a group homomorphism

$$\deg : \mathrm{Div}(S) \to \mathbb{Z}$$

We also set $\mathrm{Div}_d(S) = \{ D \in \mathrm{Div}(S) \mid \deg D = d \}$.

Rmk: The group $\mathrm{Div}(S)$ is the free abelian group on the points of S.

Example: A divisor is something as simple as
$3p_1 + 4p_2 - 5p_3$



Def: PRINCIPAL DIVISOR
Let S be a compact and connected Riemann surface and let f be a nonzero meromorphic function of S. We define the DIVISOR of f as

$$\boxed{\operatorname{div}(f) = \sum \operatorname{ord}_p(f) \cdot p}$$

This can be written as the difference of two effective divisors

$$\operatorname{div}(f) = \operatorname{div}_0(f) - \operatorname{div}_\infty(f)$$

$$\operatorname{div}_0(f) = \sum_{\substack{p \text{ zero} \\ \text{of } f}} \operatorname{ord}_p(f) \cdot p$$

$$\operatorname{div}_\infty(f) = \sum_{\substack{p \text{ pole} \\ \text{of } f}} (-\operatorname{ord}_p(f)) \cdot p$$

Such a divisor is called PRINCIPAL and the set of all principal divisor is denoted by
$\operatorname{Princ}(S) \subseteq \operatorname{Div}(S)$

Rmk: If we denote by $\mathbb{C}(S) = \left\{ f \begin{array}{l} \text{meromorphic} \\ \text{on } S \end{array} \right\}$
then this is naturally a FIELD via sum and product of meromorphic functions. Then

$$\operatorname{div}(f \cdot g) = \operatorname{div}(f) + \operatorname{div}(g) \quad f, g \in \mathbb{C}(S)^*$$

as it is easy to check, hence

$$\mathrm{div}: \mathbb{C}(S)^* \longrightarrow \mathrm{Div}(S) \text{ is a group hom.}$$

and its image $\mathrm{Princ}(S)$ is a subgroup of $\mathrm{Div}(S)$.

<u>Def</u>: LINEAR EQUIVALENCE, PICARD GROUP

Two divisors D_1, D_2 are called LINEAR EQUIVALENCE if the difference $D_1 - D_2$ is a principal divisor.

In symbols

$$\boxed{D_1 \sim D_2 \overset{\text{def}}{\iff} D_1 - D_2 = \mathrm{div}(f)} \text{ for a } f \in \mathbb{C}(S)^*$$

Equivalently D_1, D_2 are linearly equivalent if and only if they have the same class in the PICARD GROUP

$$\boxed{\mathrm{Pic}(S) = \mathrm{Div}(S)/\mathrm{Princ}(S)}$$

<u>Lemma</u>: Any principal divisor has degree 0.

In particular, if D_1, D_2 are linearly equivalent, then they have the same degree.

<u>proof</u>: $\deg(\mathrm{div}(f)) = \sum_{p \in S} \mathrm{ord}_p(f) = 0.$ $\square$

<u>Rmk</u>: This means that the degree morphism actually factors through the Picard group

$$\deg: \mathrm{Pic}(S) \to \mathbb{Z}, \quad \mathrm{Pic}_d(S) = \{[D] \mid \deg(D) = d\}$$

And $\mathrm{Pic}_0(S)$ is an abelian group.

Enough general theory, let's make some examples

<u>Prop</u>: On $\mathbb{P}^1$, two divisors are linearly equivalent if and only if they have the same degree. Equivalently
$$\deg : \mathrm{Pic}(\mathbb{P}^1) \to \mathbb{Z} \text{ is an isomorphism}$$

<u>proof</u>: Since the degree is clearly surjective, we need to prove that it is injective. Let D be a divisor of degree 0, then we can write
$$D = P_1 + \dots + P_d - Q_1 - \dots - Q_d$$
for points $P_i, Q_i \in S$ (possibly with repetitions). Then it is enough to prove that $P_i \sim Q_i$. We will prove that any point $P \in \mathbb{P}^1$ is linearly equivalent to $\infty \in \mathbb{P}^1$. We can assume that $P = a \in \mathbb{C}$ and then
$$a - \infty = \mathrm{div}(z - a). \qquad \square$$

<u>Rmk</u>: Even more explicitly, if $a_1, \dots, a_r \in \mathbb{C}$ then
$$\mathrm{div}((z-a_1)\cdot \ldots \cdot (z-a_r)) = a_1 + \dots + a_r - r \cdot \infty$$
and if we have also $b_1, \dots, b_s \in \mathbb{C}$ then
$$\mathrm{div}\left(\frac{(z-a_1)\cdot \ldots \cdot (z-a_r)}{(z-b_1)\cdot \ldots \cdot (z-b_s)} \right) =$$
$$= a_1 + \dots + a_r - b_1 - \dots - b_s + (s-r)\infty$$

Example: HYPERPLANE SECTIONS of PLANE CURVES

Let $C = \{F(X,Y,Z)=0\}$ be a smooth projective plane curve. For a line $L = \{\ell(X,Y,Z)=0\} \subseteq \mathbb{P}^2$ (different from C) we define a divisor

$$\operatorname{div}_C(L) = \sum_{P \in C \cap L} \mu_P(C,L) \cdot P$$

By BEZOUT, we know that

$$\deg(\operatorname{div}_C(L)) = \sum \mu_P(C,L)$$
$$= (\deg C)\cdot(\deg L) = \deg C.$$

We claim that all these divisors are linearly equivalent. More precisely, if

$$L_1 = \{\ell_1(X,Y,Z)=0\}, \quad L_2 = \{\ell_2(X,Y,Z)=0\}$$

are two such lines, then

$$\operatorname{div}_C(L_1) - \operatorname{div}_C(L_2) = \operatorname{div}\left(\frac{\ell_1(X,Y,Z)}{\ell_2(X,Y,Z)}\right)$$

meaning that

$$\mu_P(C,L_1) - \mu_P(C,L_2) = \operatorname{ord}_P\left(\frac{\ell_1}{\ell_2}\right) \quad \text{for all } P \in C$$

but we proved this already.