LINEAR SYSTEMS and MAPS to PROJECTIVE SPACES

S = compact connected Riemann surface.

If we want to look at S as a projective variety, we should map it to projective space.

We are looking for holomorphic maps

$$\varphi : S \longrightarrow \mathbb{P}^r$$

Consider now nonzero meromorphic functions

$$f_0, f_1, \dots, f_r \text{ on } S$$

We want to define a holomorphic map

$$[f_0, \dots, f_r] : S \longrightarrow \mathbb{P}^r$$
$$p \longmapsto [f_0(p), \dots, f_r(p)]$$

This is clearly well defined at all points p where the f_i have no poles and no common zeroes

We claim that this is defined everywhere:

Take a point $p \in S$ and a local coordinate z centered at the point. Then we can write each f_i as

$$\underbrace{f_i = z^{m_i} \cdot g_i(z)}$$

$m_i = \operatorname{ord}_p(f_i)$

$g_i(z)$ is holomorphic and $g_i(p) \neq 0$

So, in a neighborhood around p we can write

$$[f_0, \dots, f_r] = [z^{m_0} \cdot g_0(z), \dots, z^{m_r} \cdot g_r(z)]$$

suppose $m = \min\{m_i\}$

$$= [z^{m_0 - m} g_0(z), \dots, z^{m_r - m} g_r(z)]$$

now notice that $m_i \geq m$ by definition and there is at least one i s.t. $m_i = m$.

This means that all the $z^{m_i - m} g_i(z)$ are holomorphic at p and at least one of them does not vanish at p. So this is well-defined at p.

Example: Take on $\mathbb{P}^1$ the meromorphic function $\dfrac{z^2}{(z-1)(z-2)}$ where z is an affine coordinate

Then we have the map

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ z &\longmapsto \left[1, \frac{z^2}{(z-1)(z-2)}\right] \\ &= \left[(z-1)(z-2), z^2\right] \end{aligned}$$

$(u = 1/z)$

$$\begin{aligned} u &\longmapsto \left[\left(\frac{1}{u}-1\right)\left(\frac{1}{u}-2\right), \frac{1}{u^2}\right] \\ &= \left[(1-u)(1-2u), 1\right] \end{aligned}$$

Why is the previous map well defined in the intersections?

Recall for example that for us a meromorphic function f_i is a collection

$\left(\dfrac{a_{ij}}{b_{ij}}\right)$ of meromorphic functions on a collection of charts U_j

and such that on $U_j \cap U_k$ we have

$$\frac{a_{ij}}{b_{ij}} = \frac{a_{ik}}{b_{ik}}$$

Then the map of before becomes

$$[f_0, \dots, f_r] = \left[\frac{a_{0j}}{b_{0j}}, \dots, \frac{a_{rj}}{b_{rj}}\right] \text{ on } U_j$$

$$= \left[\frac{a_{0k}}{b_{0k}}, \dots, \frac{a_{rk}}{b_{rk}}\right] \text{ on } U_k$$

From this collection of meromorphic functions $f_0, \dots, f_r$ we get a map

$$[f_0, \dots, f_r] : S \longrightarrow \mathbb{P}^r$$

we should check that this map is holomorphic but this follows from what we have seen before: around each point $p \in S$ we can write the map as

$$[f_0, \dots, f_r] = [h_0(z), \dots, h_r(z)]$$

where z is a local coordinate at p

the h_i are holomorphic and at least one of these does not vanish at p.

Suppose for example that $h_0(p) \neq 0$

Then we can write this as

$$\begin{aligned} \Delta = \text{small neighborhood of } p &\longrightarrow \{x_0 \neq 0\} \subseteq \mathbb{P}^r \\ z &\longmapsto [h_0, \dots, h_r] \end{aligned}$$

same as

$$\begin{aligned} \Delta &\longrightarrow \mathbb{C}^r \\ z &\longmapsto \left(\frac{h_1}{h_0}, \frac{h_2}{h_0}, \dots, \frac{h_r}{h_0}\right) \end{aligned}$$

and the $\frac{h_i}{h_0}$ are clearly holomorphic.

So this proves that the map

$$[f_0, \dots, f_r] : S \longrightarrow \mathbb{P}^r$$

is holomorphic.

Conversely, we claim that all holomorphic maps

$$\varphi : S \longrightarrow \mathbb{P}^r$$

have this form. Consider the chart $\{x_i \neq 0\}$ in $\mathbb{P}^r$ and $U_i = \varphi^{-1}(\{x_i \neq 0\})$

then $\varphi_{|U_i} : U_i \longrightarrow \{x_i \neq 0\} = \mathbb{C}^r$

is holomorphic. So we can write

$$\begin{aligned} \varphi_{|U_i} : U_i &\longrightarrow \{x_i \neq 0\} \\ p &\longmapsto [f_{i0}(p), f_{i1}(p), \dots, f_{ir}(p)] \end{aligned}$$

where the f_{ij} are holomorphic on U_i, $f_{ii} = 1$

We can rewrite this as

$$[f_{i0}, \dots, f_{ir}] = \left[1, \frac{f_{i1}}{f_{i0}}, \dots, \frac{f_{ir}}{f_{i0}}\right] = \varphi_{|U_i}$$

Since $\varphi|U_i = \varphi|U_j$ we have

$$\left[1, \frac{f_{i1}}{f_{i0}}, \dots, \frac{f_{ir}}{f_{i0}}\right] = \left[1, \frac{f_{j1}}{f_{j0}}, \dots, \frac{f_{jr}}{f_{j0}}\right]$$

on $U_i \cap U_j$. Then

$$\frac{f_{ik}}{f_{i0}} = \frac{f_{jk}}{f_{j0}} \text{ on } U_i \cap U_j$$

but this means that the meromorphic functions $\left(\frac{f_{ik}}{f_{i0}}\right)$ on U_i glue together to a single meromorphic function f_k on S

So we can write

$$\boxed{\varphi : S \longrightarrow \mathbb{P}^r \text{ as } \varphi = [f_0, f_1, \ldots, f_r] \text{ where the } f_i \text{ are meromorphic}}$$

What is the connection to divisors?

PULLBACK of DIVISORS

Let $\varphi : S \longrightarrow \mathbb{P}^r$ be a holomorphic map and $D \subseteq \mathbb{P}^r$ be a divisor, or hypersurface

and such that $\varphi(S) \not\subseteq D$.

We want to define a divisor on S by pulling back D.

For a point $p \in S$ take an affine chart in $\mathbb{P}^r$ containing $\varphi(p)$ and a local coordinate z around p.

Then we can write

$$\varphi = (\varphi_1(z), .., \varphi_r(z)) \in \mathbb{C}^r$$

around p. In the affine chart, the divisor D is given by a polynomial equation $F(x_1, .., x_r) = 0$

Then we define

$$(\varphi^* D)_p = \operatorname{ord}_p F(\varphi_1(z), \dots, \varphi_r(z))$$

We define the pullback as

$$\boxed{\varphi^* D = \sum_{p \in S} (\varphi^* D)_p \cdot p}$$

This is an effective divisor and

$(\varphi^* D)_p \neq 0$ iff $p \in S \cap \varphi^{-1}(D)$

this is a finite set because it is discrete and S is compact.

Example: $C = \{F(X,Y,Z) = 0\}$ a smooth projective plane curve and

$j : C \hookrightarrow \mathbb{P}^2$ the embedding

Take a line $L = \{aX + bY + cZ = 0\}$ s.t. $C \not\subseteq L$. Then

$$j^*L = \sum_{p \in C \cap L} \mu_p(C,L) \cdot p$$

if in general D is another plane curve

$$\boxed{j^*D = \sum_{p \in C \cap D} \mu_p(C,D) \cdot p}$$

In particular, in this case we see that all divisors j^*L for L a line are linearly equivalent on C:

$$L_1 = \{ \ell_1(X,Y,Z) = 0 \}, \quad L_2 = \{ \ell_2(X,Y,Z) = 0 \}$$

then

$$j^*L_1 - j^*L_2$$

$$= \sum_{p \in C \cap L_1} \mu_p(C,L_1) \cdot p - \sum_{p \in C \cap L_2} \mu_p(C,L_2) \cdot p$$

$$= \operatorname{div}\left(\frac{\ell_1(X,Y,Z)}{\ell_2(X,Y,Z)} \right).$$

This holds in general

Prop: Let $\varphi = [f_0, \dots, f_r] : S \to \mathbb{P}^r$ be a holomorphic map, with the f_i meromorphic functions. Assume that the map φ is nondegenerate, meaning that $\varphi(S)$ is not contained in a hyperplane.

Take two hyperplanes

$$H_1 = \{ \sum a_i x_i = 0 \}$$

$$H_2 = \{ \sum b_i x_i = 0 \}$$

Then

$$\boxed{\varphi^* H_1 - \varphi^* H_2 = \operatorname{div}\left(\frac{\sum a_i f_i}{\sum b_i f_i} \right)}$$

Proof: we can assume that $H_2 = H_0 = \{ X_0 = 0 \}$ because if we prove it in this case then

$$\varphi^* H_1 - \varphi^* H_2 = \varphi^* H_1 - \varphi^* H_0 + \varphi^* H_0 - \varphi^* H_2$$

$$= (\varphi^* H_1 - \varphi^* H_0) - (\varphi^* H_2 - \varphi^* H_0)$$

$$= \operatorname{div}\left(\frac{\sum a_i f_i}{f_0}\right) - \operatorname{div}\left(\frac{\sum b_i f_i}{f_0}\right)$$

$$= \operatorname{div}\left(\frac{\sum a_i f_i}{\cancel{f_0}} \cdot \frac{\cancel{f_0}}{\sum b_i f_i}\right)$$

Now we need to prove that

$$\varphi^* H_1 - \varphi^* H_0 = \operatorname{div}\left(\frac{\sum a_i f_i}{f_0}\right)$$

$$= \operatorname{div}\left(a_0 \frac{f_0}{f_0} + a_1 \frac{f_1}{f_0} + \dots + a_r \frac{f_r}{f_0}\right)$$

and since

$$[f_0, f_1, \dots, f_r] = \left[1, \frac{f_1}{f_0}, \dots, \frac{f_r}{f_0}\right]$$

we can also assume that $f_0 = 1$:

$$\boxed{\varphi^* H_1 - \varphi^* H_0 = \operatorname{div}\left(\sum a_i f_i\right)}$$

Fix a point $p \in S$. Suppose first that none of the f_i has a pole at p. Then $\varphi(p)$ belongs to the chart $\{x_0 \neq 0\}$

and in there we have the affine equations

$$H_1 \cap \{x_0 \neq 0\} = \{a_0 + a_1 \cdot x_1 + \dots + a_r \cdot x_r = 0\}$$

$$H_0 \cap \{x_0 \neq 0\} = \phi \text{, so}$$

Then

$$\boxed{(\varphi^* H_1)_p - (\varphi^* H_0)_p = \operatorname{ord}_p(a_0 + a_1 f_1 + \dots + a_r f_r)}$$

Suppose one of the f_i has a pole at p and suppose f_j is the one such that

$$\operatorname{ord}_p(f_j) = \min\{\operatorname{ord}_p(f_i)\}$$

Then around p we can write

$$\varphi(z) = [f_0, f_1, \dots, f_r]$$
$$= \left[\frac{f_0}{f_j}, \frac{f_1}{f_j}, \dots, \frac{f_r}{f_j}\right]$$

and all the f_i/f_j are holomorphic around p and $\frac{f_j}{f_j} = 1$, so

$\varphi(p) \in \{x_j \neq 0\}$. In this affine chart

$$H_1 \cap \{X_j \neq 0\} = \{a_0 x_0 + \dots + a_j + \dots + a_r x_r = 0\}$$

$$H_0 \cap \{X_j \neq 0\} = \{x_0 = 0\}$$

so

$$(\varphi^* H_1)_p - (\varphi^* H_0)_p =$$

$$= \operatorname{ord}_p\left(a_0 \frac{f_0}{f_j} + \dots + a_j + \dots + a_r \frac{f_r}{f_j}\right)$$

$$- \operatorname{ord}_p\left(\frac{f_0}{f_j}\right)$$

$$= \operatorname{ord}_p\left(\frac{a_0 f_0 + a_1 f_1 + \dots + a_r f_r}{f_0}\right)$$

which is what we want. $\square$